Minimum rank and 2-separations

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September 5, 2007

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The minimum rank of a graph

Let $G = (V, E)$ be a graph with $V = \{1, 2, \ldots, n\}$ and let F be a field. Define $S(G; F)$ as the set of all symmetric $n \times n$ matrices $A = (a_{i,j})$ with

- $a_{i,j} \neq 0$ if $ij \in E$,
- $a_{i,j} = 0$ if $i \neq j$ and $ij \notin E$, and
- $a_{i,i} \in F$ for all $i \in V$.

Definition

$M(G; F)$ is the largest possible corank over all matrices $A \in S(G; F)$. We define $M(G) = M(G; \mathbb{R})$.

Definition

$\mathrm{mr}(G; F)$ is the smallest possible rank over all matrices $A \in S(G; F)$.

Observation

$\mathrm{mr}(G; F) + M(G; F) = n$

Find $M(G; F)$

Problem

Given a graph G, determine $M(G; F)$.

Problem equivalent to the determination of maximum multiplicity of an eigenvalue of a matrix in $S(G; F)$.
Generalization of the problem (if $F = \mathbb{R}$) is

Problem (Inverse Eigenvalue Problem of a Graph)

Given real numbers $\lambda_1, \dots, \lambda_n$, find a matrix $A \in S(G; \mathbb{R})$ that has $\lambda_1, \dots, \lambda_n$ as eigenvalues.

Not much known is about this problem.

Some easy facts

- $\mathrm{mr}(K_n; F) = 1$ if $n \geq 2$. $\mathrm{mr}(K_1; F) = 0$.
- if G is connected, then $\mathrm{mr}(G; F) \leq 1$ if and only if $G = K_n$.
- if $p, q \geq 1$ then $\mathrm{mr}(K_{p,q}; F) = 2$.
- if H is an induced subgraph of G, then $\mathrm{mr}(H; F) \leq \mathrm{mr}(G; F)$.
- $0 \leq \mathrm{mr}(G; F) - \mathrm{mr}(G - v; F) \leq 2$.
- $M(G - v; F) - 1 \leq M(G; F) \leq M(G - v; F) + 1$.

Known results

Theorem (Fiedler (1969))

A graph G has $M(G) \leq 1$ if and only if G is a path.

Corollary

$M(C_n) = 2$.

Theorem (Duarte and Johnson (1999))

For any tree T, $M(T)$ is equal to the minimum number of vertex-disjoint paths needed to cover all vertices of T.

Theorem (Barrett, vdHolst, Loewy (2004))

A graph G has $\mathrm{mr}(G) \leq 2$ if and only if the complement of G is of the form

$$(K_{s_1} \cup K_{s_2} \cup K_{p_1,q_1} \cup \ldots \cup K_{p_k,q_k}) \vee K_r$$

for appropriate nonnegative numbers $s_1, s_2, k, p_1, q_1, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$ for $i = 1, 2, \ldots, k$.

- Theorems were given for each field F.
- At the moment no characterization is known of the class of graphs G with $\mathrm{mr}(G) \leq 3$.
- If F is a finite field, then $\mathrm{mr}(G; F) \leq 3$ is characterized by a finite list of forbidden induced subgraphs (Ding and Kotlov).
- It is unknown whether there is a finite list of forbidden induced subgraphs for $\mathrm{mr}(G) \leq 3$.

Definition

A *k-tree* is recursively defined by:

- a complete graph on k vertices is a k-tree, and
- if $G = (V, E)$ is a k-tree and $v_1, \ldots, v_k$ form a clique in G with k vertices, then $G' = (V \cup \{v\}, E \cup \{(v_1, v) \mid 1 \leq i \leq k\})$ with v a new vertex, is a k-tree.

Definition

A *k-path* is a k-tree with exactly two vertices of degree k. A partial k-path is a subgraph of a k-path.

Lemma (Johnson and Loewy; Hogben and vdHolst)

The following are equivalent for 2-connected graphs.

- $M(G) \leq 2$,
- G *is a partial* 2*-path, and*
- G *has no* K_4*-, no* T_3*-, and no* $K_{2,3}$*-minor.*

Johnson and Loewy have given a complete description of the graphs G with $M(G) \leq 2$. Hogben and van der Holst used this theorem to give a forbidden minor characterization of the graphs with $\xi(G) \leq 2$.

Lemma (vdHolst)

The following are equivalent for 3-connected graphs.

