A note on the tight spherical 7-design in $\mathbb{R}^{23}$ and 5-design in $\mathbb{R}^7$

Hans Cuypers
Department of Mathematics
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven
The Netherlands
email: hansc@win.tue.nl

To the memory of Jaap Seidel

Abstract

In this note we prove the uniqueness of the tight spherical 7-design in $\mathbb{R}^{23}$ consisting of 4600 vectors and with automorphism group $2 \times \mathrm{Co}_2$ as well as the uniqueness of the tight spherical 5-design in $\mathbb{R}^7$ on 112 vectors and with automorphism group $2 \times \mathrm{Sp}_6(2)$.

1. Introduction

In [5], Delsarte, Goethals and Seidel introduced the concept of a spherical t-design as follows. Consider $\mathbb{R}^d$ with its standard inner product $\langle \cdot, \cdot \rangle$ and let $O(d)$ be the orthogonal group of this space. Denote by $\mathrm{Hom}_k(\Omega_d)$ the linear space of all functions $V : \Omega_d \to \mathbb{R}$ which are represented by homogeneous polynomials of degree k in the d coordinates of the elements of Ω_d, the unit sphere in $\mathbb{R}^d$.

A finite, nonempty subset Δ of the unit sphere Ω_d is called a *spherical t-design* if for all polynomials $P \in \mathrm{Hom}_k(\Omega_d)$, where $k \leq t$, and elements $T \in O(d)$ we have

$$\sum_{x \in \Delta} P(Tx) = \sum_{x \in \Delta} P(x).$$

Here Tx denotes the image of the vector x under the orthogonal transformation T.

As is shown in [5], the cardinality of a t-design is bounded from below. Indeed, if Δ is a $2e$-design, then

$$|\Delta| \geq \binom{d+e-1}{d-1} + \binom{d+e-2}{d-1}$$

and if Δ is a $2e+1$-design, then

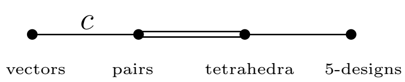
$$|\Delta| \geq 2\binom{d+e-1}{d-1}.$$

Spherical t-designs attaining the above bounds are called *tight*.

Although for any value of t there exist infinitely many spherical t-designs, see [6], tight t-designs in $\mathbb{R}^d$, with $d \geq 3$ only exist for $t \leq 11$, see [1]. For $t = 11$ and $d \geq 3$, there exists a unique design in Ω_{24}. Up to scaling it consists of the 196560 vectors of minimal norm 4 in the Leech lattice. Its uniqueness has been shown by Bannai and Sloane [2]. In this design the inner product between two vectors equals $\pm 4, \pm 2, \pm 1$, or 0. If one fixes a norm 4 vector v of the design, then there 4600 vectors having inner product 2 with v. Projection of these vectors onto the orthogonal complement of v and rescaling yields a tight spherical 7-design on 4600 vectors in $\mathbb{R}^{23}$ with automorphism group $2 \times \mathrm{Co}_2$. Bannai and Sloane [2] also showed the uniqueness of this design by embedding it into the Leech lattice and the 11-design on 196560 in $\mathbb{R}^{24}$. In the same paper, Bannai and Sloane also present a proof of the uniqueness of a tight 5-design on 112 vectors in $\mathbb{R}^7$. This design can be obtained from the E_8-root lattice. Indeed, if v denotes a vector of norm 2 in the E_8-root lattice, then projection of the 112 vectors making inner product -1 with v on the orthogonal complement of v yields 112 vectors of a tight spherical 5-design in $\mathbb{R}^7$ with automorphism group $2 \times \mathrm{Sp}_6(2)$. In this note we give a different proof of the uniqueness of both the tight 7-design in $\mathbb{R}^{23}$ and the tight 5-design in $\mathbb{R}^7$.

Theorem 1.1 *There exists (up to orthogonal transformations) a unique tight spherical* 7*-design in* $\mathbb{R}^{23}$ *as well as a unique tight spherical* 5*-design in* $\mathbb{R}^7$.

In our proof of the above theorem we exploit the geometry of the designs. The 4600 vectors of the tight spherical 7-design Δ described above come in antipodal pairs. Moreover, the inner product between two vectors from Δ equals $\pm 1, \pm 1/3$ or 0. The set Δ is tetrahedrally closed, i.e., if three vectors of Δ are the vertices of a regular tetrahedron centered at the origin, then the fourth vertex of the tetrahedron is also present. If we fix a norm 1 vector v in Δ, then the set of tetrahedra on this particular vector induces a partial linear space on the set of 891 vectors having inner product $-1/3$ with v. This partial linear space is a dual polar space of type $\mathrm{PSU}_6(2)$, see [7]. Fixing a generalized quadrangle of order $(2, 4)$ (a so-called quad, see [7]) inside this near hexagon, we obtain a set of 28 vectors, v together with the 27 vectors from the quad, which have pairwise inner product equal to $-1/3$. Together with their antipodes this yields a set of 56 vectors of norm 1 forming the tight spherical 5-design in Ω_7 as described above, see [5]. The pairs of vectors with inner product $-1/3$, the tetrahedra and spherical 5-designs on 56 vectors inside Δ form a Buekenhout geometry with diagram



Both the spherical 7-design and the 5-design are characterized by these geometric properties, as is shown in [4]. These characterizations are used to prove Theorem 1.1.

