The geometry of secants in embedded polar spaces

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June 1, 2006

Abstract

Consider a polar space $\mathcal{S}$ weakly embedded in a projective space $\mathbb{P}$. A secant of $\mathcal{S}$ is the intersection of the point set of $\mathcal{S}$ with a line of $\mathbb{P}$ spanned by two non-collinear points of $\mathcal{S}$. The geometry consisting of the points of $\mathcal{S}$ and as lines the secants is a so-called Delta space. In this paper we give a characterization of this and some related geometries.

1 Introduction

1.1 Polar spaces and Delta spaces. A *partial linear space* is a pair $\Pi = (P, L)$ consisting of a set P whose elements are called *points* and a set L of *lines* being subsets of P of size at least 2, such that any two points are on at most one line. Two points are called *collinear* whenever there is a line containing them both. A *subspace* X of Π is a subset of P with the property that any line intersecting it in at least two points is completely contained in X. Often a subspace X is identified with the partial linear space $(X, \{l \in L \mid l \subseteq X\})$. A partial linear (sub)space is called *linear* if any two points in it are collinear. Since the intersection of subspaces is again a subspace, we can, for each subset Y of P, define the subspace $\langle Y\rangle_\Pi$ *generated by* Y to be the intersection of all subspaces containing Y. The *collinearity graph* of a partial linear space (P, L) is the graph with P as vertex set and two vertices adjacent if and only if they are collinear. A partial linear space is called *connected* if its collinearity graph is connected. It is called *coconnected* if the complement of the collinearity graph is connected.

A *polar space* $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a partial linear space satisfying the *one or all* axiom:

> a point p not on a line l is collinear with one or all points of l.

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a polar space. Then for each point $x \in \mathcal{P}$ we denote by $x^\perp$ the set of all points y with y being equal to or collinear with x. We write $x \perp y$ if $y \in x^\perp$. Notice that $x^\perp$ is a subspace of the polar space $\mathcal{S}$. For any subset X of $\mathcal{P}$, we set $X^\perp$ to be $\bigcap_{x \in X} x^\perp$.

The *radical* of the polar space $\mathcal{S}$ is the set of points x with $x^\perp = \mathcal{P}$; it equals $\mathcal{P}^\perp$. The polar space is called *non-degenerate* if its radical is empty.

A partial linear space Δ is called a *Delta space* if

> for each point x and line l not on x, the point x is collinear to 0, all but one or all points of l.

Let $\Delta = (P, L)$ be a Delta space. Then for each point x the subset $x^\perp$ consists of x and all points y *not* collinear to x. We also write $x \perp y$ if $y \in x^\perp$. In this case we write $x \sim y$ whenever x and y are collinear. Again, for any subset X of P, we set $X^\perp$ to be $\bigcap_{x \in X} x^\perp$. Both the subset $x^\perp$ as well as the subset $\Delta_x := x^\perp \setminus \{x\}$ are subspaces of Δ.

Let X be a connected subspace of Δ. Then X is called a (connected) *transversal* subspace if for each line $l \subseteq X$ the sets $x^\perp \cap X$, where x runs through l, partition X. An arbitrary subspace is called *transversal* if each of its connected components is transversal. Notice that a subspace without any lines is transversal.

1.2 Weak or polarized embeddings. Suppose $\mathbb{P}_1 = (P_1, L_1)$ and $\mathbb{P}_2 = (P_2, L_2)$ are two projective spaces. A *weak embedding* of $\mathbb{P}_1$ into $\mathbb{P}_2$ is a map φ from P_1 to P_2 satisfying the following.

(a) φ is injective;

(b) $\langle \varphi(P_1) \rangle_{\mathbb{P}_2} = \mathbb{P}_2$;

(c) for every line $l \in L_1$ we have that $\langle \varphi(l) \rangle_{\mathbb{P}_2}$ is a line of $\mathbb{P}_2$ meeting $\varphi(P_1)$ in $\varphi(l)$.

Suppose $\mathbb{P}_1$ and $\mathbb{P}_2$ are both projective space of (projective) dimension at least 3. Then, by Theorem 2.4 of [8] , a weak embedding of $\mathbb{P}_1$ into $\mathbb{P}_2$ is induced by an injective semi-linear mapping between the underlying vector spaces. In particular, if $\mathbb{P}_1 = \mathbb{P}(V_1)$ and $\mathbb{P}_2 = \mathbb{P}(V_2)$ for some vector spaces V_1 and V_2 over some skew fields K_1 and K_2, respectively, then there is an embedding of K_1 into K_2 and a map $\Phi : V_1 \to V_2$, which is semi-linear with respect to the embedding of K_1 into K_2, such that $\varphi(\langle v_1 \rangle) = \langle \Phi(v_1) \rangle$ for all non-zero $v_1 \in V_1$.

We generalize this concept of weak embeddings to both polar spaces and Delta spaces, see also [12, 13, 14, 15, 16, 17, 18].

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a polar space. A *weak* or *polarized embedding* of $\mathcal{S}$ is a map φ from $\mathcal{P}$ into a projective space $\mathbb{P}$ satisfying the following.

(a) φ is injective;

(b) $\langle \varphi(\mathcal{P}) \rangle_{\mathbb{P}} = \mathbb{P}$;

(c) for every line $l \in \mathcal{L}$ we have that $\langle \varphi(l) \rangle_{\mathbb{P}}$ is a line of $\mathbb{P}$ meeting $\varphi(\mathcal{P})$ in $\varphi(l)$;

(d) for every point $x \in \mathcal{P}$ we have that $\langle \varphi(x^\perp) \rangle_{\mathbb{P}} \cap \varphi(\mathcal{P}) \subseteq \varphi(x^\perp)$.

Similarly we can define polarized embeddings of Delta spaces. Let $\Delta = (P, L)$ be a Delta space and $\mathbb{P}$ a projective space. A map φ from P into the point set of $\mathbb{P}$ is called a *weak* or *polarized embedding* of Δ if the following hold:

(a) φ is injective;

(b) $\langle \varphi(P) \rangle_{\mathbb{P}} = \mathbb{P}$;

(c) for every line $l \in L$ we have that $\langle \varphi(l) \rangle_{\mathbb{P}}$ is a line of $\mathbb{P}$ meeting $\varphi(P)$ in $\varphi(l)$;

(d) for every point $x \in P$ we have that $\langle \varphi(\Delta_x) \rangle_{\mathbb{P}} \cap \varphi(P) \subseteq \varphi(x^\perp)$.

A polar space or Delta space is called *weakly* or *polarized embedded* in a projective space $\mathbb{P}$ if its point set is a subset of the point set of $\mathbb{P}$, and the identity map on this point set is a polarized embedding of the space into $\mathbb{P}$.

Suppose φ and ψ are two weak embeddings of a polar or Delta space into a projective space $\mathbb{P}_1$ and $\mathbb{P}_2$, respectively. Then we say ψ is *induced by* φ if there exists a weak embedding χ of $\mathbb{P}_1$ into $\mathbb{P}_2$ with $\psi = \chi \circ \varphi$.

Weakly embedded polar spaces received a lot of attention, see for example [12, 13, 14, 15, 16, 17, 18]. In particular, the results of Steinbach and Van Maldeghem [13, 14, 15, 16] describe the weak embeddings of the non-degenerate classical polar spaces of rank at least 2.

1.3 Weak embeddings of Delta spaces. In this paper, we will investigate weakly embedded Delta spaces. Examples of weakly embedded Delta spaces can be obtained from weakly embedded polar spaces. Indeed, suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a polar space weakly embedded into some projective space $\mathbb{P}$. A *secant* of $\mathcal{S}$ is the intersection of $\mathcal{P}$ with a line of $\mathbb{P}$ spanned by two non-collinear points in $\mathcal{P}$. If we set P to be the set of all points of $\mathcal{S}$ outside the radical, and L the set of secants of $\mathcal{S}$, then we claim $\Delta = (P, L)$ to be a Delta space weakly embedded in $\langle P \rangle_{\mathbb{P}}$. We only have to check that a point $x \in P$ is collinear to 0, all or all but one of the points of a secant line l not on p. So indeed, if $y \neq z \in l$ are in $x^\perp$, then $y, z \in \langle x^\perp \rangle_{\mathbb{P}}$. But then $l \subseteq \langle x^\perp \rangle_{\mathbb{P}} \cap P = x^\perp$, proving that Δ is a Delta space. The space Δ obtained from $\mathcal{S}$ is called the *geometry of secants* of the embedded polar space $\mathcal{S}$.

A second class of weakly embedded Delta spaces can be obtained from unitary spaces over the field $\mathbb{F}_{2^2}$ in their natural embedding in a projective space $\mathbb{P}$ of order 4. This time however, the points of the Delta space are the non-isotropic points of $\mathbb{P}$ and as lines we take the sets of non-isotropics in a tangent line, i.e., a line of $\mathbb{P}$ containing a single isotropic point. This yields a Delta space in which each line contains 4 points. It is called the *geometry of tangents* to the unitary polar space.

