The one million dollar quiz.

What type of programs did they run on the first computers?

1. They played pacman.
2. They cooked eggs on them.
3. They run the simplex method.

Simplex method

Each *LP* can be transformed to standard formulation by introducing slack variables and by substitution.

$$\begin{array}{lll} \min & c^T x & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

where A is an $m \times n$ matrix. The corners (hoekpunten) of the polyhedron

$$P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$$

are vectors in $\mathbb{R}^n$, each with at least $n - m$ zeroes.

- Simplex method needs a start solution that is a corner (hoekpunt) of P. This need not be an optimal solution.
- Usually easy to find.

Example: start solution

$$
\begin{array}{rrrrl}
\min f(x_1, x_2) & = & -120x_1 & -80x_2 & \\
\text{s.t.} & & x_1 & & \leq 40 \\
 & & & x_2 & \leq 10 \\
 & & 20x_1 & +10x_2 & \leq 500 \\
 & & & x_1, x_2 & \geq 0
\end{array}
$$

Standard formulation after introduction of slack variables.

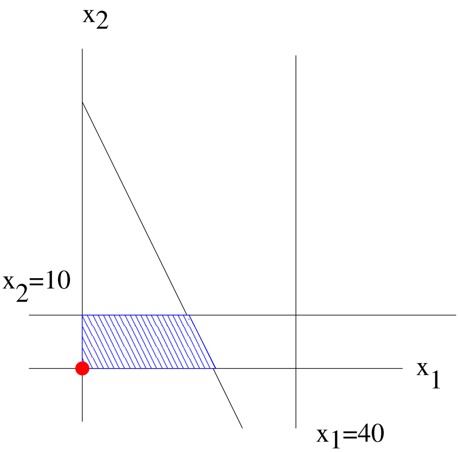
$$
\begin{array}{rrrrrrrcl}
\min f(x_1, x_2) & = & -120x_1 & -80x_2 & & & & & \\
\text{s.t.} & & x_1 & & +s_1 & & & = & 40 \\
 & & & x_2 & & +s_2 & & = & 10 \\
 & & 20x_1 & +10x_2 & & & +s_3 & = & 500 \\
 & & x_1, & x_2, & s_1, & s_2, & s_3 \geq & 0 &
\end{array}
$$

Start solution easy to read now: take $x_1 = x_2 = 0$ and $s_1 = 40$, $s_2 = 10$, and $s_3 = 500$.

In general: if

- each constraint is with $\leq$,
- non-negative right hand sides,

then easy to find start solution. Make each slack variable equal to the value of its right hand side and make all other variable $= 0$.



In this picture we see 5 lines: $x_1 = 0$, $x_2 = 0$, $x_1 = 40$, $x_2 = 10$, and $20x_1 + 10x_2 = 500$. The last three lines correspond to $s_1 = 0$, $s_2 = 0$, and $s_3 = 0$.

What does the simplex method?

For $x_1 = x_2 = 0$, $s_1 = 40$, $s_2 = 10$, and $s_3 = 500$, we get $f(x_1, x_2) = 0$.

- Let's increase x_1. We must decrease s_1 and s_3. How much can we increase x_1 without making s_1 and s_3 negative?

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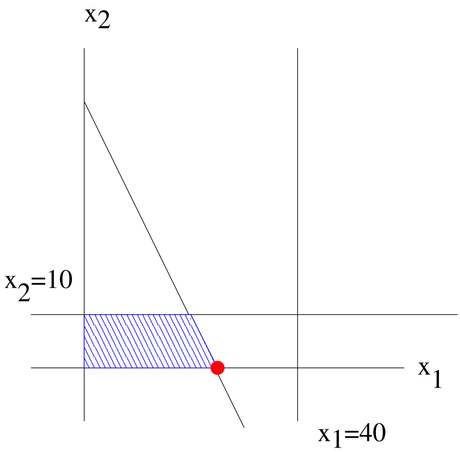
- Let's increase x_1. We must decrease s_1 and s_3. How much can we increase x_1 without making s_1 and s_3 negative?
- From $x_1 + s_1 = 40$ it follows that x_1 can be at most 40. From $20x_1 + 10x_2 + s_3 = 500$ it follows that x_1 can be at most 25.

