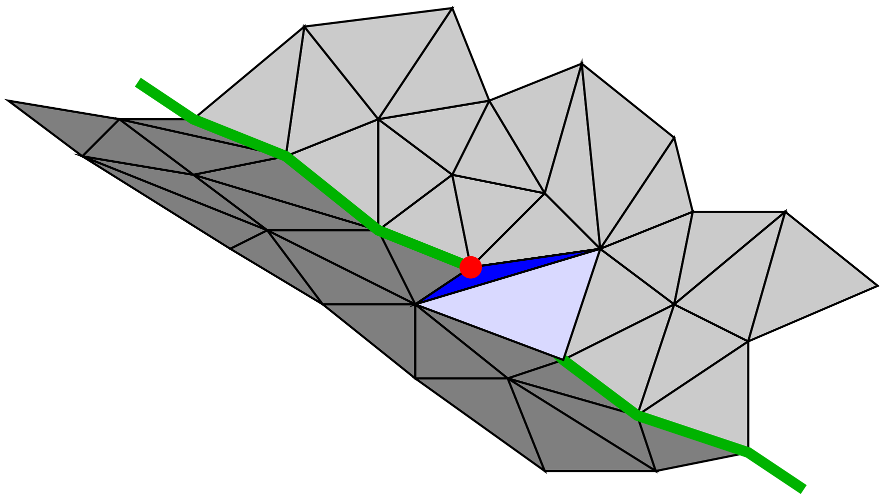
# Constraint Higher-Order Delaunay Triangulations



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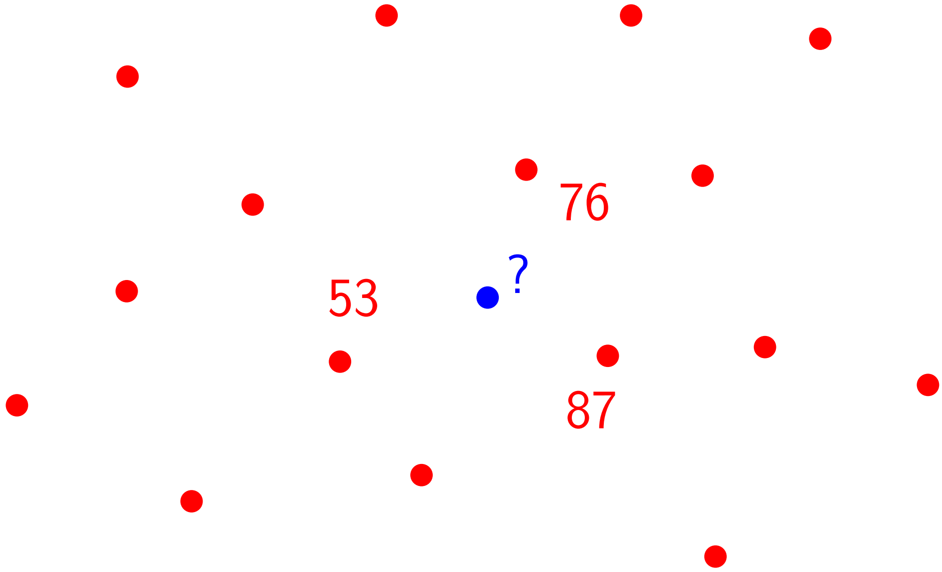
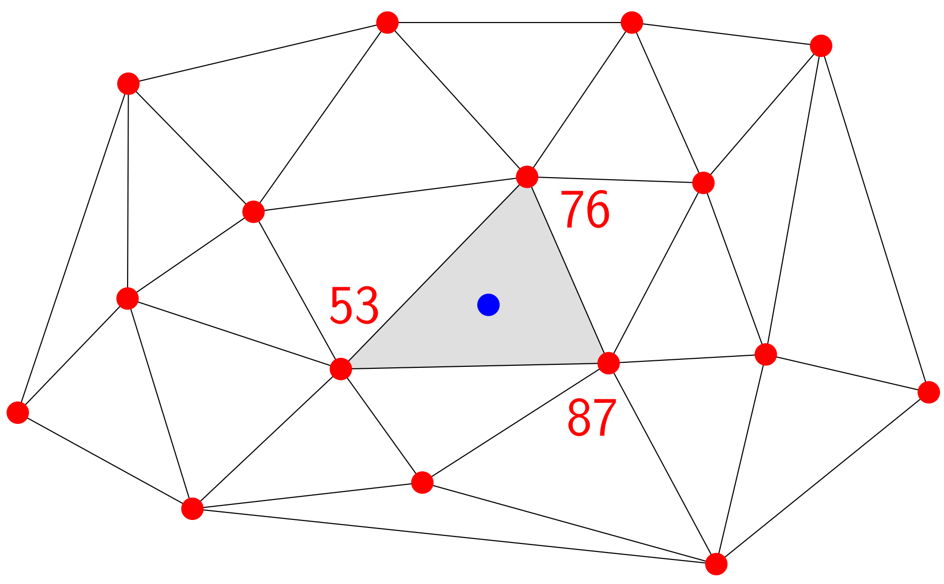
# Motivation