Due to the exceptional isomorphism of the groups $\mathrm{PSp}_4(3)$ and $\mathrm{SU}_4(2)$ we encounter the Delta space of secants of the symplectic polar space $\mathrm{Sp}(4, 3)$ also as the geometry of tangents of the unitary space $\mathrm{SU}(4, 2)$. However, this geometry admits many more weak embeddings, all related to the existence of the representation of the group $3 \times \mathrm{Sp}_4(3)$ as a (complex) reflection group; see [1].

Indeed, consider a 4-dimensional (left) vector space V over a skew field $\mathbb{K}$ containing some element ω of order 3. Then the projective points

$$\langle(1,0,0,0)\rangle, \langle(0,1,0,0)\rangle, \langle(0,0,1,0)\rangle, \langle(0,0,0,1)\rangle,$$

$$\langle(0,1,-\omega^i,\omega^j)\rangle, \langle(-\omega^i,0,1,\omega^j)\rangle, \langle(\omega^i,-\omega^j,0,1)\rangle, \langle(1,\omega^i,\omega^j,0)\rangle$$

with $i, j \in \{0, 1, 2\}$ form the point set of a Delta space isomorphic to the space of secants of the symplectic polar space $\mathrm{Sp}(4, 3)$. The lines of this Delta space are the 4-tuples of points inside a projective line.

The subspace consisting of the 12 points with last coordinate equal to 0 provides a weak embedding of the geometry of secants of $\mathrm{Sp}(3, 3)$, which is isomorphic to the dual affine plane of order 3.

In this paper we provide a geometric characterization of an important subset of these examples.

1.4 Main Theorem. *Let $\Delta = (P, L)$ be a connected Delta space with at least 4 points per line and at least 2 lines per point, weakly embedded in a Desarguesian projective space $\mathbb{P}$ such that the following conditions hold.*

(a) *Suppose $x, y \in P$. If $x^\perp \subseteq y^\perp$ then $x^\perp = y^\perp$.*

(b) *For any two intersecting lines l and m in L the intersection of P with $\langle l, m\rangle_{\mathbb{P}}$ is either a linear subspace of Δ or a transversal subspace of Δ.*

(c) *Δ is not transversal.*

(d) *For any two lines $l, m \in L$ in a linear subspace of Δ there is a point $x \in P$ with $x^\perp$ containing l but not m.*

Then we have one of the following three possibilities.

(a) *Δ is the geometry of secants of a polar space $\mathcal{S}$ weakly embedded into $\mathbb{P}$.*

(b) *Δ is isomorphic to the geometry of tangents to a unitary space over $\mathbb{F}_4$, the embedding into $\mathbb{P}$ is induced from the natural embedding.*

(c) *Δ is isomorphic to the geometry of secants of* $\mathrm{Sp}(4, 3)$. *Moreover, there exists a basis of the underlying vector space of $\mathbb{P}$, and an element ω of order 3 in the underlying skew field, such that the embedding of Δ is as described in* (1.3).

In the proof of the above theorem we distinguish between the geometries in the three different parts of the conclusion in the following way. If there is a point $x \in P$ with $x \in \langle \Delta_x\rangle_{\mathbb{P}}$, then we find the geometries in part (a) of the conclusion of Theorem 1.4. In the two remaining cases (b) and (c) the geometries contain points $x \in P$ with $x \notin \langle \Delta_x\rangle_{\mathbb{P}}$; they are distinguished by the existence of two intersecting lines in a linear subspace (leading to case (b)) or the non-existence of such pair of lines (leading to case (c)). Each of these cases is dealt with in one of the following three theorems.

1.5 Theorem. *Let $\Delta = (P, L)$ be a connected and coconnected Delta space with at least 4 points per line, weakly embedded in a projective space $\mathbb{P}$ such that the following conditions hold.*

(a) *Every triple of distinct points x, y and z with $x \sim y \sim z \perp x$ generates a projective subspace of $\mathbb{P}$ intersecting Δ in a transversal subspace.*

(b) *For all $x, y \in P$ with $x^\perp \subseteq y^\perp$ we have $x^\perp = y^\perp$.*

(c) *There exist non-collinear x, y with $x^\perp \neq y^\perp$.*

(d) *There is a point $x \in P$ with $x \in \langle \Delta_x \rangle_{\mathbb{P}}$.*

Then Δ is the geometry of secants of a polar space $\mathcal{S}$ weakly embedded into $\mathbb{P}$.

The above theorem can be viewed as a geometric version of the results of Cuypers and Steinbach [9] on linear groups generated by transvections.

In the next two results we consider the case where there is a point $x \in P$ with $x \notin \langle \Delta_x \rangle_{\mathbb{P}}$.

1.6 Theorem. *Let $\Delta = (P, L)$ be a connected Delta space with at least 4 points per line, weakly embedded in a projective space $\mathbb{P}$ such that the following conditions hold.*

(a) *Any two intersecting lines of L generate a subspace of $\mathbb{P}$ meeting Δ in a linear or transversal subspace.*

(b) *For any two lines l, m of L in a linear subspace of Δ there is a point $x \in P$ with $x^\perp$ containing l but not m.*

(c) *There exist two intersecting lines in some linear subspace of Δ.*

(d) *There is a point $x \in P$ with $x \notin \langle \Delta_x \rangle_{\mathbb{P}}$.*

Then Δ is isomorphic to the geometry of tangents to a unitary polar space of (projective) dimension at least 4 over $\mathbb{F}_4$. The embedding into $\mathbb{P}$ is induced from the natural embedding of the unitary polar space.

Finally we have to consider the third case.

1.7 Theorem. *Suppose $\Delta = (P, L)$ is a connected Delta space with at least two lines weakly embedded into the Desarguesian projective space $\mathbb{P}$. If every pair of intersecting lines is contained in a transversal subspace of Δ and there exists a point $x \in P$ with $x \notin \langle \Delta_x \rangle_{\mathbb{P}}$, then Δ is isomorphic to the geometry of hyperbolic lines of the symplectic space* $\mathrm{Sp}(4, 3)$ *or of* $\mathrm{Sp}(3, 3)$. *The embedding into $\mathbb{P}$ is as described in* (1.3).

If lines contain 3 points, then the above results as well as our methods of proof fail. Various examples of weakly embedded Delta spaces with three points per line can be found among the so called Fischer spaces; we refer the reader to [3, 5].

The remainder of this paper is organised as follows. In Section 2 we show that the geometry of secants of an important class of weakly embedded polar spaces does satisfy the hypotheses of Theorem 1.4 and 1.5. Section 3 is devoted to Delta spaces in general and transversal subspaces in particular. The proof of Theorem 1.5 is given in the Sections 4. The exceptional case leading to Theorem 1.7 is handled in Section 5, while Section 6 contains a proof of Theorem 1.6. In the final section we show how Theorem 1.4 follows from the Theorems 1.5, 1.6 and 1.7 .

Acknowledgment. The author would like to thank Antonio Pasini for various useful remarks concerning the topic of this paper and pointing out some mistake in an earlier version of this paper.

2 Secants of embeddable polar spaces

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a thick (i.e., lines contain at least 3 points) polar space of rank at least 2 weakly embedded into a projective space $\mathbb{P}$. Let $\mathcal{R}$ be the radical of $\mathcal{S}$ and R the subspace of $\mathbb{P}$ generated by this radical. We assume $\mathcal{S}$ to be distinct from $\mathcal{R}$. Denote by $\Delta = (P, L)$ the geometry of secants of $\mathcal{S}$ in $\mathbb{P}$. The lines in $\mathcal{L}$ will be called *singular* lines, those in L *secants.*

2.1 (a) *If x and y are points of Δ, then $x^\perp \subseteq y^\perp$ implies $x^\perp = y^\perp$.*

(b) *Δ is a connected Delta space weakly embedded in $\mathbb{P}$.*

(c) *For each $x \in P$ the space $\langle \Delta_x \rangle_{\mathbb{P}}$ is a hyperplane of $\mathbb{P}$ containing x.*

Proof. (a). This is a well known property of polar spaces.
(b). That Δ is a Delta space weakly embedded in $\mathbb{P}$ was already proved in (1.3). It remains to show connectedness. So, let $x, y \in P$ be non-collinear points. Then x and y are collinear in $\mathcal{S}$. If $x^\perp = y^\perp$, then, as x and y are not in $\mathcal{R}$, there is a point $z \in P \setminus x^\perp$ which is collinear in Δ to both x and y.