What does the simplex method?

For $x_1 = x_2 = 0$, $s_1 = 40$, $s_2 = 10$, and $s_3 = 500$, we get $f(x_1, x_2) = 0$.

- Let's increase x_1. We must decrease s_1 and s_3. How much can we increase x_1 without making s_1 and s_3 negative?
- From $x_1 + s_1 = 40$ it follows that x_1 can be at most 40. From $20x_1 + 10x_2 + s_3 = 500$ it follows that x_1 can be at most 25.
- We can increase x_1 not beyond 25. If $x_1 = 25$, then $s_3 = 0$.

We obtain $x_1 = 25$, $x_2 = 0$, $s_1 = 15$, $s_2 = 10$, and $s_3 = 0$.



The red point is in the intersection of the lines $x_2 = 0$ and $20x_1 + 10x_2 = 500$.

In general, if we have two lines

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 &= b_2, \end{aligned}$$

the intersection is the vector

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

But what happens algebraically?

In the beginning we have:

$$\begin{array}{rcrrrrrcl} f(x_1, x_2) & = & -120x_1 & -80x_2 & & & & & \\ & & x_1 & & +s_1 & & & = & 40 \\ & & & x_2 & & +s_2 & & = & 10 \\ & & 20x_1 & +10x_2 & & & +s_3 & = & 500 \\ & & x_1, & x_2, & s_1, & s_2, & s_3 & \geq & 0 \end{array}$$

Rewrite the last equality as

$$x_1 + \frac{1}{2}x_2 + \frac{1}{20}s_3 = 25$$

We can now immediately read that x_1 can be at most 25. Subtract this equation from

$$x_1 + s_1 = 40.$$

And then?

Substitute $x_1 = 25 - \frac{1}{2}x_2 - \frac{1}{20}s_3$ in the objective function. We obtain:

$$\begin{array}{rcccccccl} f(x_1,x_2) & = & & -20x_2 & & & 6s_3 & - & 3000 \\ & & & -\frac{1}{2}x_2 & +s_1 & & -\frac{1}{20}s_3 & = & 15 \\ & & & x_2 & & +s_2 & & = & 10 \\ & & x_1 & +\frac{1}{2}x_2 & & & +\frac{1}{20}s_3 & = & 25 \\ & & x_2, & s_1, & s_2, & s_3 & \geq & 0 & \end{array}$$

Make $x_2 = s_3 = 0$, and $x_1 = 25$, $s_2 = 10$, and $s_1 = 15$. This yields $f(x_1,x_2) = -20x_2 + 6s_3 - 3000 = -3000$. We have a next vertex of the polyhedron described by the constraints.
From $f(x_1,x_2) = -20x_2 + 6s_3 - 3000$ we read that $f(x_1,x_2)$ will be decreased if we increase x_2. Let's do that. By how much can we increase x_2.

How much can we increase x_2?

The equality

$$-\frac{1}{2}x_2 + s_1 - \frac{1}{20}s_3 = 15$$

is equal to the equality

$$s_1 - \frac{1}{20}s_3 = 15 + \frac{1}{2}x_2$$

If we increase x_2, we have to increase s_1. We are allowed to do this. Remember that $s_3 = 0$. So no restriction from this equality.
From

$$x_2 + s_2 = 10$$

follows that x_2 cannot be larger than 10. From

$$x_1 + \frac{1}{2}x_2 + \frac{1}{20}s_3 = 25$$

follows that x_2 cannot be larger than 50. So x_2 can be increased to 10, and s_2 will decrease to 0, and x_1 will decrease to 20.