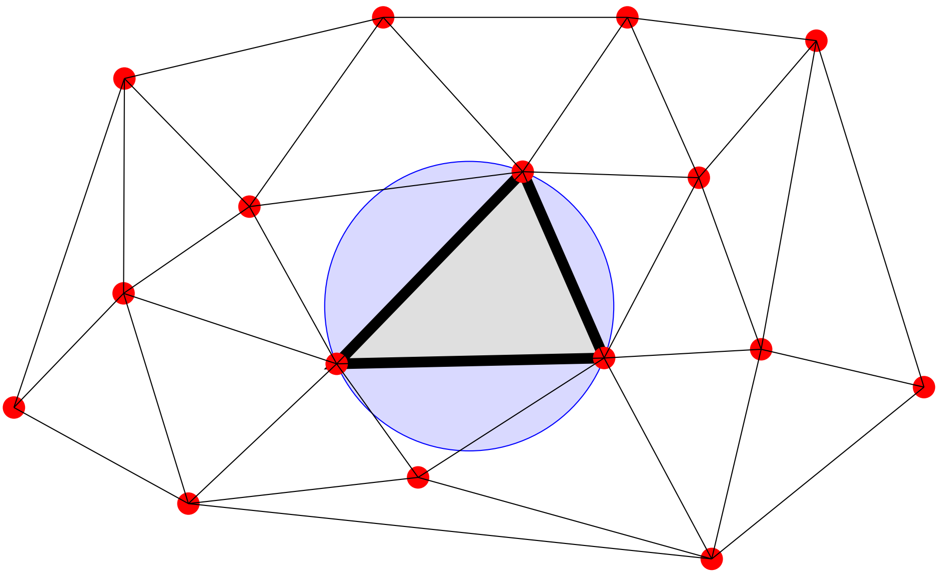
Interpolation of elevation data for terrain reconstruction.