Now suppose $x^\perp \neq y^\perp$. As $\mathcal{S}$ is thick, we can find a third point $z \in P$ on the line of $\mathcal{S}$ through x and y. But then $z^\perp \not\subseteq x^\perp$, for otherwise by (a) $x^\perp = z^\perp \cap x^\perp \subseteq y^\perp$ which implies, again using (a), that $x^\perp = y^\perp$, which contradicts our assumption. Thus there exists a point $u \in z^\perp$ which is collinear in Δ to x and hence also to y. This proves connectedness of Δ.
(c). Let $x, y \in P$ with $y \not\perp x$. Since there is a thick line of $\mathcal{S}$ on x, it is clear that $x \in \langle \Delta_x \rangle_{\mathbb{P}}$. The polar space $\mathcal{S}$ is generated by $x^\perp$ and y, see Cohen and Shult [6]. Thus $\mathbb{P}$ is generated by $\langle \Delta_x \rangle_{\mathbb{P}}$ and y, proving $\langle \Delta_x \rangle_{\mathbb{P}}$ to be a hyperplane. $\square$

Condition (b) of Theorem 1.4 is not automatically satisfied in the space Δ. However, under some additional assumptions condition (b) holds true.

2.2 *Let $l, m \in L$ be two intersecting secant lines. Suppose that singular lines of $\mathcal{S}$ are full lines of $\mathbb{P}$ or that the subspace $\langle l, m \rangle_{\mathbb{P}}$ misses R. Then the subspace $P \cap \langle l, m \rangle_{\mathbb{P}}$ is either linear or transversal.*

Proof. Let $l, m \in L$ be two intersecting secant lines of Δ. Then $\langle l, m \rangle_{\mathbb{P}}$ is a projective plane π. Clearly, $\pi \cap P$ is a subspace of Δ. This subspace $\pi \cap P$ is not linear if and only if it contains two non-collinear points and hence a singular line. Suppose that there is a singular line n inside this plane π. Let $x \in l$ be a point not on n and let z be the unique point of n in $x^\perp$. Then $\pi = \langle n, x \rangle_{\mathbb{P}} \subseteq \langle \Delta_z \rangle_{\mathbb{P}}$.

Consider a secant line $k \in L$ containing two points u, v of π (being different from z) and suppose s is a singular line in π (necessarily on z). Without loss of generality we can assume $v \notin s$.

As we can assume that $z \notin \mathcal{R}$, we can find a point $w \not\perp z$ inside $u^\perp \cap v^\perp$. (Indeed, let a be a point not in $z^\perp$, then there is a singular line s' on a meeting s nontrivially. The point v is on a singular line meeting s' in a point outside $z^\perp$. Now we can take w to be the unique point on the latter singular line collinear to u.) Now $\langle w^\perp \rangle_{\mathbb{P}}$ meets π in a line containing k. As each singular line in π through z contains a point in $w^\perp$, each singular line in π meets k.

If all singular lines are full projective lines, then it is obvious that each singular line in π meets k.

In any case, since the singular lines on z induce a partition of the point set of $\pi \cap P \setminus \{z\}$, the sets $x^\perp \cap \pi \setminus \{z\}$, with $x \in k$ partition the set $\pi \cap P \setminus \{z\}$. This proves $\pi \cap P$ to be transversal. $\square$

The above shows that the geometry of secants of a weakly embedded non-degenerate polar space of rank at least 2 gives rise to a geometry satisfying the hypothesis of Theorem 1.5. For a classification of weakly embedded non-degenerate polar spaces of rank at least 2 we refer to the work of Steinbach and Van Maldeghem [13, 15, 16].

3 Transversal spaces

In this section we give some general results on Delta spaces and their transversal subspaces. Let $\Delta = (P, L)$ be a Delta space with at least 3 points per line.

3.1 *Every connected subspace of* Δ *has diameter at most* 2.

Proof. Suppose v, x, y, z is a path of length 3 in the collinearity graph of Δ. By the Delta space property the points v and z are collinear to all but one of the points on the line $l = xy$ on x and y. Since l contains at least 3 points, there is a point $u \in l$ collinear to both v and z. This clearly implies that the diameter of any connected component is at most 2. □

3.2 *If X is a connected transversal subspace of Δ and $x \in X$, then $x^\perp \cap X$ is a maximal coclique of X. Moreover, each line of X meets $x^\perp \cap X$ in a unique point.*

Proof. If X is a connected transversal subspace, then for each $x \in X$ the set $x^\perp \cap X$ is a coclique. Indeed, if $x^\perp \cap X$ contains collinear points y and z then $x \in y^\perp \cap z^\perp \cap X$, contradicting that $u^\perp \cap X$ with u running through yz is a partition of X. The coclique $x^\perp \cap X$ is certainly a maximal coclique.

Now suppose l is a line of X. Then $x \in y^\perp$ for some point $y \in l$. But then $l \cap x^\perp = \{y\}$. □

3.3 *If X is a connected transversal subspace containing at least two lines, then for each $x \in X$ there is a point $y \in \Delta_x \cap X$.*

Proof. Suppose l, m are two lines in X. By (3.2) both lines meet $x^\perp \cap X$ in a point. If these intersection points are different from x, we are done. Thus assume that $x \in l \cap m$. Then let u be a point on l different from x and $z \neq x$ a point on m not in $u^\perp$. The line on u and z meets $x^\perp$ in a point $y \in X$ different from x. This point y is in $\Delta_x \cap X$. □

3.4 *Let X and Y be subspaces of Δ. If X is a transversal subspace, then $X \cap Y$ is transversal.*

Proof. Without loss of generality we can assume X to be connected. If $X \cap Y$ is a coclique, there is nothing to prove. So, suppose l is a line in $X \cap Y$. As X is partitioned by the sets $x^\perp \cap X$, with x running through l, the set $X \cap Y$ is partitioned by the sets $x^\perp \cap X \cap Y$ where $x \in l$. □

3.5 *Let x, y and z be three distinct points of Δ with $x \sim y \sim z \perp x$ contained in a transversal subspace. Then the subspace of Δ generated by x, y and z is a transversal subspace.*

Proof. This is straightforward by (3.4). □

From now on we suppose $\Delta = (P, L)$ is a Delta space satisfying the hypothesis (a) of Theorem 1.5. Let x, y, z be three non-collinear points from P with $x \sim y \sim z \perp x$. Then the transversal subspace of Δ generated by x, y and z will be called a *transversal plane* of Δ. Any maximal coclique of a transversal plane will be called a *transversal*.

3.6 *Let π be a transversal plane and l a line in π. Then each transversal contains at least $|l| - 1$ points.*

Proof. Suppose l is a line and T a transversal of π. Since π contains non-collinear points, there is a point $x \in \pi$ not on l.

First suppose x is not in T. Since $x^\perp$ meets l in just one point, there are at least $|l| - 1$ lines on x, each meeting T, see (3.2). Thus T contains at least $|l| - 1$ points.

If $x \in T$, then there is a line on x meeting l. As this line contains at least three points, it also contains a point x' not in l nor in T. Now we can apply the above with x' instead of x and find at least $|l| - 1$ points in T. □

4 The geometry of secants

In this section we start with the proof of Theorem 1.5. However, we first consider a more general situation as described in the setting below.

4.1 Setting. Let $\Delta = (P, L)$ be a connected and coconnected Delta space weakly embedded into the projective space $\mathbb{P}$. Suppose the following holds:

(a) all lines in L contain at least 4 points;

(b) if $x, y \in P$ with $x^\perp \subseteq y^\perp$, then $x^\perp = y^\perp$;

(c) there is a point $x \in P$ for which the set Δ_x is not empty;

(d) any three points x, y, z in P, with $x \sim y \sim z \perp x$ generate a subspace of $\mathbb{P}$ meeting Δ in a transversal subspace.

Notice that since Δ is assumed to be connected and coconnected, it is certainly not transversal.

If x is a point in P, then the subspace $\langle \Delta_x \rangle_{\mathbb{P}}$ of $\mathbb{P}$ will be denoted by H_x.

4.2 *If $x \neq y \in P$, then $\langle x, y \rangle_{\mathbb{P}} \cap P$ is either a line of Δ or a set of non-collinear points.*

Moreover, if $\langle x, y \rangle_{\mathbb{P}} \cap P$ contains more than 2 non-collinear points then $x \in H_x$.

Proof. If $\langle x, y \rangle_{\mathbb{P}} \cap P$ contains two collinear points, then condition (c) of (1.2) implies that $\langle x, y \rangle_{\mathbb{P}} \cap P$ is a line of Δ.

Suppose y, z are two non-collinear points in $\langle x, y \rangle_{\mathbb{P}} \cap P$, both different from x. Then $x \in \langle y, z \rangle_{\mathbb{P}} \subseteq H_x$. $\square$

4.3 *For all $x \in P$ we have $\Delta_x \neq \emptyset$.*

Proof. Suppose $x \in P$ with Δ_x empty. By condition 4.1(c) there exist $y \neq z \in P$ with $y \perp z$. Now x is collinear to both y and z. But then x, y, z are contained in a transversal subspace of Δ and (3.3) implies that Δ_x is not empty. This contradiction proves the statement. $\square$

4.4 *Each point x is in a transversal plane.*

Proof. Suppose $x \in P$ and $y \in \Delta_x$. Then, by (3.1) there is a point z collinear with both x and y. So, the subspace generated by x, y and z is a transversal plane on x. $\square$

4.5 *Let π be a transversal plane of Δ. Suppose x and y are non-collinear points of π. Then $\langle x, y \rangle_{\mathbb{P}}$ meets π in at least all but one of the points of the transversal of π on x and y.*

If a transversal of Δ contains more than 3 points, it is contained in a line of $\mathbb{P}$.

