1 How to find a start solution for the simplex method

1. How to find a start solution for the simplex method
2. What to do if it is not easy to find a start solution: 2-phase method

If

- each constraint is with $\leq$, and
- non-negative right handsides,

then easy to find basic feasible solution to start the simplex method with.

If each constraint ends with $\leq b_i$, where $b_i \geq 0$. Adding slack variables yields a standard form Example:

$$\begin{array}{rcrrl} \max f(x_1, x_2) & = & -4x_1 & +3x_2 & \\ & & x_1 & +x_2 & \leq 40 \\ & & 2x_1 & +x_2 & \leq 60 \\ & & x_1 & , x_2 & \geq \quad 0 \end{array}$$

becomes in standard form:

$$\begin{array}{rcllllcl} \min -f(x_1, x_2) & = & +4x_1 & -3x_2 & & & & \\ & & x_1 & +x_2 & +s_1 & & = & 40 \\ & & 2x_1 & +x_2 & & +s_2 & = & 60 \\ & & x_1 & , x_2 & , s_1 & , s_2 & \geq & 0 \end{array}$$

Otherwise apply the 2-phase method.

The 2-phase method – phase 1

Step 1. Make each $b_i \geq 0$. If it was negative, multiply both sides of the constraint by -1 and switch the inequality.

Step 2. Put the LP-program in standard form by adding slack or surplus variables. (Add $+s_i$ if slack variable and $-e_i$ if surplus variable.)

Step 3. Add an artificial variable a_i to each constraint that contains no slack variable.

Step 4. Replace the objective function by

$$\min g(a_1, \ldots, a_k) = \sum a_i.$$

If $\min g(a_1, \ldots, a_k) = 0$, we have found a solution to start the second phase with. The a_i's have done their job, remove them.

Initial BFS for phase 1

Choose as basic variables

- the artificial variables a_i, and
- the slack variables s_i.

Warning: we must make sure that the reduced cost coefficients of basic variables are equal to 0.

Phase 1: example

$$\begin{array}{llllll} \max & 3x_1 & +2x_2 & -x_3 & & \\ \text{s.t.} & 2x_1 & +x_2 & +x_3 & = & 8 \\ & x_1 & +x_2 & & \leq & 10 \\ & -x_1 & -3x_2 & & \leq & -20 \\ & x_1 & , x_2 & , x_3 & \geq & 0 \end{array}$$

Step 1: Each number of the right hand side must be non-negative.

$$\begin{array}{lllll} 2x_1 & +x_2 & +x_3 & = & 8 \\ x_1 & +x_2 & & \leq & 10 \\ x_1 & +3x_2 & & \geq & 20 \\ x_1 & , x_2 & , x_3 & \geq & 0 \end{array}$$

Step 2: Introduction of slack and surplus variables.

$$\begin{array}{llllllll} 2x_1 & +x_2 & +x_3 & & & = & 8 \\ x_1 & +x_2 & & +s_1 & & = & 10 \\ x_1 & +3x_2 & & & -e_1 & = & 20 \\ x_1 & ,x_2 & ,x_3 & ,s_1 & ,e_1 & \geq & 0 \end{array}$$

Step 3: Introduction of artificial variables.

$$\begin{array}{llllllllll} 2x_1 & +x_2 & +x_3 & & & +a_1 & & = & 8 \\ x_1 & +x_2 & & +s_1 & & & & = & 10 \\ x_1 & +3x_2 & & & -e_1 & & +a_2 & = & 20 \\ x_1 & ,x_2 & ,x_3 & ,s_1 & ,e_1 & ,a_1 & ,a_2 & \geq & 0 \end{array}$$

Step 4: Replace objective function by min $a_1 + a_2$.

$$\begin{array}{lllllllllll} \min & a_1 & +a_2 & & & & & & & \\ \text{s.t.} & 2x_1 & +x_2 & +x_3 & & & +a_1 & & = & 8 \\ & x_1 & +x_2 & & +s_1 & & & & = & 10 \\ & x_1 & +3x_2 & & & -e_1 & & +a_2 & = & 20 \\ & x_1 & ,x_2 & ,x_3 & ,s_1 & ,e_1 & ,a_1 & ,a_2 & \geq & 0 \end{array}$$