2. Tight spherical designs in $\mathbb{R}^7$ and $\mathbb{R}^{23}$

Let Δ_7 be a tight spherical 5-design in $\mathbb{R}^7$, and Δ_{23} a tight spherical 7-design in $\mathbb{R}^{23}$. Then

$$|\Delta_7| = 2 \cdot \binom{7+3-1}{6} = 112,$$

and

$$|\Delta_{23}| = 2 \cdot \binom{23+3-1}{22} = 4600.$$

After rescaling we may (and do) assume that all vectors in Δ_7 and Δ_{23} have norm 3.

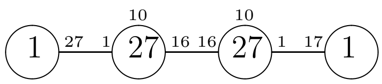
Lemma 2.1 *(i) If $v, w \in \Delta_7$, then $\langle v, w\rangle = \pm 3$ or ± 1. Moreover, $-v \in \Delta_7$.*

(ii) If $v, w \in \Delta_{23}$, then $\langle v, w\rangle = \pm 3,\ \pm 1$ or 0. Moreover, $-v \in \Delta_{23}$.

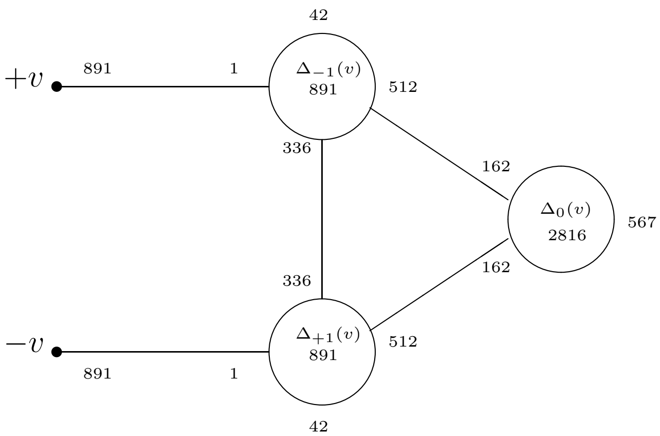
Proof. This follows by Theorem 5.12 of [5]. $\square$

Let d be 7 or 23. Suppose $\alpha \in \mathbb{R}$, then by $\Delta_{d,\alpha}$ we denote the set of all pairs of vectors (v, w) from Δ_d with inner product α. If $v \in \Delta_d$, then $\Delta_{d,\alpha}(v)$ denotes the set of vectors in Δ_d having inner product α with v. Of course, the relevant values for α are $\pm 3, \pm 1$ and 0.

Consider the graph Γ_d with vertex set Δ_d and and edge set $\Delta_{d,-1}$. The distance distribution diagram of Γ_d with respect to a vector v is uniquely determined, see [5, Theorem 7.4] and its proof. The distribution diagrams for both $d = 7$ and $d = 23$ are displayed below.



$d = 7$



$d = 23$

The next proposition is crucial in our proof of Theorem 1.1, as it reveals the geometry of the designs. Its proof relies heavily on the basic results on spherical designs obtained by Delsarte, Goethals and Seidel, see [5]. (For the notion and theory of near hexagons, the reader is referred to [7].)

Proposition 2.2 *Let v be a vector in Δ_d. Then the subgraph of Γ_d induced on $\Delta_{d,-1}(v)$ is the collinearity graph of a generalized quadrangle of order $(2,4)$ when $d = 7$, and, of a regular near hexagon of order $(2,4,20)$ when $d = 23$.*

Proof. First consider the case $d = 7$. Fix a vector v in Δ_7. The projection of $\Delta_{7,-1}(v)$ onto the orthogonal complement of v yields a set of 27 vectors in $\mathbb{R}^6$. As is shown in [5, Theorem 8.2], these vectors form a tight spherical 4-design on 27 vectors in $\mathbb{R}^6$. After rescaling the vectors to norm 4, the inner product between any two vectors in this 4-design equals $4, -2, 1, -1$. Such a design carries a 2-class association scheme, cf. [5, Theorem 7.5]. In particular, by [5, Theorem 7.4], the intersection numbers are uniquely determined. But that implies that the subgraph of Γ_d induced on $\Delta_{7,-1}(v)$ by taking as edges the pairs of vectors with inner product -1, is distance regular graph with intersection array $\{10, 4; 5, 1\}$. (Indeed, in the known example, the local graph is the collinearity graph of a classical generalized quadrangle of order $(2,4)$, which is distance regular with the above intersection array.) In particular, as any distance regular graph with intersection array $\{10, 4; 5, 1\}$ is the collinearity graph of a generalized quadrangle of order $(2, 4)$, we have proved the lemma for $d = 7$.