Proof. Let T be the transversal coclique of π on x and y and let z be a third point on T. Then $T\setminus\{z\}$ is contained in $H_z\cap\langle\pi\rangle_{\mathbb{P}}$. Since $\pi\not\subseteq H_x$, the intersection $H_z\cap\langle\pi\rangle_{\mathbb{P}}$ is the line $\langle x,y\rangle_{\mathbb{P}}$. So, $T\setminus\{z\}$ is contained in $\langle x,y\rangle_{\mathbb{P}}$.

If $z'\in T$ is a fourth point of T, then $z\in T\setminus\{z'\}$. Moreover, as above we find that $T\setminus\{z'\}$ is contained in $\langle x,y\rangle_{\mathbb{P}}$. This implies z and hence all of T being in $\langle x,y\rangle_{\mathbb{P}}$. So, if $|T|\geq 4$, then T is contained in the projective line spanned by any two of its elements. $\square$

4.6 *If a transversal coclique T of Δ is contained in a line of $\mathbb{P}$, then for every $x\in T$ we have $x\in H_x$.*

On the other hand, if x is a point with $x\in H_x$, then T is contained in a projective line for every transversal T on x.

Proof. Suppose T is contained in a line and x is a point of T, then, as $|T|\geq 3$, we have $x\in\langle T\setminus\{x\}\rangle_{\mathbb{P}}\subseteq H_x$.

Now suppose $x\in H_x$ for some point $x\in P$. Consider a transversal coclique T on x and a transversal plane π containing T. Then for any two points $y,z\in T$ different from x we find that $T\subseteq H_x\cap\langle\pi\rangle_{\mathbb{P}}=\langle y,z\rangle_{\mathbb{P}}$. In particular, T is contained in a line of $\mathbb{P}$. $\square$

4.7 *Suppose $x\notin H_x$, and π is a transversal plane on x. Then π is isomorphic a dual affine plane of order 3, the field $\mathbb{K}$ of coordinates of $\mathbb{P}$ contains an element ω of (multiplicative) order 3 and there is a basis of $\langle\pi\rangle_{\mathbb{P}}$ such that the coordinates of the points of π are*

$$\langle(1,0,0)\rangle,\ \langle(0,1,0)\rangle,\ \langle(0,0,1)\rangle,$$

and

$$\langle(1,\omega^i,\omega^j)\rangle,$$

where $i,j\in\{0,1,2\}$.

Proof. Suppose $x\notin H_x$, then (4.5) and (4.6) imply that all transversals on x contain exactly 3 points. Let π be a transversal plane on x and T the transversal on x inside π. By (3.6), all lines in the transversal plane contain at most 4 points, and therefore exactly 4 points. As each line in π on a point y of $\pi\setminus T$ meets T nontrivially, such point y is on 3 lines inside π. Fix such a point y and a point z on the line through x and y. Then the above also implies that the transversal coclique on y inside π, which meets each line on z nontrivially, has size 3. Thus π contains $4\cdot 3=12$ points, each being on 3 lines of size 4 and on a unique transversal of 3 points. This clearly implies π to be a dual affine plane of order 3.

Fix the three points x, y and z of a transversal coclique on x. As they are not collinear in $\mathbb{P}$, they generate $\langle \pi \rangle_{\mathbb{P}}$. Now let v be a point collinear to x, y and z. We can fix a basis $\mathcal{B}$ of $\langle \pi \rangle_{\mathbb{P}}$ such that $x = \langle (1,0,0) \rangle$, $y = \langle (0,1,0) \rangle$, $z = \langle (0,0,1) \rangle$, and $v = \langle (1,1,1) \rangle$. If u is a third point on the line through x and v, then with respect to $\mathcal{B}$ we find $u = \langle (1, \omega, \omega) \rangle$ for some $\omega \in \mathbb{K}$, where $\mathbb{K}$ is the skew field of coordinates of $\mathbb{P}$. The coordinates of all other points are uniquely determined. The result follows now by straightforward calculation of these coordinates. See also [2]. $\square$

4.8 *Suppose there is a point $x \in P$ with $x \notin H_x$. Then for all $x \in P$ we have $x \notin H_x$.*

Proof. Let $y \in P$ be a point different from x. If $y \perp x$, then as the diameter of Δ is 2, there is a transversal plane, and hence a transversal on x and y. Since $x \notin H_x$ we also have $y \notin H_y$, see (4.6).

If y is collinear to x, and there is a transversal plane on x and y, then, using (4.7), one can check easily that $y \notin H_y$.

So assume that there is no transversal plane on x and y. Let π be a transversal plane on x (which exists by (4.4)) and l a line on x inside π and m a line in π missing x. Then y is collinear with all points of l, for otherwise $\langle y, l \rangle_\Delta$ is a transversal plane on x and y. But then y is collinear to all points of m that are collinear to x and therefore to at least all but one of the points of m. This implies that there is a transversal T in π containing two points u and v collinear with y. By the above applied to the points of π we have $u \notin H_u$. Now we can also apply the above to the transversal plane generated by u, v and y and find $y \notin H_y$. $\square$

From now on we add the following condition to our Setting 4.1.

Setting 4.1(e) there is a point $x \in P$ with $x \in H_x$.

This implies the following.

4.9 *Any transversal T is contained in a projective line of $\mathbb{P}$.*

In particular, if $x, y \in P$ are distinct points with $x \perp y$, then $\langle x, y \rangle_{\mathbb{P}} \cap P$ contains at least 3 points.

Proof. As follows from (4.8) we find that $x \in H_x$ for all points $x \in P$. Moreover, by (4.6) each transversal coclique T of Δ is contained in a line of $\mathbb{P}$.

Now suppose $x \perp y$ for distinct points x, y. Then by (3.1) there is a point z collinear with both x and y. Inside the transversal plane generated by x, y and z one finds a transversal on x and y. This transversal contains at least 3 points and is completely contained in $\langle x, y \rangle_{\mathbb{P}}$. □

4.10 *Let π be a transversal plane of Δ and x a point not in π. Then $x^\perp \cap \pi$ is empty, a point, a line, a transversal coclique or π.*

Proof. The subspace H_x of $\mathbb{P}$ either misses $\langle \pi \rangle_{\mathbb{P}}$ or meets it in a point, a line or $\langle \pi \rangle_{\mathbb{P}}$ itself. But then $x^\perp \cap \pi$ is empty, a point, a line, a transversal coclique or π, see (4.5). □

The above lemma implies that we are in a similar setting as in Chapter 3 and 4 of [10], except for the fact that transversal planes are not necessarily dual affine planes. Many arguments from [10] carry over to the present situation. We continue our proof of Theorem 1.5 following the line of [10].

4.11 *Let π be a transversal plane and l a line meeting π in a point x. If for some $y \in l \setminus \{x\}$ we have $y^\perp \cap \pi$ is a transversal coclique (or a line) of π, then for all $y \in l \setminus \{x\}$ the intersection $y^\perp \cap \pi$ is a transversal coclique (or a line, respectively) of π.*

Proof. Suppose y and z are two points of $l \setminus \{x\}$ with $y^\perp \cap \pi$ being a transversal coclique T_y of π. Let m be a line of π on x meeting T_y in a point in $y^\perp$. Then $\langle m, l \rangle_\Delta$ is a transversal plane. So m contains a point different from x which is non-collinear to z. As the above is true for every line m in π on x, we find that $z^\perp \cap \pi$ is either a line or a transversal coclique. Suppose $n = z^\perp \cap \pi$ is a line. Then this line n meets T_y in some point, u say. Inside the transversal plane π we see that all points of T_y, and hence in particular u, are collinear to x. However, $u^\perp$ contains y and z and, as Δ is a Delta space, all points of l including x. Thus $z^\perp$ meets π in a transversal coclique.

Now assume that $y^\perp$ meets π in a line n. Then again every line m on x in Δ meeting n contains a point in $z^\perp$. This implies that $z^\perp$ meets π either in a line or a transversal coclique. However, in the latter case we get a contradiction when applying the above with the role of y and z switched. So, in this case $z^\perp$ meets π also in a line. □

4.12 *Let π be a transversal plane containing two non-collinear points x and y. Suppose z is a point collinear to both x and y such that $z^\perp \cap \pi$ is a line. Then for every point $u \in \langle \pi \rangle_{\mathbb{P}} \cap P$ we have either have $u \in \langle x, y \rangle_{\mathbb{P}}$ or $H_u \cap \langle x, y, z \rangle_{\mathbb{P}}$ contains a line from Δ.*

Proof. Let l be the line in π which is contained in $z^\perp$. Since π is transversal, the line l meets $x^\perp$ in a point, say v. Then v is the intersection point of l with $\langle x, y \rangle_{\mathbb{P}}$. Moreover, $x, y \in H_v$.

