The hyperplanes of the M_{24} near polygon

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Abstract

We give a complete description of the geometric hyperplanes of the 759-point near hexagon belonging to M_{24}.

1. Introduction

Consider the unique Steiner system $S(5, 8, 24)$ on a set Ω of 24 symbols. By $\Gamma = (P, L)$ we denote the partial linear space (with points set P and line set L) obtained by taking the 759 octads as points, and the triples of pairwise disjoint octads as lines. The space Γ is the (unique) regular *near hexagon* of order $(s, t, t_2) = (2, 14, 2)$ on 759 points; its automorphism group is isomorphic to the Mathieu group M_{24}, see [2, 3, 7].

A *geometric hyperplane*, or hyperplane for short, of a partial linear space is a subset of the point set which meets each line in one or all points of the line. In this note we determine the M_{24}-orbits on the hyperplanes of Γ, completing, correcting and simplifying the results of Chapter 4 of [5]. This classification is used in the study of circular extensions of Γ, see [4].

2. The hyperplanes of the near hexagon Γ

Let $\Gamma = (P, L)$ be the near hexagon on 759 points related to M_{24}. (We use the notation introduced in the previous section.) In this section we determine the M_{24}-orbits on the geometric hyperplanes of Γ. But first we discuss some results on embeddings of Γ.

Projective embeddings. A (*projective*) *embedding* of $\Gamma = (P, L)$ is an injective map ϕ of P into the point set of a projective space $\mathbb{P}$ such that lines are mapped onto lines and $\phi(P)$ generates $\mathbb{P}$. The near hexagon Γ admits three different group-admissible embeddings, i.e., embeddings into a projective space for which $\mathrm{Aut}(\Gamma) \simeq M_{24}$ acts as a subgroup of the projective linear group on the embedded near hexagon.

First of all, there is the universal embedding into $P(U)$, where U is the quotient of $\mathbb{F}_2 P$ by the subspace K spanned by the various vectors $\Sigma_{x \in l}\, x$, where $l \in L$. Here a point p is mapped to $p + K$. By [6], the dimension of U is 23. Any other embedding is a quotient of this universal embedding.

For every point $p \in P$, the points at distance at most 2 from p form a hyperplane which we denote by $p^\perp$. The map $p \in P \mapsto p^\perp$ yields an embedding into the subspace of

$P(U^*)$, where U^* is the dual of U. This subspace is 22-dimensional and the embedding is called the *near polygon embedding*, see [3]. Finally, we can embed Γ into the 11-dimensional Golay code module for M_{24}, see [1]. Indeed, if we define the *extended binary Golay code* $\mathcal{G}$ to be the subspace of the 24-dimensional space $\mathbb{F}_2\Omega$ generated by $\Sigma_{x\in B}\ x$, where B runs through the set of octads of the Steiner system $S(5,8,24)$, then the *Golay code module* is the space $\mathcal{G}/\langle v\rangle$ where $v=\Sigma_{x\in\Omega}\ x$, and the map $p\in P\mapsto p+\langle v\rangle$ is an embedding of Γ into this 11-dimensional Golay code module, see [1].

Hyperplanes. The M_{24}-orbits on the geometric hyperplanes of the near hexagon Γ are listed in the table below. We explain the notation of this table and how to find the various M_{24}-orbits on the set hyperplanes.

Let H_1 and H_2 be two hyperplanes of Γ. Then it is easily seen that the complement H_1+H_2 of the symmetric difference of H_1 and H_2 is also a geometric hyperplane. So, the hyperplanes form a $\mathbb{F}_2$ vector space, V say, which is isomorphic to the dual of the universal embedding space U of Γ, see [6]. Below we will analyze the action of M_{24} on this space of geometric hyperplanes. In particular, we determine the orbits of M_{24} on this module.

A 12-dimensional subspace. First we describe some particular hyperplanes of Γ, i.e., elements of V. The trivial hyperplane consisting of all octads is denoted by 0, as it is the zero element in V. If one fixes a symbol o from Ω, then each line of Γ contains a unique octad containing o. Thus the set of 253 octads containing this symbol is a hyperplane without any lines, i.e., it is an *ovoid* of Γ. This ovoid will also be denoted by the symbol o. As there are 24 elements in Ω, this yields 24 hyperplanes.