Now assume $d = 23$. As before, fix a vector v in Δ_{23}. The projection of $\Delta_{23,-1}(v)$ onto the orthogonal complement of v now yields a set of 891 vectors in $\mathbb{R}^{22}$. As is shown in [5, Theorem 8.2], these vectors form a spherical 5-design on 891 vectors in $\mathbb{R}^{22}$. After rescaling the vectors to norm 8, the inner product between any two vectors in this 5-design equals $8, -4, 2, -1$. Such a design carries a 3-class association scheme, cf. [5, Theorem 7.5]. In particular, by [5, Theorem 7.4], the intersection numbers are uniquely determined. But that implies that the graph induced on $\Delta_{23,-1}(v)$ is distance regular graph with intersection array $\{42, 40, 32; 1, 5, 21\}$. (Indeed, in the known example, the local graph is the collinearity of the classical near hexagon related to $\mathrm{PSU}_6(2)$, a distance regular with the above intersection array.) Since any two adjacent vertices of this local graph have a unique common neighbour, we see that this graph carries the structure of a partial linear space with lines of size 3. Now it is easy to check that this partial linear space is a regular near hexagon of order $(2, 4, 20)$, see [7]. This proves the lemma. □

We notice that there exists, up to isomorphism, a unique generalized quadrangle of order $(2, 4)$ and a unique regular near hexagon of order $(2, 4, 20)$, see for example [7].

The above lemma implies that if u, v and w are three vectors of Δ_d with $\langle u, v\rangle = \langle v, w\rangle = \langle w, u\rangle = -1$, then there is a fourth vector $x \in \Delta$ with $\langle x, y\rangle = -1$ for y equal to u, v or w. But then $\langle x+u+v+w, x+u+v+w\rangle = 0$, so $x = -(u+v+w)$. In other words, if u, v and w are three vectors of Δ forming three vertices of a regular tetrahedron centered at the origin, then the fourth vertex of that tetrahedron (i.e., $-(u + v + w)$) is also present in Δ_d. We say that Δ_d is *tetrahedrally closed*.

The above implies that the lines spanned by the vectors of Δ_d form a *regular line system* in $\mathbb{R}^d$, in the sense of [4]. By Theorem 1.1 of [4], this line system, and hence Δ_d is unique up to orthogonal transformations. This of course, proves Theorem 1.1.

For convenience of the reader, we sketch a proof of the uniqueness in case $d = 7$.

Assume $d = 7$ and let $\mathcal{L}$ be the set of 28 lines spanned by the vectors of Δ_7. A tetrahedron (of lines) is considered to be the set of 4 lines spanned by 4 vectors of Δ_7 forming the vertices of a regular tetrahedron centered at the origin. If we fix a line $l \in \mathcal{L}$, then l is contained in 45 tetrahedra. These tetrahedra induce the structure of a generalized quadrangle of order $(2, 4)$ on the remaining 27 lines different from l, see Proposition 2.2. If T_1 and T_2 are two tetrahedra on l meeting in a second line, then the symmetric difference $T_1 \cup T_2 \setminus (T_1 \cap T_2)$ is again a tetrahedron. Actually, any tetrahedron not on l can be obtained as such a symmetric difference. As, up to isomorphism, there is a unique generalized quadrangle of order $(2, 4)$, this implies that the Gram matrix of Δ_7 is uniquely determined. Indeed, let $+l$ and $-l$ be the two norm 1 vectors on l, then we can choose on each line $m \in \mathcal{L}$ different from l the two vectors $+m$ and $-m$ of norm 1 in such a way that $\langle +l, -m\rangle = -1/3$ and $\langle +l, +m\rangle = +1/3$. But then $\langle -m, -n\rangle = -1/3$ if and only if l, m and n are three lines in a tetrahedron, or equivalently, m and n are collinear in the generalized quadrangle induced on $\mathcal{L} \setminus \{l\}$.

The uniqueness of the Gram matrix of Δ_7 clearly implies the uniqueness of the design. So we have obtained a proof of Theorem 1.1 in the case that d equals 7.

Although the uniqueness proof in case $d = 23$ is much harder, it is along the same lines. For details we refer the reader to [4].

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