Let u be a point in $\langle \pi \rangle_{\mathbb{P}} \cap P$. If u is a point in $\langle \pi \rangle_{\mathbb{P}} \cap P$ not collinear to v, then it is contained in H_v, which meets $\langle \pi \rangle_{\mathbb{P}}$ in the line $\langle x, y \rangle_{\mathbb{P}}$.

So, from now on we can assume that u is collinear v. Moreover, assume $u \notin l$. On u there are at least 2 lines meeting l in a point not in $\langle x, y \rangle_{\mathbb{P}}$. Inside the transversal subspace on $\langle \pi \rangle_{\mathbb{P}}$ we see that these lines both meet $\langle x, y \rangle_{\mathbb{P}}$ in a point of P. Without loss of generality we can assume these two points to be x and y.

As $v^\perp$ contains x, y and z , the point v is contained in H_r for every point $r \in \langle x, y, z \rangle_\Delta$.

Now both $\langle u, x, z \rangle_\Delta$ and $\langle u, y, z \rangle_\Delta$ are transversal planes. This implies that there are points x' on xz and y' on yz that are both in $u^\perp$. Notice $x' \neq x, z$ and $y' \neq y, z$. If $x' \perp y'$, then $H_{x'}$ contains $\langle x', y' \rangle_{\mathbb{P}}$. Moreover, as $H_{x'}$ also contains v, we can conclude that $v \in \langle x', y' \rangle_{\mathbb{P}} \subseteq H_u$, which however contradicts v and u to be collinear. Hence x' and y' are collinear and they generate a line in $u^\perp$.

The lemma follows now also for points u on l by applying (4.11). □

4.13 $(P, \perp)$ *has diameter* 2.

Proof. Since Δ is coconnected, $(P, \perp)$ is connected. Suppose the point d is at distance 3 from the point a in $(P, \perp)$ and let $a \perp b \perp c \perp d$ be a path in $(P, \perp)$. Then $\langle a, c, d \rangle_\Delta$ is a transversal plane. Fix a point e on the transversal coclique on c in $\langle a, c, d \rangle_\Delta$ different from c and d. Then b and e are collinear. Since b is not collinear with the points on the line ac we can apply the above Lemma 4.12. In particular, there is a line $m \in L$ contained in $H_a \cap \langle b, d, e \rangle_{\mathbb{P}}$. As $\langle b, d, e \rangle_{\mathbb{P}}$ meets Δ in a transversal subspace, there is a point $f \in m$ that is not collinear to d. But then $d \perp f \perp a$. Thus we have found a path of length 2 from a to d. This contradicts our assumption and shows that the diameter of $(P, \perp)$ is at most 2 and hence equal to 2. □

On the point set P of Δ we can define the relation $\equiv$ by

$$x \equiv y \Leftrightarrow x^\perp = y^\perp.$$

This relation is clearly an equivalence relation on P. The $\equiv$-equivalence class of a point x is denoted by $[x]$.

4.14 *Suppose $x, y \in P$. Then $x \perp y$ if and only if $x' \perp y'$ for all $x' \in [x]$ and $y' \in [y]$.*

The above Lemma 4.14 justifies the following definitions. If $x, y \in P$, then we write $[x] \perp [y]$ if and only if $x \perp y$. Moreover, $[x]^\perp$ denotes the set of all classes $[y]$ with $x \perp y$.

4.15 *Let $x \in P$. The set $x^\perp \setminus [x]$ is a connected subspace of Δ; its diameter is 2.*

Proof. Let $y, z \in x^\perp \setminus [x]$ be two non-collinear points. Then there are transversal cocliques $T_{x,y}$ on x and y, $T_{y,z}$ on y and z and $T_{x,z}$ on x and z. Suppose $u \in x^\perp$ but not in $y^\perp$. Then $u \notin [x]$. If u is collinear to z we have found a path inside $x^\perp \setminus [x]$ from y to z. Thus assume $u \perp z$. Next fix an element $v \in x^\perp$ but not in $z^\perp$. Then $v \notin [x]$. As above we can assume $v \perp y$. But both v and u are collinear to all but one of the points of $T_{y,z}$. As $T_{y,z}$ contains at least 3 points there is a point $w \in T_{y,z}$ collinear to both u and v. The point $w \in x^\perp$. As it is collinear to v it is not in $[x]$. Hence, again we have found a path from y to z inside $x^\perp \setminus [x]$. So $x^\perp \setminus [x]$ is a connected Delta space, and its diameter is two as follows by (3.1). □

4.16 *Suppose $x, y \in P$ are collinear, then $\langle y, \Delta_x \rangle_\Delta = P$.*

Proof. Consider the subspace $X := \langle y, \Delta_x \rangle_\Delta$ of Δ. Let z be a point in Δ_y. If $x \perp z$, then $z \in X$. If z is collinear to x, then x, y and z are in a transversal plane and we can find points $x_1, x_2 \in X$ such that $z \in \langle x_1, x_2, y \rangle_\Delta \subseteq X$. Thus X contains Δ_y. Now suppose u is a point collinear to y and contained in $x^\perp \setminus [x]$. Then applying the above to $X_1 := \langle u, \Delta_y \rangle_\Delta$ yields that $\Delta_u \subseteq X_1$. But, $X_1 \subseteq X$, so Δ_u is contained in X. By connectedness of $x^\perp \setminus [x]$, see (4.15), the above yields that for every element $u \in x^\perp \setminus [x]$ we have $\Delta_u \subseteq X$. But then connectedness of $(P, \perp)$ yields that $X = P$. □

4.17 *For each $x \in P$ the space H_x is a hyperplane of $\mathbb{P}$.*

Proof. As Δ is generated by Δ_x and one extra point $y \not\perp x$, see (4.16), we find that $\mathbb{P}$ is generated by H_x and y. Hence, H_x is a hyperplane of $\mathbb{P}$. □

4.18 *Let x and y be points with $y \in x^\perp \setminus [x]$ and suppose z is collinear to both x and y. Then $\langle x, y\rangle_{\mathbb{P}}$ contains a unique point in $z^\perp$.*

Proof. Let x and y be two non-collinear points and z a point collinear to both of them. Such a point exists by (3.1). Then $\pi_1 = \langle x, y, z\rangle_\Delta$ is a transversal plane. By (4.13) there is a point u in $x^\perp \cap z^\perp$. Notice that the point u is in $x^\perp \setminus [x]$.

First suppose that u and y are collinear. Then $u^\perp \cap \langle x, y, z\rangle_\Delta$ is the line xz. Fix a point w in the transversal coclique on x in $\langle x, y, z\rangle_\Delta$ different from x and y, and let π_2 be the tranversal plane $\langle u, w, y\rangle_\Delta$. As $xz \subseteq u^\perp$, (4.12) implies that there is a unique line on u in the plane π_2 that is contained in $z^\perp$. The intersection point v of l with $z^\perp$ is on $\langle y, w\rangle_{\mathbb{P}} = \langle x, w\rangle_{\mathbb{P}} = \langle x, y\rangle_{\mathbb{P}}$ and is thus the point we are looking for. Clearly v is the unique point in $z^\perp \cap \langle x, y\rangle_{\mathbb{P}}$.

Now assume that $u \perp y$. By (4.15) there is a point w in $x^\perp \setminus [x]$ collinear to both u and y. If $w \perp z$ we are in the above situation with w instead of u. Thus we can assume that w and z are collinear. By the arguments in the preceeding paragraph with w replacing y, we find a point on $u' \in \langle x, w\rangle_{\mathbb{P}} \cap z^\perp$. Since y and w are collinear but y and x are not, we have that y and u' are collinear. So we can apply the above with u' instead of u. □

4.19 *Suppose $x \neq y \in P$ are collinear points. Then $H_x = \langle x^\perp \cap y^\perp, x\rangle_{\mathbb{P}}$.*
Moreover, if z is a point on xy then $H_x \cap H_y \subseteq H_z$.

Proof. Suppose $z \in \Delta_x \setminus [x]$. Then by (4.18), there is a point $u \in x^\perp \cap y^\perp$ with $u \in \langle x, z\rangle_{\mathbb{P}}$. Hence $\Delta_x \setminus [x]$ is contained in $\langle x^\perp \cap y^\perp, x\rangle_{\mathbb{P}}$.