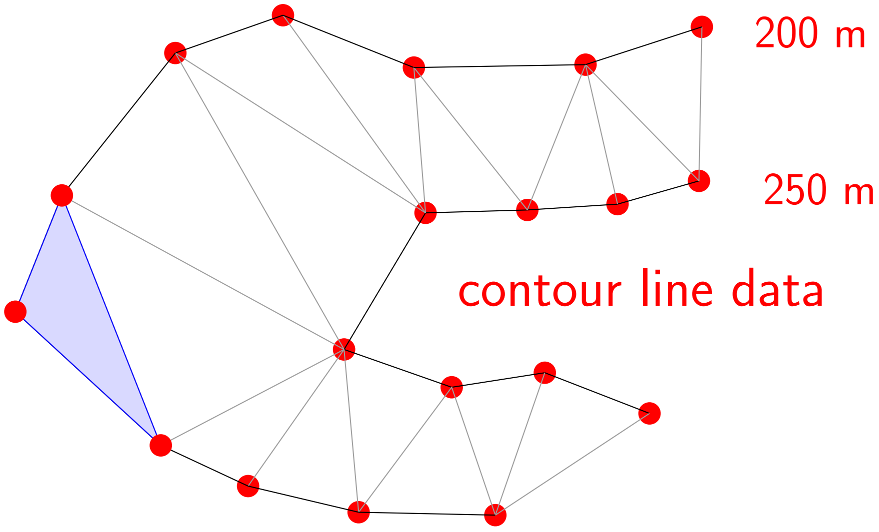



# Delaunay Triangulation: Artefacts

Delaunay triangulation: well-shaped triangles because of empty circle property

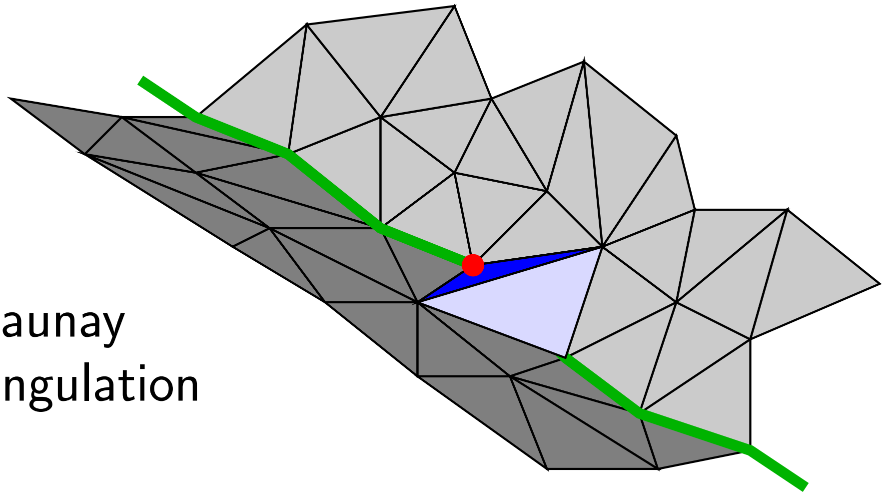


But: doesn't consider values/elevations to choose which triangulation → artefacts like plateaus and local minima

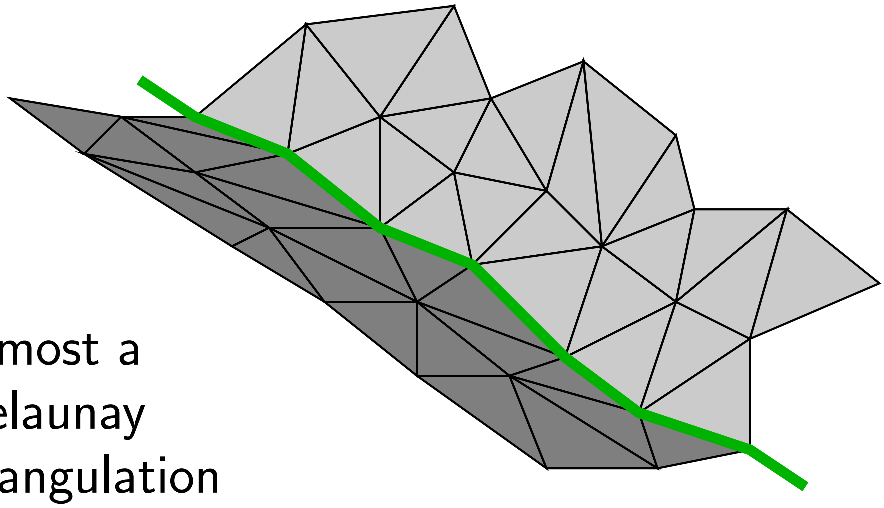


# Artificial dams

Artificial dam =
Artificial local minimum =
interrupted valley line



Delaunay triangulation



Almost a Delaunay triangulation

# Probem

Given a set of points in the plane with elevation data, find a well-shaped tirangulation that optimizes some criterion, e.g.:

- few local extrema
- few plateaus
- small degree of vertices
- few obtuse angles
- small area of largest triangle
- small total edge length
- ...