Algebraically, we did

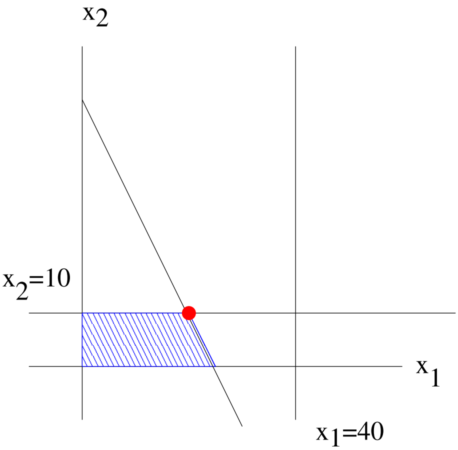
First we had

$$\begin{array}{rcrrrrcl}
f(x_1,x_2) & = & & -20x_2 & & & 6s_3 & - & 3000 \\
& & & -\frac{1}{2}x_2 & +s_1 & & -\frac{1}{20}s_3 & = & 15 \\
& & & x_2 & & +s_2 & & = & 10 \\
& & x_1 & +\frac{1}{2}x_2 & & & +\frac{1}{20}s_3 & = & 25 \\
& & & x_2, & s_1, & s_2, & s_3 & \geq & 0
\end{array}$$

Now we have

$$\begin{array}{rrrrrrcl}
f(x_1,x_2) = & & & & +20s_2 & +6s_3 & - & 3200 \\
& & & s_1 & +\frac{1}{2}s_2 & -\frac{1}{20}s_3 & = & 20 \\
& & x_2 & & +s_2 & & = & 10 \\
& x_1 & & & -\frac{1}{2}s_2 & +\frac{1}{20}s_3 & = & 20 \\
& x_1, & x_2, & s_1, & s_2, & s_3 & \geq & 0
\end{array}$$

Make $s_2 = 0$ and $s_3 = 0$. Then $s_1 = 20$, $x_2 = 10$, and $x_1 = 20$. The value of $f(x_1, x_2)$ is now -3200.


x2
x2=10
x1
x1=40

We must do this systematically.

We have

$$\begin{array}{rrrrrrrrr} \min f(x_1,x_2) & = & -120x_1 & -80x_2 & & & & & \\ \text{s.t.} & & x_1 & & +s_1 & & & = & 40 \\ & & & x_2 & & +s_2 & & = & 10 \\ & & 20x_1 & +10x_2 & & & +s_3 & = & 500 \\ & & x_1, & x_2, & s_1, & s_2, & s_3 \geq & 0 & \end{array}$$

The feasible solutions satisfy:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 20 & 10 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \\ 500 \end{bmatrix}$$

We can write this as:

$$Ax = b,$$

where

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \end{bmatrix}$$

and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

1. If a set of columns of A form a basis, then we call the corresponding set of variables a basis.
2. Variables in the basis are called basic variables.
3. The other variables are called non-basic variables.

So s_1, s_2, s_3 form a basis. Then x_1, x_2 are non-basic variables.
Also x_2, s_1, s_2 form a basis. Then x_1, s_3 are non-basic variables.

Make non-basic variables zero

If we make non-basic variables zero, then we can solve the basic variables. (Because the corresponding columns form a basis.)

If we take s_1, s_2, s_3 as basis, then we make $x_1 = 0$ and $x_2 = 0$. We can solve s_1, s_2, s_3 in

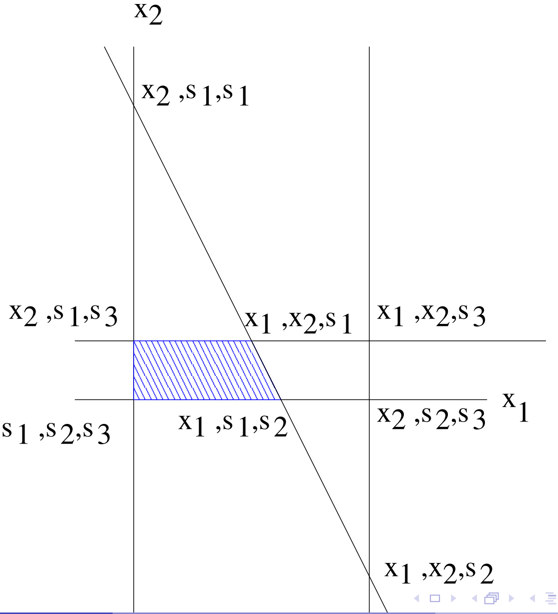
$$A \begin{bmatrix} 0 \\ 0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = b.$$