If $x' \in [x]$ is different from x, then pick an element $z \in \Delta_x \setminus [x]$ collinear to y. By (4.9) there is a third point z' on the line $\langle x', z\rangle_{\mathbb{P}}$ which is in $x^\perp \cap y^\perp$. So, the point z' is in $\Delta_x \setminus [x]$. By the above we find z' but then also x' in $\langle x^\perp \cap y^\perp, x\rangle_{\mathbb{P}}$. Thus Δ_x is contained in $\langle x^\perp \cap y^\perp, x\rangle_{\mathbb{P}}$ which therefore equals H_x.

Since $x^\perp \cap y^\perp = x^\perp \cap z^\perp$ for any $z \neq x$ on xy, the second part of the lemma follows immediately. □

Let R be the subspace $\bigcap_{x\in P} H_x$ of $\mathbb{P}$. Then R is called the *radical* of Δ.

4.20 *If $x, y \in P$ then $\langle R, x\rangle_{\mathbb{P}} = \langle R, y\rangle_{\mathbb{P}}$ if and only if $x \equiv y$ if and only if $\langle x, y\rangle_{\mathbb{P}}$ meets R in a point.*

Proof. Suppose z is a point collinear with x. Then by (4.16), $x^\perp$ and z generate Δ. So, by (4.19), the space R equals $(\bigcap_{u\in x^\perp} H_u) \cap H_z$. Each element $y \in [x]$ is contained in $\bigcap_{u\in x^\perp} H_u$, but not in H_z, so R is a hyperplane in $\bigcap_{u\in x^\perp} H_u$. Moreover, for each $y \in [x]$ we find $\langle y, R\rangle_{\mathbb{P}}$ to be equal to $\bigcap_{u\in x^\perp} H_u$. In particular, $\langle R, x\rangle_{\mathbb{P}} = \langle R, y\rangle_{\mathbb{P}}$.

On the other hand if $y \in P$ and $\langle R, x\rangle_{\mathbb{P}} = \langle R, y\rangle_{\mathbb{P}}$, then $y \in \bigcap_{u \in x^\perp} H_u$. The rest follows easily. $\square$

Suppose $x, y \in P$ with $x \equiv y$, then by (4.20) the line $\langle x, y\rangle_{\mathbb{P}}$ meets R in a point. This point is called a *radical point*. By $\mathcal{R}$ we denote the set of all radical points. Let $\mathcal{P}$ denote the set $P \cup \mathcal{R}$. We extend the relation $\perp$ to a symmetric relation on $\mathcal{P}$ by the rule that $x \perp y$ for all $x \in \mathcal{R}$ and $y \in P \cup \mathcal{R}$. By $\mathcal{L}$ we denote the set of all intersections $\langle x, y\rangle_{\mathbb{P}} \cap \mathcal{P}$ where $x, y \in \mathcal{P}$ are distinct points with $x \perp y$.

Now we are able to state the main result of this section, which holds under the assumptions of Setting 4.1(a) to (e). This theorem implies Theorem 1.5.

4.21 Theorem. *Suppose $\Delta = (P, L)$ is a connected and coconnected Delta space weakly embedded into the projective space $\mathbb{P}$ such that*

(a) *all lines in L contain at least 4 points;*

(b) *if $x, y \in P$ with $x^\perp \subseteq y^\perp$, then $x^\perp = y^\perp$;*

(c) *any three points x, y, z in P, with $x \sim y \sim z \perp x$ generate a subspace of $\mathbb{P}$ meeting Δ in a transversal subspace.*

(d) *there is a point $x \in P$ with $x \in \langle \Delta_x\rangle_{\mathbb{P}}$.*

Then the space $\mathcal{S} := (\mathcal{P}, \mathcal{L})$ is a polar space with radical $\mathcal{R}$ weakly embedded into $\mathbb{P}$.

Moreover, the space Δ is the geometry of secants of $\mathcal{S}$.

Proof. Notice that condition (d) implies that there is a point x with Δ_x not empty. Hence Δ satisfies the Setting (4.1)(a)-(e).

The space $(\mathcal{P}, \mathcal{L})$ satisfies the 'one-or-all' axiom. Indeed, suppose $z \in \mathcal{P}$ and $l \in \mathcal{L}$. If $z \in \mathcal{R}$, then clearly, it is collinear to all points on l. Thus assume $z \in P$. If $l \subseteq H_z$ then z is collinear to all points of l. Suppose $l \not\subseteq H_z$. If l meets $\mathcal{R}$ in a point x, then x is the unique point of l in H_z. Thus suppose that l does not meet $\mathcal{R}$. Then it contains at least two points x and y of P with $y \in x^\perp \setminus [x]$, both collinear to z. But then (4.18) implies that l contains a unique point in $z^\perp$. This proves the 'one or all' axiom. So $\mathcal{S}$ is a polar space.

Clearly $\mathcal{R}$ is the radical of $\mathcal{S}$.

It remains to show that $\mathcal{S}$ is weakly embedded in $\mathbb{P}$. But this follows by definition of the lines of $\mathcal{S}$ and the fact that for each $x \in P$ the space $H_x \cap P = x^\perp$.

By construction Δ is the geometry of secants of $\mathcal{S}$. $\square$

5 The exceptional case

Suppose that $\Delta = (P, L)$ is a connected Delta space with at least two lines weakly embedded into the projective space $\mathbb{P}$ and all lines of L contain at least 4 points. We will prove Theorem 1.7. So, suppose that there is a point x in P with $x \notin H_x$, the subspace of $\mathbb{P}$ generated by Δ_x, and that any two intersecting lines are contained in a transversal subspace. Then by (4.8) we find that $x \notin H_x$ for all points $x \in P$. Moreover, by (4.7), all transversal planes of Δ are isomorphic to a dual affine plane of order 3. But this implies that we can apply the results of Cuypers [4] and find that Δ is isomorphic to the geometry of hyperbolic lines of a (possibly degenerate) symplectic space $\mathcal{S}$ over the field with 3 elements. Let π be a transversal plane and suppose y is a point not in π with $y^\perp \cap \pi$ being a transversal coclique T. Then H_y meets $\langle \pi \rangle_{\mathbb{P}}$ in a line containing T. In particular, T is contained in a line and (4.6) contradicts that $x \notin H_x$ for all $x \in P$. Hence in Δ there is no point y outside π with $y^\perp \cap \pi$ being a transversal coclique. This implies that Δ is isomorphic to a dual affine plane of order 3, also denoted by $\mathrm{Sp}(3,3)$, or to the geometry of hyperbolic lines in the symplectic polar space $\mathrm{Sp}(4,3)$. Embeddings of dual affine planes of order 3 are considered in (4.7). So, Theorem 1.7 follows from (4.7) and the following result.

5.1 *Suppose Δ is the geometry of secants of* $\mathrm{Sp}(4,3)$*. Then $\mathbb{P}$ is isomorphic to $\mathbb{P}(\mathbb{K}^4)$ for some skew field $\mathbb{K}$ containing an element of order* 3.

After choosing an appropriate basis of $\mathbb{P}$, the points of Δ are

$$\langle(1,0,0,0)\rangle, \langle(0,1,0,0)\rangle, \langle(0,0,1,0)\rangle, \langle(0,0,0,1)\rangle,$$

$$\langle(0,1,-\omega^i,\omega^j)\rangle, \langle(-\omega^i,0,1,\omega^j)\rangle, \langle(\omega^i,-\omega^j,0,1)\rangle, \langle(1,\omega^i,\omega^j,0)\rangle$$

where $\omega \in \mathbb{K}$ is an element of order 3 *and $i, j \in \{0,1,2\}$. The lines of Δ are the* 4*-sets of points collinear in $\mathbb{P}$.*

Proof. Suppose Δ is the geometry of secants of $\mathrm{Sp}(4,3)$. As Δ can be generated by 4 points, we find $\mathbb{P} = \mathbb{P}(\mathbb{K}^4)$ for some skew field $\mathbb{K}$. Take a maximal coclique x_1, x_2, x_3, x_4 inside Δ. Then for each $i \in \{1, \ldots, 4\}$ the space Δ_{x_i} is a transversal plane.

Now, by (4.7), we can choose a basis $\mathcal{B} = (b_1, b_2, b_3, b_4)$ of $\mathbb{K}^4$ such that the 12 points of Δ_{x_4} are $x_1 = \langle(1,0,0,0)\rangle$, $x_2 = \langle(0,1,0,0)\rangle$, $x_3 = \langle(0,0,1,0)\rangle$ and $\langle(1,\omega^i,\omega^j,0)\rangle$, with ω some fixed element of order 3 in $\mathbb{K}$ and $i, j \in \{0,1,2\}$. Denote by x the point $\langle(1,1,1,0)\rangle$. Now fix a point y in Δ collinear to x_3 but not to x_1 and x.

Let z be the point on the line through x_3 and y that is not collinear to $\langle(1, \omega, 1, 0)\rangle$. After replacing b_4 in $\mathcal{B}$ by an appropriate vector b_4', we can assume y to be the point $\langle(0, 1, -1, 1)\rangle$ and z to be the point $\langle(0, 1, -\omega^2, 1)\rangle$.