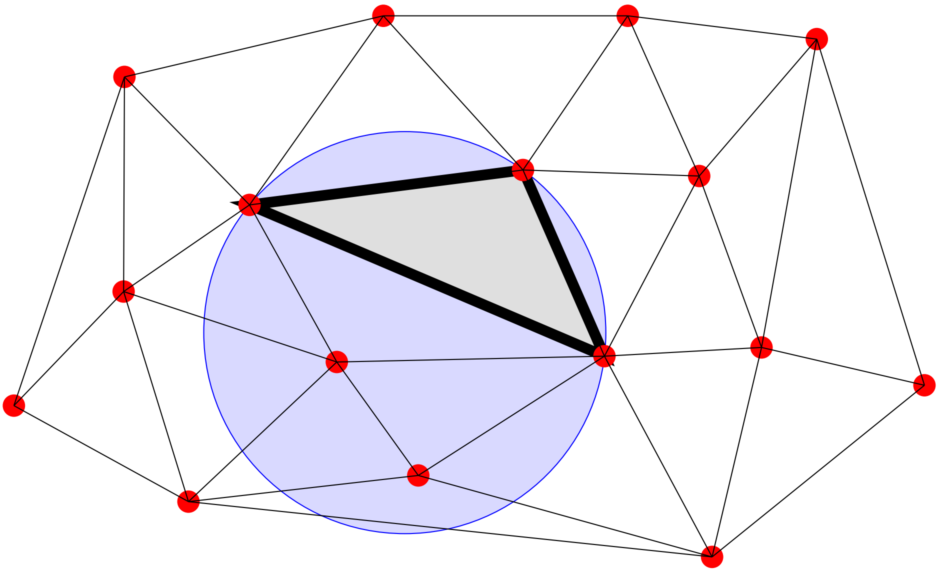
Applications of triangulations with properties:

- interpolation, terrain reconstruction, hydrological modelling etc.
- mesh generation for finite element methods

See also:
Marshall Bern: Triangulations, in: *Handbook of Discrete and Computational Geometry*, 1997.

# Higher-Order Delaunay Triangulation

Relax empty circle property:
allow at most $k$ points inside circle through triangle vertices
$\rightarrow$ $k$-order Delaunay triangulation



A 2nd-order Delaunay triangle in
a 2nd-order Delaunay triangulation.

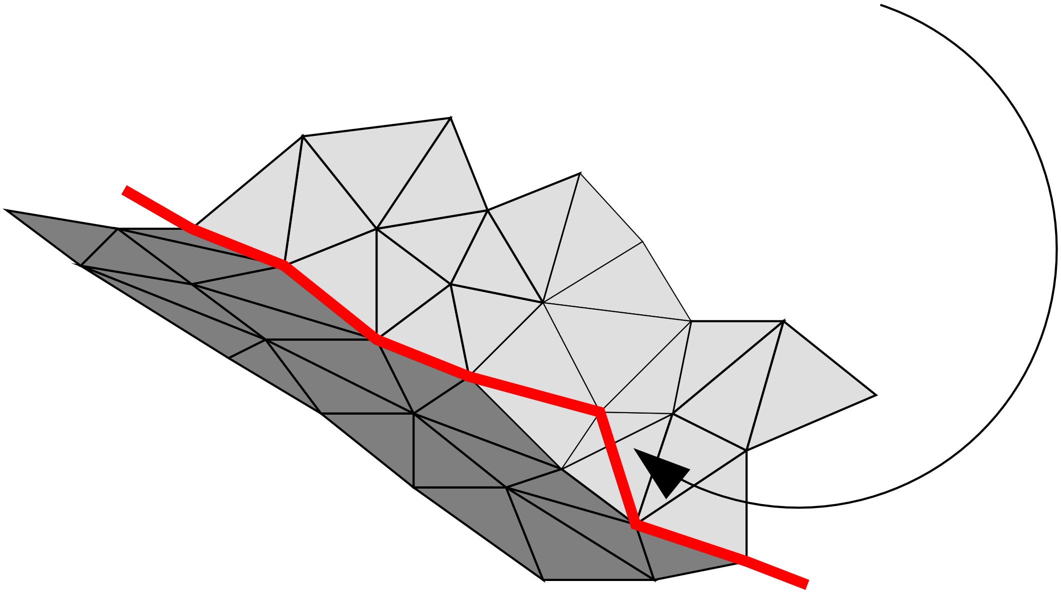
**New problem statement:**

Given a set of points in the plane and a 'well-shapedness' parameter $k$,
find a $k$-ODT that optimizes some criterion