- $M(G) \leq 3$,
- G *is a partial* 3*-path, and*
- G *has no* K_5*-, no* $K_{2,2,2}$*-, no* $K_{3,3}$*-, no* Q_3*-, and no* $Q_3 Y\Delta$*-minor.*

separations of graphs

Definition

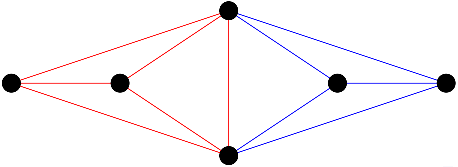
A separation (G_1, G_2) of a graph G is a pair of subgraphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ such that

1. $V_1 \cup V_2 = V$,
2. $E_1 \cup E_2 = E$, and
3. $E_1 \cap E_2 = \emptyset$.

The order of a separation is $|V_1 \cap V_2|$.

Definition

A k-separation is a separation of order k.



0- and 1-separations

Theorem (Homework exercise)

Let (G_1, G_2) *be a* 0*-separation of* G*. Then*

$$M(G;F) = M(G_1;F) + M(G_2;F).$$

Theorem (Barioli, Fallat, and Hogben; and Hsieh)

Let (G_1, G_2) *be a* 1*-separation of* G*. Let* r *be the common vertex of* G_1 *and* G_2*. Then*

$$M(G;F) = \max\{M(G_1;F)+M(G_2;F)-1, M(G_1-r;F)+M(G_2-r;F)-1\}.$$

multigraphs

We extend $S(G;F)$ to multigraphs G.
If F is a field unequal to F_2, define $S(G;F)$ as the set of all symmetric $n \times n$ matrices $A = (a_{i,j})$ with

- $a_{i,j} \neq 0$ if $i \neq j$ and i and j are connected by a single edge,
- $a_{i,j} \in F$ if $i \neq j$ and i and j are connected by multiple (parallel) edges,
- $a_{i,j} = 0$ if $i \neq j$ and $ij \notin E$, and
- $a_{i,i} \in F$ for all $i \in V$.

Let G be a graph in which i and j are connected by multiple edges. Let G' and G'' be obtained from G by removing one edge between i and j, and removing all edges between i and j, resp.

Lemma

$M(G;F) = \max\{M(G';F), M(G'';F)\}.$

So we can reduce computation of $M(G;F)$ for multigraphs to simple graphs.

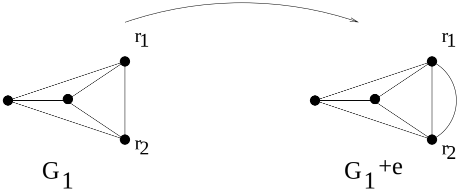
For F_2, we define $S(G; F_2)$ as the set of all symmetric $n \times n$ matrices $A = (a_{i,j})$ with

- $a_{i,j} \neq 0$ if $i \neq j$ and i and j are connected by an odd number of parallel edges,
- $a_{i,j} = 0$ if $i \neq j$ and i and j are connected by an even number of parallel edges,
- $a_{i,j} = 0$ if $i \neq j$ and $ij \notin E$, and
- $a_{i,i} \in F_2$ for all $i \in V$.

Let (G_1, G_2) be a 2-separation of G. Let r_1, r_2 be the common vertices of G_1 and G_2.

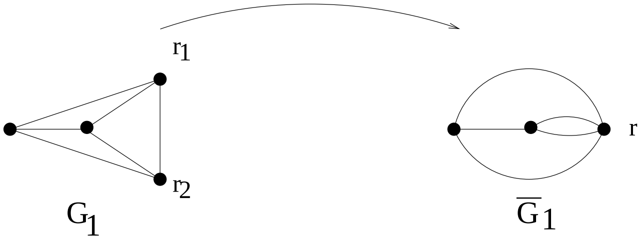
Definition

$G_1 + e$ and $G_2 + e$ denote the graphs obtained from G_1 and G_2, resp. by adding an edge between r_1 and r_2.



Definition

$\overline{G_1}$ and $\overline{G_2}$ denote the graphs obtained from G_1 and G_2 by identifying r_1 and r_2.