Now all other coordinates are uniquely determined and the result follows. $\square$

6 The geometry of tangents

This section is devoted to the proof of Theorem 1.6. We assume Δ to be a Delta space embedded in a Desarguesian projective space $\mathbb{P}$ as in the hypothesis of Theorem 1.6. By (4.7) all transversal planes are dual affine planes of order 3.

We keep the notation of the previous sections.

6.1 *Let π be a transversal plane and $u \in P$ a point, then $|u^\perp \cap \pi| \neq 1$.*

Proof. Suppose $u^\perp \cap \pi$ consists of a single point x. Let y and z be the two points different from x in the transversal coclique T of π containing x. The three points u, y, z generate a transversal plane ρ. This plane is contained in $x^\perp$. So $H_x \cap \langle \pi, u\rangle_{\mathbb{P}}$ equals $\langle \rho\rangle_{\mathbb{P}}$.

Suppose v is a point in π but not in T. Then u is collinear to all points on the line yv, which implies that v is collinear to all points on the line uy. But that implies that $v^\perp \cap \rho$ contains at most 2 points. Now fix v in such a way that it is collinear to all three points of the transversal coclique on u in ρ. Let u_1 and u_2 be the two points of that transversal different from u. Then $H_x \cap H_u \cap \langle \pi, u\rangle_{\mathbb{P}} = H_u \cap \langle \rho\rangle_{\mathbb{P}} = \langle u_1, u_2\rangle_{\mathbb{P}}$.

Now x, v and u generate a transversal plane containing a point $u' \neq u$ in $x^\perp \cap u^\perp$. This point is on the projective line $\langle u_1, u_2\rangle_{\mathbb{P}}$. By (4.2) we find that u' has to be equal to u_1 or u_2. Without loss of generality assume that $u' = u_1$. But then $u_2 \notin \langle v, u, x\rangle_{\mathbb{P}} = \langle v, u_1, x\rangle_{\mathbb{P}}$. Now consider the transversal plane generated by x, v and u_2. Inside this plane we find a point $u_2' \neq u_2$ inside H_x. As $\langle x, v, u_2\rangle_{\mathbb{P}} = \langle x, v, u_2'\rangle_{\mathbb{P}}$, the point u_2' is not inside the plane $\langle x, v, u\rangle_{\mathbb{P}}$. This implies that we have found a point u_2' in $\langle \rho\rangle_{\mathbb{P}}$ distinct from u, u_1 but inside $H_{u_2} \cap \langle \rho\rangle_{\mathbb{P}} = \langle u, u_1\rangle_{\mathbb{P}}$. This contradicts (4.2). $\square$

6.2 *Suppose l and m are two intersecting lines in a linear subspace. Then $\langle l, m\rangle_\Delta$ is an affine plane of order* 4.

Proof. Denote by π the linear subspace of Δ generated by l and m. Let $x \in P$ be a point in P with $x^\perp$ containing l but not m. Such point exists by condition

(b) of Theorem 1.6. Then x and m generate a transversal plane ρ. Suppose y is a point on m collinear to x and z a third point on the line through x and y. Then each line on y meeting l generates with x a transversal plane. So such line contains a point in $z^\perp$. Thus $z^\perp$ meets π also in a line. Varying x through the set of points of the transversal plane ρ but not on m, we find, using (6.1), exactly 4 lines $l_1 = l, l_2, l_3$ and l_4 in π occurring as intersections of the form $x^\perp \cap \pi$. Without loss of generality we can assume that $y \in l_4$. These 4 lines do not meet, but each l_i meets m is a unique point called x_i. Moreover, if $u \in l_i$ distinct from x_i, then the line through u and x_j is contained in $A := \bigcup_{i=1}^4 \ l_i$. Thus for each point $u \in A \setminus m$ the 4 lines on u meeting m are contained in A.

Now consider a line m_1 meeting m in y and l in a point y_1 distinct from x_1. Then we can repeat the above with m_1 instead of m. This gives rise to the same lines $l_1, \dots, l_4$ and set A. As above we see that a point $z_1 \in l_1$ distinct from x_1 and y_1 is on at least 4 lines. We find again that the 4 lines on z_1 meeting m_1 are inside A. However, repeating the above, with l and m permuted, shows that at least one of these 4 lines will not meet m. This implies that z_1 is on 5 lines inside A. Applying the same arguments to any line in A, we obtain that each point of A is on exactly 5 lines contained in A. This proves that A is a subspace of Δ isomorphic to an affine plane of order 4. □

6.3 *Each line is contained in an affine plane.*

Proof. By assumption there is a linear (and hence affine) plane π in Δ. Now let x be a point in that plane and l a line on x but not in π. Since $y^\perp \cap \pi$ is at most a line for any point $y \in l$, there is at least one line in π on x which together with l generates an affine plane. Thus any line on x is contained in an affine plane.

By connectedness of Δ we find that every point of Δ is contained in an affine plane. But then the above argument proves that also every line is in an affine plane. □

By the above we have now that Δ has the property that any pair of intersecting lines of Δ generates a subspace isomorphic to either a dual affine or an affine plane with 4 points per line. Thus Δ is a *generalized Fischer space* of order 3 in terms of Cuypers [5].

Using [5] we obtain the following.

6.4 Proposition. Δ *is isomorphic to the geometry of tangent lines of a non-degenerate unitary polar space of projective dimension at least* 4 *over the field* $\mathbb{F}_4$.

Proof. We only have to show that Δ is reduced to apply the results of [5] (see [5] for definitions). That means we have to show that the following two implications hold:

$$x^\perp = y^\perp \Rightarrow x = y,$$

$$\Delta_x = \Delta_y \Rightarrow x = y.$$

The first implication follows from [6]. Indeed, suppose x and y are two points with $x^\perp = y^\perp$. Then by connectedness of Δ, they are contained in a dual affine plane π. Fix a line l on x inside π. Then by (6.3) there is an affine plane ρ on l. Now [6] implies that $y^\perp$ meets ρ in at least a line on x, contradicting that $x^\perp = y^\perp$.

Now we consider the second implication. Suppose $\Delta_x = \Delta_y$ for some distinct points $x, y \in P$. Then x and y are collinear. Let l be the line through x and y. By (6.3) there is an affine plane on l. Pick a line $m \neq l$ in this plane on x. By condition (b) of (1.6) there is a point v in $m^\perp$ but not $l^\perp$. So $v \in \Delta_x$ but not in Δ_y, which contradicts again the assumption that $\Delta_x = \Delta_y$. This proves Δ to be reduced. □

If A is a subspace of Δ isomorphic to an affine plane, then parallelism defines an equivalence relation on the lines in A. The transitive closure of this relation on L is an equivalence relation $||$ on L, which restricted to any affine plane induces the natural relation of being parallel inside that plane, see [7]. Suppose $l \in L$, then by $[l]$ we denote the $||$-equivalence class. On the set of $||$-equivalence classes $L/||$ we can define a unitary polar space $\mathcal{S}$, where the lines are sets of 5 classes $[l]$, where l runs through the line set of an affine plane in Δ. Moreover, the parallel classes together with the points from P form the point set of a projective space of order 4 in which this unitary polar space embeds. This follows from (6.4); see also [7]. The lines of this projective space are the lines of $\mathcal{S}$, the sets $l \cup \{[l]\}$, where $l \in L$ and sets consisting of three equivalence classes of lines on a fixed point x inside a dual affine plane of Δ together with the two points of this plane not collinear with x. We will recover this projective space of order 4, which we denote by $\mathbb{P}_4$, inside $\mathbb{P}$.

6.5 *Let $\mathbb{K}$ be the underlying (skew) field of coordinates of $\mathbb{P}$. Then $\mathbb{K}$ has characteristic* 2.