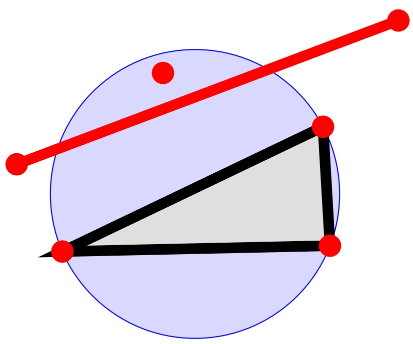
(studied by Hammar et al., 2000)

# Constraint Triangulations

Some edges may be known that must be included in the triangulation, e.g. valley lines or drainage network. Delaunay triangulation might not work:



Constrained Delaunay triangulation:
includes all constraining (given) edges,
circle through triangle vertices may contain points hidden behind a constraining edge



# Constraint Higher-Order DT

Accommodate constraining edges in higher-order Delaunay triangulations.

**Questions:**

1. Given a set $P$ of points and a set $E$ of edges, what is the lowest order $k$ of a triangulation of $P$ that includes the edges of $E$?
2. What $k$-ODT includes $E$ **and** optimizes some other criterion?
3. How may additional points on the edges of $E$ reduce the lowest possible order to $k'$?

# Order of a Constraint Triangulation

**Definition:** An edge is $k$-useful iff there is a $k$-ODT that includes it.

**Question:** Given a set $P$ of points and a set $E$ of $k$-useful edges, what is — in the worst case — the lowest order $k_T$ of a triangulation of $P$ that includes the edges of $E$?

By definition: if $|E| = 1$, then $k_T = k$.

Known: if $k = 1$, then $k_T = k = 1$
(if all constraining edges are 1-useful, a 1-ODT can be computed in $O(n \log n)$ time)

**What if $|E| > 1$ and $k > 1$?**

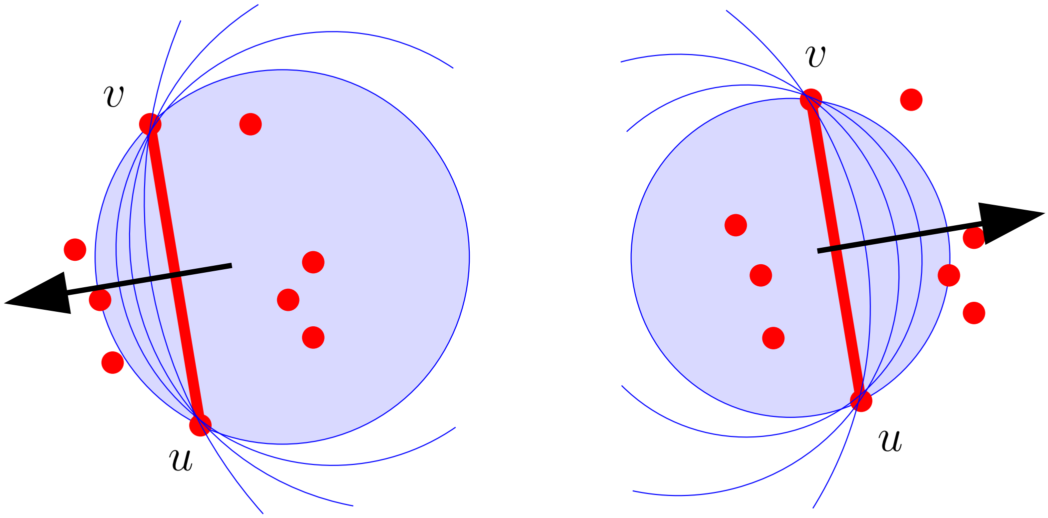
**Theorem:**
Given a set $P$ of points in the plane and a set of $k$-useful Delaunay edges. Then:

- The standard constraint Delaunay triangulation has order at most $2k - 2$.
- For some input, the order is at least $2k - 2$ for any constraint triangulation.