We will now determine all the hyperplanes in the subspace $V_{\mathcal{T}}$ of V generated by the 24 symbols in Ω. The sum of i different symbols, $i\leq 24$, say $o_1+\cdots+o_i$, is the geometric hyperplane consisting of the octads missing an even number of symbols from $o_1,\ldots,o_i$. If B is an octad, then every other octad meets B in an even number of points. Thus the hyperplane $\Sigma_{o\in B}\ o$ contains every octad and is equal to 0. Since a hyperplane which is the sum of at most 7 symbols is easily seen to be nontrivial, we find that the submodule of V generated by the hyperplanes o, $o\in\Omega$, consists of sums of 0, 1, 2, 3 or 4 symbols. Moreover, since every *tetrad* (set of 4 symbols) is in exactly 5 octads, we can describe a hyperplane, which is the sum of 4 symbols, in exactly 6 different ways as the sum of 4 symbols. The corresponding six tetrads form a partition of Ω, also called a *sextet*. We notice that the $\binom{6}{2}$ octads that are the union of two tetrads in the sextet form a quad in Γ. Hence, the group M_{24} has five orbits on $V_{\mathcal{T}}$, of size 1, $\binom{24}{1}$, $\binom{24}{2}$, $\binom{24}{3}$ and $\frac{1}{6}\binom{24}{4}$, and denoted by 0, o, oo, ooo and oooo, respectively. The submodule $V_{\mathcal{T}}$ is isomorphic to the 12-*dimensional Todd module* for M_{24}. Its 11-dimensional submodule consisting of the hyperplanes in the orbits 0, oo and oooo is the 11-*dimensional Todd module*, see [1].

The remaining hyperplanes. The near polygon embedding in 22-space is self dual and admits a quotient isomorphic to the Golay code module for M_{24}. The kernel of this quotient map is then the 11-dimensional Todd module described above. This implies that the 23-dimensional space of hyperplanes V admits an 11-dimensional quotient isomorphic to the Golay code module with kernel the 12-dimensional Todd module $V_{\mathcal{T}}$ described above.

The group M_{24} has two nontrivial orbits on the Golay code module, one of size 759 corresponding to the octads of the Steiner system, and one of size 1288 corresponding to duums. A *duum* is a partition of Ω in two *dodecads* (i.e., sets of size 12 that are the symmetric difference of two octads meeting in 2 points). To each octad B we can associate the hyperplane, labeled by B, consisting of all octads at distance at most 2 from B in Γ, i.e., octads meeting B in 8, 4 or 0 points. The M_{24}-orbit of hyperplanes labeled by an octad is denoted by B. If D is a dodecad then each octad meets D in 2, 4 or 6 points. This implies that either one or all three octads on a line of Γ meets D in 4 points. So to D we can associate the hyperplane, labeled by D, of all the octads meeting D in 4 points. The dodecad $\Omega \setminus D$ determines the same hyperplane as D. The M_{24}-orbit of size 1288 on hyperplanes labeled by dodecads is denoted by D. The hyperplanes labeled by octads and dodecads are the representatives of the various nontrivial cosets of $V_{\mathcal{T}}$ in V.

The set of hyperplanes that are a sum of a hyperplane labeled by an octad B and a hyperplane labeled by up to 3 symbols in or outside the octad is denoted Bi, Bo, Bii, etc. Since the stabilizer of an octad B induces the full alternating group on the symbols in B and $2^4 : L_4(2)$ on the 16 symbols outside B, we easily see that the group M_{24} is transitive on the sets Bi, Bo, ..., Booo. If we fix an octad B and a sextet S, then B is either the union of two tetrads of S, or it meets 4 tetrads in 2 points, or it meets each tetrad from S in one or three points. In other words, the point B of Γ is either in the quad of Γ determined by the sextet S, at distance 1 or at distance 2 from this quad. This leads to three M_{24}-orbits on hyperplanes denoted by Biiii, Biioo and Biiio, respectively.

The set of hyperplanes that are a sum of a hyperplane labeled by a dodecad D and a hyperplane labeled by up to 3 symbols in or outside the dodecad is denoted Di, Do, Dii, etc. The stabilizer of a dodecad D induces the group M_{12} on D in one of its 5-transitive actions. On the complement of D it also induces the group M_{12} on D in one of its 5-transitive actions, but the two actions on D and its complement are not the same.