Proof. By (6.2) we find that Δ contains an affine plane with 4 points per line which is weakly embedded into a subspace of $\mathbb{P}$. An easy computation with coordinates of the points of this affine plane shows that the underlying (skew) field of coordinates of $\mathbb{P}$ has characteristic 2. See also [2]. □

6.6 *If l is a line, and $x, y \in l$ then $H_x \cap \langle l \rangle_{\mathbb{P}} = H_y \cap \langle l \rangle_{\mathbb{P}}$*

Proof. Suppose π is transversal plane on l. (Such plane exists as can be checked using (6.4).) If $x, y \in l$ and x_1, x_2 and y_1, y_2 are the two other points inside the transversal on x or y, respectively, in π, then $H_x \cap \langle l \rangle_{\mathbb{P}}$ is the intersection point of l and $\langle x_1, x_2 \rangle_{\mathbb{P}}$ and $H_y \cap \langle l \rangle_{\mathbb{P}}$ is the intersection point of l and $\langle y_1, y_2 \rangle_{\mathbb{P}}$. Since the characteristic of the skew field of coordinates of $\mathbb{P}$ is 2, (4.7) implies that these intersection points are the same. □

6.7 *Suppose $l, m \in L$ with $l||m$. Then $\langle l \rangle_{\mathbb{P}} \cap H_x = \langle m \rangle_{\mathbb{P}} \cap H_y$ for any point $x \in l$ and $y \in m$.*

Proof. First assume that l and m are two parallel lines inside an affine plane π of Δ. Let n and n' be two more lines in the affine plane π parallel to both l and m. Then by (6.4) we can find a line k which generates an affine plane with all three of l, m and n but is contained in $n'^{\perp}$. The intersection point p of $\langle k \rangle_{\mathbb{P}}$ with $\langle l \rangle_{\mathbb{P}}$ equals the intersection point of $\langle k \rangle_{\mathbb{P}}$ with $\langle m \rangle_{\mathbb{P}}$ and of $\langle k \rangle_{\mathbb{P}}$ with $\langle n \rangle_{\mathbb{P}}$ and is contained in H_x for all $x \in n'$. So, by varying n and n' we see that the lines in π parallel to l all generate a projective line meeting $\langle l \rangle_{\mathbb{P}}$ in the intersection point p of $\langle l \rangle_{\mathbb{P}}$ with $\langle m \rangle_{\mathbb{P}}$ and that that point is in H_x for all $x \in \pi$.

Now for any two lines l and m with $l||m$, we either find l and m inside an affine plane or there is a line $n \in [l]$ with both the pairs l, n and n, m in a subspace of Δ isomorphic to an affine plane; see (6.4) and also [5]. But then applying the above to the two planes $\langle l, n \rangle_{\Delta}$ and $\langle n, m \rangle_{\Delta}$ yields the lemma. □

Let $\mathcal{P}$ be the set of points of $\mathbb{P}$ which are of the form $\langle l \rangle_{\mathbb{P}} \cap H_x$ for some line $l \in L$ and point $x \in l$. Such a point we call a *singular point*. It is also called the singular point *at infinity* of l.

Let φ be the map from $P \cup L/||$ into the point set of $\mathbb{P}$ which is the identity on P and maps each parallel class $[l]$, with $l \in L$, to the singular point $l \cap H_x$ for $x \in l$. By (6.7) the map φ is well-defined.

6.8 *Let l be a line in L with singular point p at infinity. If $x \in P$ with $p \in H_x$, then there is a line l' parallel with l containing x.*

Proof. This can be easily checked inside a unitary space. See also [7, 4.1]. □

6.9 *The map φ is a weak embedding of $\mathbb{P}_4$ into $\mathbb{P}$.*

Proof. First we prove φ to be injective. Clearly we only need to prove that any two lines $l, m \in L$ with the same singular point at infinity are parallel.

So suppose $l, m \in L$ do have the same point at infinity. Fix $x \in l$. Then, as H_x contains the point at infinity of m, we can replace m by a line m' parallel to it containing x; see (6.8). But then $l = m'$ and $[l] = [m]$. So indeed, φ is injective. We identify the point set of $\mathbb{P}_4$ with its image under φ.

Next we will show that a line of $\mathbb{P}_4$ is inside a line of $\mathbb{P}$ containing exactly 5 points from $P \cup \mathcal{P}$.

We consider the three types of lines from $\mathbb{P}_4$.

If such a line consists of the 4 points of a line l of Δ together with its singular point p at infinity, then this line is contained in the projective line $\langle l \rangle_{\mathbb{P}}$ of $\mathbb{P}$. As Δ is weakly embedded, the projective line $\langle l \rangle_{\mathbb{P}}$ does not contain any points from P except for those in l. Now suppose m is a line in L whose singular point q at infinity is on $\langle l \rangle_{\mathbb{P}}$ but different from p. Let $x \in l$. Then x is collinear with a point of m. So, the subspace $\langle x, m \rangle_{\Delta}$ is either affine or dual affine. If this subspace is affine, we can replace m by a parallel line m' on x. But then l equals m' and $[l] = [m]$, which contradicts $p \neq q$. If $\langle x, m \rangle_{\Delta}$ is a dual affine plane, then m contains a point y in $x^{\perp}$. The projective line $H_y \cap \langle l, m \rangle_{\mathbb{P}}$ contains x, q and hence l, as well as two non-collinear points of the transversal on y in $\langle x, m \rangle$, which contradicts Δ being weakly embedded in $\mathbb{P}$. Thus $\langle l \rangle_{\mathbb{P}}$ meets $P \cup \mathcal{P}$ in 5 points of a line of $\mathbb{P}_4$.

Next consider an affine plane π. The 5 parallel classes of π determine 5 singular points which are all in the line $\langle \pi \rangle_{\mathbb{P}} \cap H_x$ for $x \in \pi$. If $p \in P$ is a point of $\langle \pi \rangle_{\mathbb{P}} \cap H_x$, then $p \in H_y$ for all points $y \in \pi$, but H_p does meet $\langle \pi \rangle_{\mathbb{P}}$ only in a line. This yields a contradiction. If p is a singular point in $\langle \pi \rangle_{\mathbb{P}} \cap H_x$ different from the 5 points at infinity of the lines from π, then, by (6.8), we can find a line $m \in L$ meeting π in a point y and having p as point at infinity. Using our knowledge of the structure of Δ, there is a point $z \in P$ with $z \in m^{\perp}$ but not $z \in \pi^{\perp}$. That implies that H_z contains at least 2 but not all singular points on $\langle \pi \rangle_{\mathbb{P}} \cap H_x$, which is impossible. So $\langle \pi \rangle_{\mathbb{P}} \cap H_x$ contains exactly 5 singular points forming a line from $\mathbb{P}_4$.

Finally suppose π is a dual affine plane and fix a point $x \in \pi$. The three points at infinity of the three lines in π through x together with the two points in $\Delta_x \cap \pi$, called y and z, form a line of $\mathbb{P}_4$ and are contained in the projective line $\langle \pi \rangle_{\mathbb{P}} \cap H_x$. Now assume u is a point of $P \cup \mathcal{P}$ in $\langle \pi \rangle_{\mathbb{P}} \cap H_x$ different from these 5 points. Then by (4.2) we find u to be singular. As $u \in H_x$, there is a line m on x with u at infinity; see (6.8). But then m is in the transversal subspace $\langle \pi \rangle_{\mathbb{P}} \cap P$ but not in π. Let x' be a point in π different from but collinear to x. Then $H_{x'}$ contains a point on m and two points from π, which contradicts the above applied to $\langle \pi \rangle_{\mathbb{P}} \cap H_{x'}$ instead of $\langle \pi \rangle_{\mathbb{P}} \cap H_x$.

This final contradiction implies that the map φ is a weak embedding of $\mathbb{P}_4$ into $\mathbb{P}$. $\square$

By the above results we find that the embedding of Δ is induced from the embedding into $\mathbb{P}_4$. This proves Theorem 1.6.

7 Proof of the main theorem

In this final section we show how Theorem 1.4 follows from the results obtained so far. So, assume $\Delta = (P, L)$ is a partial linear space satisfying the hypotheses of Theorem 1.4.

Since Δ is not transversal, there is a line $l \in L$ and a point $x \in P \setminus l$ with $x \in l^{\perp}$ or $x^{\perp} \cap l = \emptyset$. In the latter case, $\langle x, l \rangle_{\Delta}$ is contained in a linear subspace of Δ. But then (1.4)(d) implies that there is a point in $l^{\perp}$. So, assume $x \in l^{\perp}$. This implies that there are two points $a, b \in P$ with $a^{\perp} \neq b^{\perp}$. Indeed, take $a \in l$ and $b = x$. It also implies that Δ is coconnected, as can be seen by the following arguments. Suppose y is a point in P not in the component of $(P, \perp)$ of x and l. Then y is collinear to all points of l. In particular, $\langle y, l \rangle_{\Delta}$ is a linear subspace. Suppose m is a line on y meeting l in a point v. By condition (1.4)(d) there is a point $u \in m^{\perp}$. But then $y \perp u \perp v \perp x$ is a path from x to y, contradicting our assumption. This contradiction proves coconnectedness of Δ.

So, if there is a point $x \in P$ with $x \in H_x$, then Δ is a partial linear space satisfying the hypotheses of Theorem 1.5. By this theorem we find Δ to be the geometry of secants of a weakly embedded polar space as in case (a) of the conclusion of Theorem 1.4.

If for all points $x \in P$ we have $x \notin H_x$, then either there are two intersecting lines $l, m \in \Delta$ contained in some linear subspace of Δ and we can apply Theorem 1.6, or any two intersecting lines are contained in a transversal plane, a case which has been covered by Theorem 1.7. So, in this case we find Δ to be as in case (b) or (c) of the conclusion of Theorem 1.4.

This finishes the proof of Theorem 1.4.

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