# Worst-Case Lower Bound

**Theorem:** For some sets of points and constraining edges, no triangulation has order less than $2k - 2$.

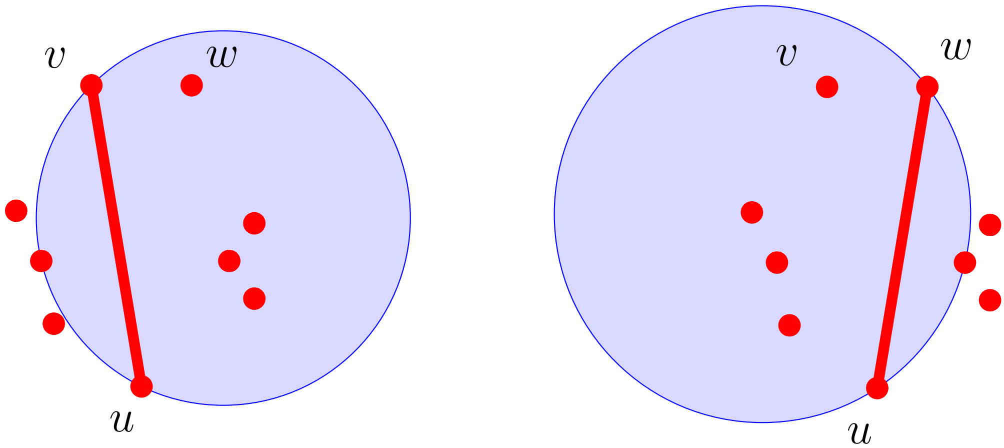
**Example:** The useful order of an edge is determined by pushing a circle between the endpoints until it hits another point.
We do this in both directions:



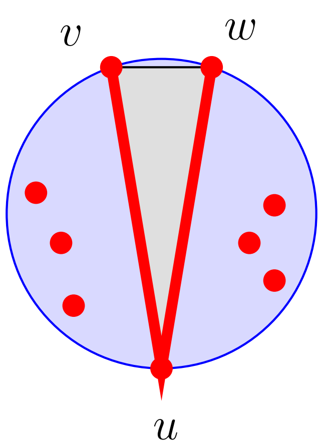

Useful order = maximum number of points in these two circles (Hammar *et al.*, 2000), $\rightarrow$ $\overline{uv}$ is 4-useful.

# Worst-Case Lower Bound (2)

$\overline{uv}$ and $\overline{uw}$ are 4-useful edges:




Any triangulation with constraining edges $\overline{uv}$ and $\overline{uw}$ contains $\Delta uvw$, which has order $6 = 2k - 2$.