This yields $s_1 = 40$, $s_2 = 10$, $s_3 = 500$.

Definition

A basic solution is a solution of $Ax = b$ where we have made non-basic variables equal to zero. If the basic solution satisfies $x_1 \geq 0$, $x_2 \geq 0$, $s_1 \geq 0$, $s_2 \geq 0$, $s_3 \geq 0$, then we call x a basic feasible solution.

Each basic solution corresponds to the intersection of two lines. (In general: to the intersection of n hyperplanes.)



- If we take x_1, s_1, s_2 as basis, then we make $x_2 = 0$ and $s_3 = 0$. We can solve x_1, s_1, s_2 and find that $x_1 = 25$, $s_1 = 15$, $s_2 = 10$. Then $x_1 = 25$, $x_2 = 0$, $s_1 = 15$, $s_2 = 10$, and $s_3 = 0$ is a basic feasible solution.
- If we take x_1, x_2, s_2 as basis, then we make $s_1 = 0$ and $s_3 = 0$. We find that $x_1 = 40$, $x_2 = -30$, $s_2 = 40$. Then $x_1 = 40$, $x_2 = -30$, $s_1 = 0$, $s_2 = 40$, and $s_3 = 0$ is a basic solution which is not feasible.

Steps of simplex method

The simplex method goes from one basic feasible solution to another basic feasible solution, each time make one basic variable non-basic and one non-basic variable basic.

The augmented matrix is

a_1	a_2	a_3	a_4	a_5	b
1	0	1	0	0	40
0	1	0	1	0	10
20	10	0	0	1	500

We place the objective function as the last row in the augmented matrix.

	a_1	a_2	a_3	a_4	a_5	b
	1	0	1	0	0	40
	0	1	0	1	0	10
	20	10	0	0	1	500
c^T	-120	-80	0	0	0	0

As basic variables we take s_1, s_2, s_3. In the simplex method we can find canonical basis vectors under the columns corresponding to the basic variables.

What now?

The elements in the row of c^T are called the reduced cost coefficients. (Except for the last element, which is the negative of $f(x_1, x_2)$.)

The reduced cost coefficients of basic variables must always be 0.

Is any of reduced cost coefficients negative?

- No. We have found an optimal solution.
- Yes. We can make the solution we have found sofar better. We change the basis: one basic variable will leave the basis and one non-basic variable will enter the basis.

- We see that the column a_1 has a negative number in the row of c^T.
- We make x_1 the entering variable. Previously it was a non-basic variable, now it becomes a basic variable.
- Apply minimum ratio test: Find in the column of a_1 (the entering variable) the row i such that entry $a_{i,1}$ is positive and $b_i/a_{i,1}$ is minimal.

a_1	b	$b_1/a_{i,1}$
1	40	$b_1/a_{1,1} = 40$
0	10	N.A
20	500	$b_3/a_{3,1} = 25$

  Minimum is attained in the third row.
- s_3 is the leaving variable. This variable will leave the basis.

- The column of the entering variable will be called the pivot column. The row of the leaving variable will be called the pivot row. The entry that is in the pivot column and pivot row is the pivot element.
- We have a new basis. Must make the matrix of columns of the basic variables the identity matrix (up to column permutation).
- Perform Gaussian elimination to accomplish this.