From this one easily derives that the group M_{24} is transitive on the sets Di, Do, ..., Diio. (Notice that by interchanging the rôle of D and its complement we may assume that the majority of the symbols is inside D.) It remains to describe the M_{24}-orbits on hyperplanes that are sums of hyperplanes labeled by a dodecad and hyperplanes labeled by four symbols. Fix a dodecad D. Then have one of the following situations for a sextet S. One tetrad of S is in D, one misses D and the other four meet D in two points; since the stabilizer of D induces the 5-transitive group M_{12} on D, this leads to an orbit on hyperplanes of size $1288 \times \binom{12}{4}$ denoted by Diiii. There are three tetrads in S meeting D in 3 points and three meeting it in 1 point. This leads to an orbit of hyperplanes of size $\binom{12}{3} \times 12 \times \frac{1}{3} \times 1288$ denoted by Diiio. Finally, the remaining 396 sextets S only have tetrads meeting D in 2 points. This leads to an M_{24}-orbit of hyperplanes of size 396×1288 denoted by Diioo.

The table. In the table below, we give for each M_{24}-orbit on the hyperplanes, its name (explained above), the universal embedding dimension of the partial linear space induced on a hyperplane in the orbit (found by computer), the size of the hyperplanes in the orbit (viewed as collection of octads), the size of the orbit, and the *line distribution* of a hyperplane H in the orbit: in i^{a_i} the number a_i is the number of points of H that are on precisely i lines entirely contained within H. Moreover the name used in [5] is also given in the column EWL name. Under the header submodule we describe in which submodule an orbit can be found, the 11-dimensional Todd module 11-*Todd*, the 12-dimensional Todd module 12-*Todd*, the near polygon embedding module *Near Polygon* and the dual universal space *Universal*. (Notice that 11-Todd is contained in 12-Todd and Near Polygon and that all are contained in Universal.) Since the orbit sizes add up to the right number 2^{23}, the list is complete. The line distribution helps to identify a given orbit with one from the list.

M_{24}-orbits on the hyperplanes of Γ

EWL name	name	udim	size	orbit size	submodule	line distribution
(X)	0	23	759	1	11-Todd	15^{759}
(H13)	oo	22	407	276	11-Todd	$7^{330}15^{77}$
(H4)	oooo	23	375	1771	11-Todd	$7^{360}15^{15}$
	o	253	253	24	12-Todd	0^{253}
	ooo	43	381	2024	12-Todd	$0^{21}8^{360}$
(H1)	B	51	311	1×759	Near Polygon	$3^{280}15^{31}$
(H14)	Bii	23	343	28×759	Near Polygon	$3^{60}5^{192}7^{30}11^{60}15^{1}$
(H7)	Bio	23	351	128×759	Near Polygon	$3^{35}5^{126}7^{120}9^{70}$
(H10)	Boo	23	407	120×759	Near Polygon	$3^{14}7^{120}9^{224}11^{42}15^{7}$
(H2)	Biiii	23	439	35×759	Near Polygon	$3^{8}7^{24}9^{256}11^{144}15^{7}$
(H9)	Biiio	23	383	896×759	Near Polygon	$3^{5}5^{78}7^{120}9^{150}11^{30}$
(H5)	Biioo	23	375	840×759	Near Polygon	$3^{6}5^{96}7^{126}9^{128}11^{18}15^{1}$
	D	22	495	1×1288	Near Polygon	11^{495}
(H3)	Dii	23	367	132×1288	Near Polygon	$3^{30}5^{72}7^{180}9^{40}11^{45}$
(H11)	Dio	23	407	144×1288	Near Polygon	$5^{22}7^{165}9^{110}11^{110}$
(H12)	Diiii	23	367	495×1288	Near Polygon	$3^{24}5^{64}7^{192}9^{64}11^{23}$
(H6)	Diiio	23	375	880×1288	Near Polygon	$3^{18}5^{54}7^{189}9^{78}11^{36}$
(H8)	Diioo	23	399	396×1288	Near Polygon	$5^{24}7^{180}9^{120}11^{75}$
	Bi	23	477	8×759	Universal	$0^{1}10^{336}12^{140}$
	Bo	37	365	16×759	Universal	$0^{15}6^{280}12^{70}$
	Biii	24	349	56×759	Universal	$0^{1}4^{120}6^{160}10^{48}12^{20}$
	Biio	23	397	448×759	Universal	$4^{15}6^{100}8^{135}10^{132}12^{15}$
	Bioo	23	381	960×759	Universal	$4^{21}6^{140}8^{129}10^{84}12^{7}$
	Booo	26	365	560×759	Universal	$0^{3}4^{36}6^{168}8^{108}10^{48}12^{2}$
	Di	23	341	24×1288	Universal	$4^{165}6^{110}10^{66}$
	Diii	23	405	440×1288	Universal	$4^{9}6^{54}8^{216}10^{90}12^{36}$
	Diio	23	373	1584×1288	Universal	$4^{55}6^{90}8^{180}10^{38}12^{10}$