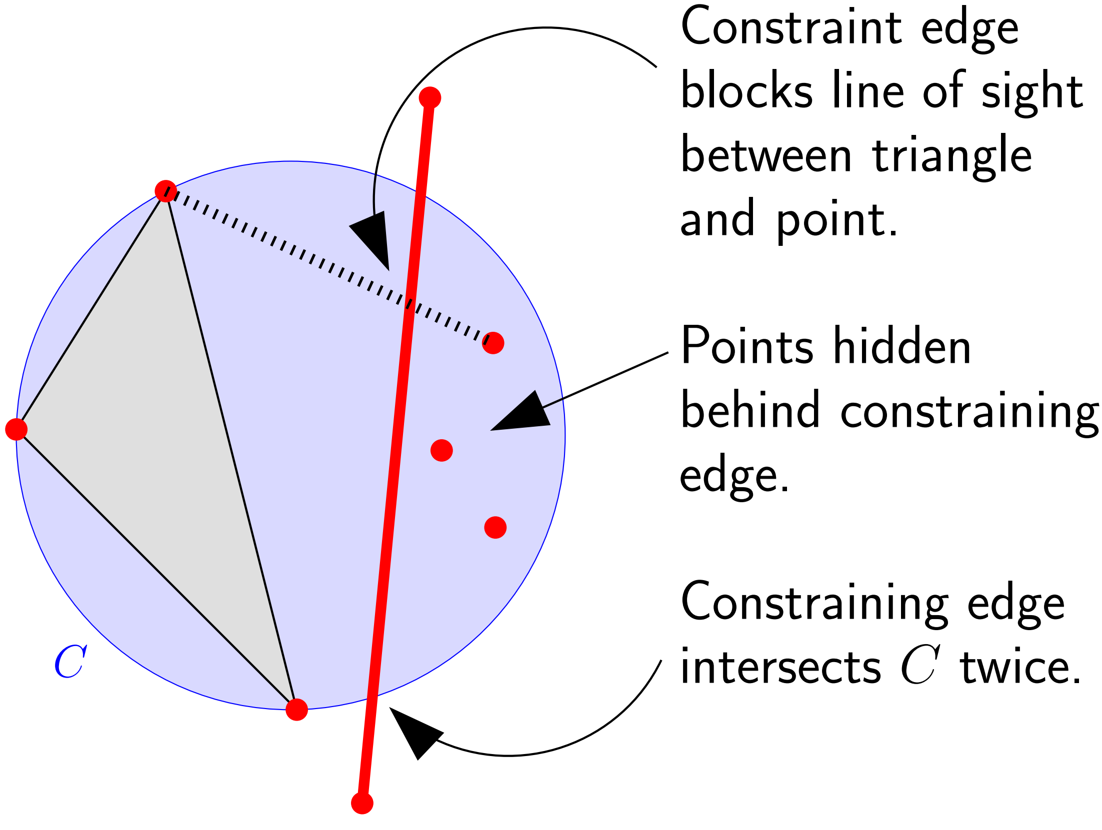



Generalizes to $(2k - 2)$-order triangulation when edges are $k$-useful.

# Order of Standard CDT

**Theorem:** The standard constrained Delaunay triangulation has order $2k - 2$.

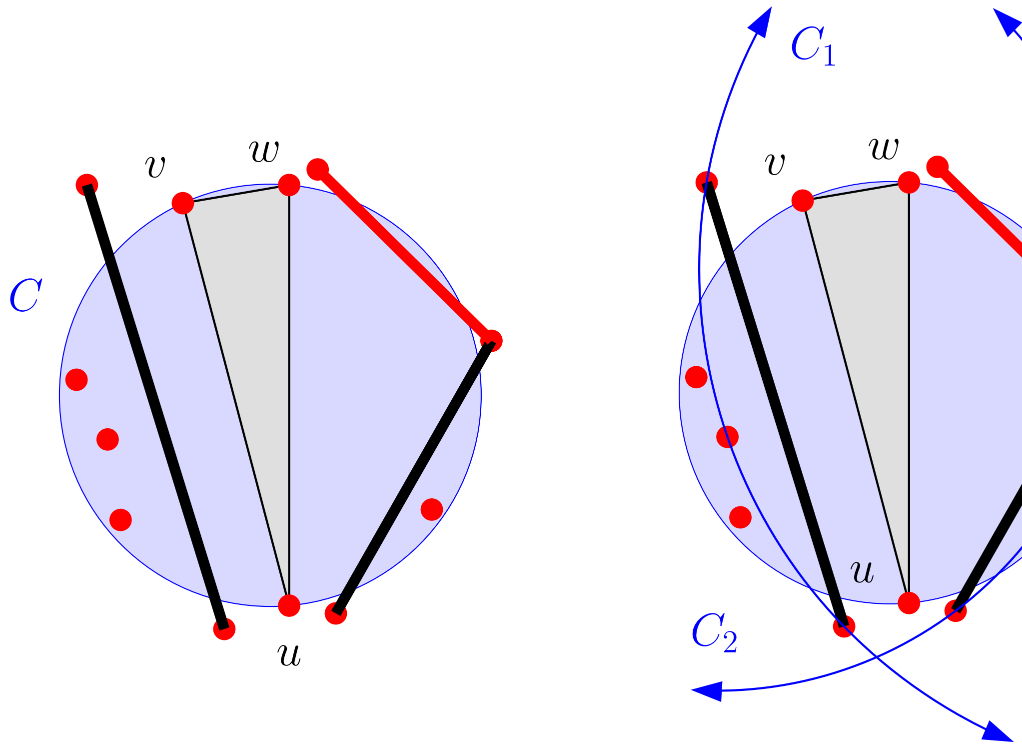
**Proof:** Circle $C$ through vertices of any CDT-triangle only contains points that cannot be seen from the triangle.