First we have:

	a_1	a_2	a_3	a_4	a_5	b
	1	0	1	0	0	40
	0	1	0	1	0	10
	20	10	0	0	1	500
c^T	-120	-80	0	0	0	0

After Gauss we obtain

	0	$-\frac{1}{2}$	1	0	$-\frac{1}{20}$	15
	0	1	0	1	0	10
	1	$\frac{1}{2}$	0	0	$\frac{1}{20}$	25
c^T	0	-20	0	0	6	3000

In the columns corresponding to the basic variables we find canonical basis vectors. The last column shows their values. Non-basic variables have value 0. So $f(x_1, x_2) = -3000$, $s_1 = 15$, $s_2 = 10$, $x_1 = 25$, $x_2 = 0$, and $s_3 = 0$.

Next tableau is:

$$\left|\begin{array}{ccccc|c} 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{20} & 20 \\ 0 & 1 & 0 & 1 & 0 & 10 \\ 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{20} & 20 \\ 0 & 0 & 0 & 20 & 6 & 3200 \end{array}\right.$$

No negative number in the row of z. Optimum. Optimal solution is $s_1 = 20$, $x_2 = 10$, and $x_1 = 20$. The optimal value is $f(x_1, x_2) = -3200$. The optimal value is the negative of the right lower element in the

tableau.

Remember: In each tableau we see on the right side of the augmented matrix non-negative numbers. (Except of course for the lower right element, which is $-f(x_1, x_2)$.)

Unbounded

Minimum ratio test: Find in the column of the entering variable x_j the row i such that entry a_{i,x_j} is positive and $b_i/a_{i,x_j}$ is minimal. What if there is no positive a_{i,x_j}.

Unbounded

If there there is a column a_j such that all entries are non-positive (≤ 0), then the optimal value of the LP is unbounded.

Remember this rule.

$$
\begin{array}{rcllll}
\max f(x_1, x_2) & = & x_1 & +2x_2 & & \\
& & -x_1 & +2x_2 & \leq & 1 \\
& & & x_2 & \leq & 1 \\
& & x_1 & , x_2 & \geq & 0
\end{array}
$$

We translate this to

$$
\begin{array}{rcllllll}
\min -f(x_1, x_2) & = & -x_1 & -2x_2 & & & & \\
& & -x_1 & +2x_2 & +s_1 & & = & 1 \\
& & & x_2 & & +s_2 & = & 1 \\
& & x_1 & , x_2 & , s_1 & , s_2 & \geq & 0
\end{array}
$$

	a_1	a_2	a_3	a_4	b
	-1	2	1	0	1
	0	1	0	1	1
c^T	-1	-2	0	0	0

In column a_1 we see only non-positive elements. Unbounded.

degenerate

What if the b_i of the leaving variable is zero?

- There is no decrease in the value of $f(x_1, x_2)$.
- Might be a problem. The simplex method could run forever.
- There are methods to circumvent this. You need not know these methods in this course.

Multiple optimal solutions

What if in the last row the entry of the entering variable is zero.

- There are multiple solutions.

$$
\begin{array}{rclll}
\max f(x_1, x_2) & = & 3x_1 & +2x_2 & \\
 & & 3x_1 & +2x_2 & \leq & 3 \\
 & & x_1 & +3x_2 & \leq & 3 \\
 & & x_1 & -4x_2 & \leq & 2 \\
 & & x_1 & , x_2 & \geq & 0
\end{array}
$$

	a_1	a_2	a_3	a_4	a_5	b
	3	2	1	0	0	3
	1	3	0	1	0	3
	1	-4	0	0	1	2
c^T	-3	-2	0	0	0	0

An optimal simplex tableau is:

	x_1	x_2	s_1	s_2	s_3	b
	1	$\frac{2}{3}$	$\frac{1}{3}$	0	0	1
	0	$2\frac{1}{3}$	$-\frac{1}{3}$	1	0	2
	0	$-4\frac{2}{3}$	$-\frac{1}{3}$	0	1	1
	0	0	1	0	0	3