Subspaces. A subset S of a partial linear space is called *subspace* if it contains every line that meets it in at least two points. The partial linear space Γ has many subspaces, and one may wonder whether any reasonable classification is possible.

A way to obtain many subspaces is the following: Let G be an abelian group, and $u = (u_\omega)$ a vector indexed by Ω with elements in G such that $\sum u_\omega = 0$. Then $S(u) := \{B | \sum_{\omega \in B} u_\omega = 0\}$ is a subspace. If we call such subspaces *abelian*, then the intersection of two abelian subspaces is again abelian.

For example, let us take $G = \mathbb{Z}_{16}$, then, in obvious notation, we have: oo $= S(8^2 0^{22})$, B $= S(4^8 0^{16})$, D $= S(4^{12} 0^{12})$, o $= S(7^1 (-1)^{23})$. We see that there is an embedding (of abelian groups) of V into the quotient of $\mathbb{Z}_{16}^{24}$ by the subgroups spanned by twice the all-1 vector and 8 times the extended binary Golay code; in the image all coordinates are congruent mod 4 and sum to zero.

A few more large subspaces (that are not hyperplanes) are found in the same way. For example, the subspace $S(u)$ where $u = (2^4 (-2)^4 0^{16})$ (with the $4+4$ positions forming an octad) is a subspace on 431 points and udim 22, contained in the hyperplane $S(2u)$ of type Biiii with 439 points. Similarly, the subspace $S(u)$ with $u = (1^{11} (-3)^1 (-1)^{11} 3^1)$ (with the $11+1$ positions forming a dodecad) is a subspace on 385 points and udim 22 contained in the hyperplane $S(2u)$ of type Dio with 407 points.

In this way it happens that most hyperplanes have udim 23 again—they are of the form $S(2u)$ and have hyperplanes $S(u)$ not obtained by intersection with a hyperplane in the entire space. (Note that u is not determined by $2u$.) Of course, isolated points, visible in the line distribution as points on zero lines, each add 1 to udim. Finally hyperplanes of type B are unions of a bouquet of 35 quads on a point, and visibly have udim $1+15+35 = 51$.

Spanning. A subset A is said to *span* a subspace S when S is the smallest subspace containing A. Clearly, this implies that $|A| \geq \mathrm{udim}(S)$. Cooperstein asked whether Γ is spanned by 23 points, and this is indeed the case, as computer calculation reveals. More generally, each of the hyperplanes H listed in the table can be spanned by $\mathrm{udim}(H)$ points. Of course there do exist partial linear spaces S with lines of size 3, for which one needs more than $\mathrm{udim}(S)$ points to span. For example, the affine plane $AG(2,3)$ on 9 points is spanned by 3 points but has no hyperplanes so that its udim is 0.

One can manufacture a less trivial example S with $\mathrm{udim}(S) = 6$ that requires 7 points to span as follows. Let a *tripod* with *feet* p, q, r be a set of seven points, say x, a, b, c, p, q, r, and four lines, namely xap, xbq, xcr, abc. Let S be the partial linear space with 21 points and 15 lines obtained by taking the union of three tripods with feet p_i, q_i, r_i $(i = 1, 2, 3)$ and adding the three lines $p_1p_2p_3, q_1q_2q_3, r_1r_2r_3$. One easily checks that at least 7 points are required to span S.

A tripod admits 8 geometric hyperplanes, each determined by the feet it contains. So, S has exactly 2^6 hyperplanes, each uniquely determined by its intersection with the three lines $p_1p_2p_3, q_1q_2q_3, r_1r_2r_3$. As we have seen before, these hyperplanes form an $\mathbb{F}_2$ vector space, V say. For any pair of points we can find a hyperplane containing one and not the other point. So, mapping a point of S to the set of hyperplanes containing it, yields an embedding into $P(V^*)$. This embedding is universal and $\mathrm{udim}(S) = 6$.

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