Phase 1: example → tableau

	x_1	x_2	x_3	s_1	e_1	a_1	a_2	b
	2	1	1	0	0	1	0	8
	1	1	0	1	0	0	0	10
	1	3	0	0	-1	0	1	20
c^T	0	0	0	0	0	1	1	0

Simplex tableau not in right form. The reduced cost coefficients of the basic must be made equal to 0.

x_1	x_2	x_3	s_1	e_1	a_1	a_2	b
2	1	1	0	0	1	0	8
1	1	0	1	0	0	0	10
1	3	0	0	-1	0	1	20
-3	-4	-1	0	1	0	0	-28

Now we are in position to apply the simplex method to find a basic feasible solution for the original LP.

We choose x_1 as entering variable. Then a_1 is leaving variable. New basis is w, x_1, s_1, a_2.

x_1	x_2	x_3	s_1	e_1	a_1	a_2	b
1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	4
0	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	6
0	$2\frac{1}{2}$	$-\frac{1}{2}$	0	-1	$-\frac{1}{2}$	1	16
0	$-2\frac{1}{2}$	$\frac{1}{2}$	0	1	$1\frac{1}{2}$	0	-16

We must choose x_2 as entering variable. a_2 is leaving variable. New basis is w, x_1, s_1, x_2.

x_1	x_2	x_3	s_1	e_1	a_1	a_2	b
1	0	$\frac{3}{5}$	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	$\frac{4}{5}$
0	0	$-\frac{2}{5}$	1	$\frac{1}{5}$	$-\frac{2}{5}$	$-\frac{1}{5}$	$2\frac{4}{5}$
0	1	$-\frac{1}{5}$	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$	$6\frac{2}{5}$
0	0	0	0	0	1	1	0

We see that a basic feasible solution of the original LP is $x_1 = \frac{4}{5}$, $x_2 = 6\frac{2}{5}$, $x_3 = 0$, $s_1 = 2\frac{4}{5}$, $e_1 = 0$.

The second phase

Depending on the optimal solution found in the first phase do:

- min > 0. The original LP is non-feasible.
- min $= 0$, and no a_i is in the basis. Forget the a_i's. Replace in the optimal tableau the zeroth row with the original objective function. Second phase: Eliminate basic variables from objective function and apply simplex method.
- min $= 0$ and at least one of the $a_i = 0$ is in the basis. We are in a rare situation. You need not know what to do in this case.

Example for phase 2.

Remove the column of the artificial variables and replace the last row with the objective function.

$$\min -3x_1 - 2x_2 + x_3$$

	x_1	x_2	x_3	s_1	e_1	b
	1	0	$\frac{3}{5}$	0	$\frac{1}{5}$	$\frac{4}{5}$
	0	0	$-\frac{2}{5}$	1	$\frac{1}{5}$	$2\frac{4}{5}$
	0	1	$-\frac{1}{5}$	0	$-\frac{2}{5}$	$6\frac{2}{5}$
c^T	-3	-2	1	0	0	0

(This is not in right form yet. Why?) The basic variables are the same basic variables as those we ended the first phase with.

The right simplex tableau is

x_1	x_2	x_3	s_1	e_1	b
1	0	$\frac{3}{5}$	0	$\frac{1}{5}$	$\frac{4}{5}$
0	0	$-\frac{2}{5}$	1	$\frac{1}{5}$	$2\frac{4}{5}$
0	1	$-\frac{1}{5}$	0	$-\frac{2}{5}$	$6\frac{2}{5}$
0	0	$2\frac{2}{5}$	0	$-\frac{1}{5}$	$15\frac{1}{5}$

x_1	x_2	x_3	s_1	e_1	b
5	0	3	0	1	4
-1	0	-1	1	0	2
0	1	1	0	0	8
5	0	3	0	0	16

We have reached the optimal simplex tableau. An optimal solution is $e_1 = 4$, $s_1 = 2$, $x_2 = 8$, $x_1 = x_3 = 0$. The optimal value is -16.