If there is one such edge, with order $k$, only $k$ points can be hidden inside $C$ [Hammar, 2000] $\rightarrow$ triangle has order $k$.

# Order of Standard CDT (2)

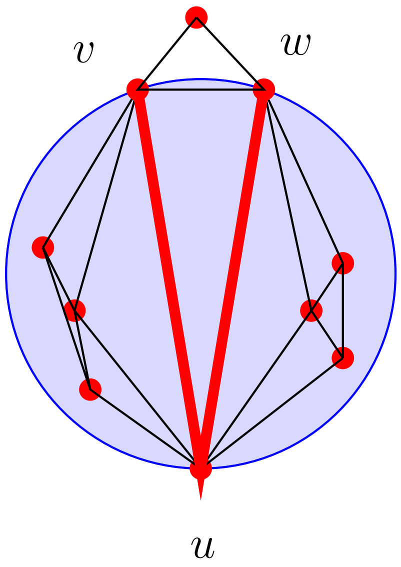
If there are more $k$-useful edges hiding points, choose any two of them (e.g. the black edges)



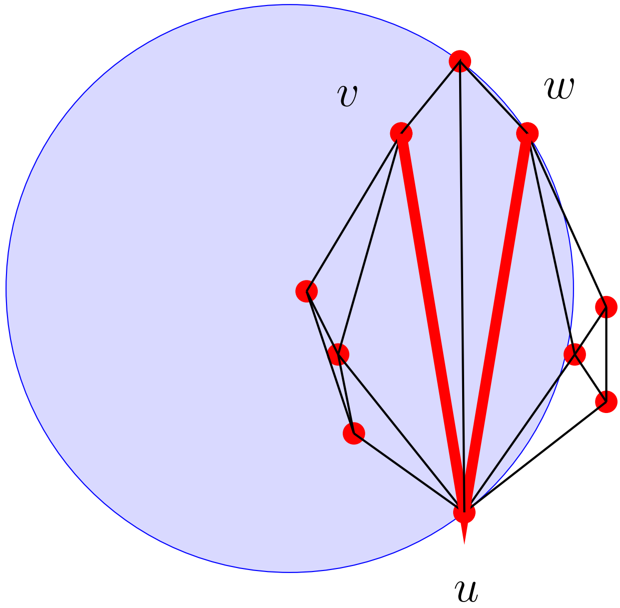

Since all constraining edges are $k$-useful, circles $C_1$ and $C_2$ contain at most $k$ points each, that is: $u$, $v$ or $w$, and at most $k-1$ other points, together $2k-2$ points other than $u$, $v$ and $w$.

$C_1$ and $C_2$ together cover $C$,
Hence, $\Delta uvw$ has order $2k-2$.

# CDT Always Optimal Order?




CDT has order $2k - 2$




CHODT of order $k$

**Theorem:**
Given a set $P$ of points in the plane and a set of $k$-useful Delaunay constraining edges. Then:

- The standard constraint Delaunay triangulation has order at most $2k - 2$.
- For some input, the order is at least $2k - 2$ for any constraint triangulation.
- For some input, the CDT has order $2k - 2$, but some other CT has order $k$.

# Conclusions

We extended concepts and results about higher-order Delaunay triangulations to constrained higher-order Delaunay triangulations.

**Open problem:**

Given a set of points and a set of edges, compute the lowest-order Delaunay triangulation that includes these edges
(CDT gives 2-approximation)

**Subject for further research:**

integrate other criteria for realistic terrain modeling by optimizing over CHODT's;
find efficient algorithms.