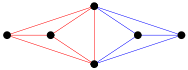


The Formula

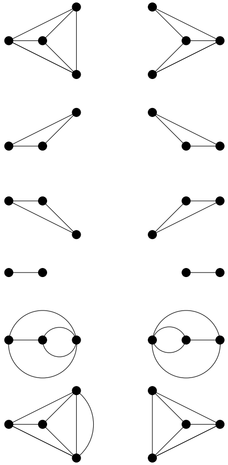
Theorem (vdHolst)

Let (G_1, G_2) be a 2-separation of G. Let r_1 and r_2 be the common vertices of G_1 and G_2. Then

$$\begin{aligned} M(G;F) = \max\{ \quad & M(G_1;F) + M(G_2;F) - 2, \\ & M(G_1 - r_1;F) + M(G_2 - r_1;F) - 2, \\ & M(G_1 - r_2;F) + M(G_2 - r_2;F) - 2, \\ & M(G_1 - \{r_1, r_2\};F) + M(G_2 - \{r_1, r_2\};F) - 2, \\ & M(G_1 + e;F) + M(G_2 + e;F) - 2, \\ & M(\overline{G_1};F) + M(\overline{G_2};F) - 2\} \end{aligned}$$



Maximum corank of this graph is 4.



3+2−2=3

2+2−2=2

2+2−2=2

1+1−2=0

2+2−2=2

3+3−2=4

All six terms are needed in the formula

Here are examples for three of the terms.

- Take $G_1 = G_2 = \overline{K_2}$. Then $G = \overline{K_2}$. We have $M(G_1; F) = M(G_2; F) = M(G; F) = 2$. This is attained only by $M(G; F) = M(G_1; F) + M(G_2; F) - 2$.
- Take $G_1 = G_2 = K_2$. Then G consists of two vertices and multiple edges. We have $M(G_1; F) = M(G_2; F) = 1$ and $M(G; F) = 2$. Only attained by $M(G; F) = M(G_1 + e; F) + M(G_2 + e; F) - 2$.
- Take $G_1 = G_2 = K_{2,2}$ with r_1 and r_2 nonadjacent. Then $G = K_{2,4}$. We have $M(G_1; F) = M(G_2; F) = 2$ and $M(G; F) = 4$. Only attained by $M(G; F) = M(\overline{G_1}; F) + M(\overline{G_2}; F) - 2$.

First step is to show that each of the following holds:

- $M(G; F) \geq M(G_1; F) + M(G_2; F) - 2$,
- $M(G; F) \geq M(G_1 + e; F) + M(G_2 + e; F) - 2$,
- $M(G; F) \geq M(\overline{G_1}; F) + M(\overline{G_2}; F) - 2$,
- $M(G; F) \geq M(G_1 - r_1; F) + M(G_2 - r_1; F) - 2$,
- $M(G; F) \geq M(G_1 - r_2; F) + M(G_2 - r_2; F) - 2$, and
- $M(G; F) \geq M(G_1 - \{r_1, r_2\}; F) + M(G_2 - \{r_1, r_2\}; F) - 2$.

Important lemma

Lemma

Let (G_1, G_2) be a 2-separation of G with G_2 a path connecting r_1 and r_2 of length two. Let u be the vertex of degree 2 in G_2. Then

$$\max\{corank(A) \mid A = (a_{i,j}) \in S(G;F), a_{u,u} = 0\} = M(\overline{G_1};F).$$

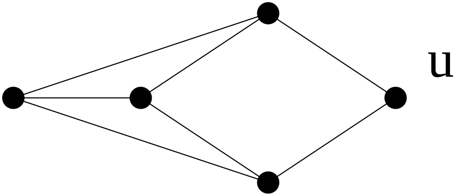


Figure: An example

Important lemma

Lemma

Let (G_1, G_2) be a 2-separation of $G = (V, E)$ with G_2 a path connecting r_1 and r_2 of length two. Let u be the vertex of degree 2 in G_2. Then

$$\max\{corank(A) \mid A = (a_{i,j}) \in S(G; F), a_{u,u} = 0\} = M(\overline{G_1}; F). \quad (1)$$

- Take a matrix $B = (b_{i,j})$ that attains the maximum on the left side of (1).
- Let $W = (w_{i,j}) = B/B[\{u, r_1\}]$.
- Then $\text{corank}(W) = \text{corank}(B)$ and $W \in S(\overline{G_1}; F)$.
- So $M(\overline{G_1}; F) \geq \text{corank}(B)$.
- Conversely, given $W \in S(\overline{G_1}; F)$, we can construct a $B \in S(G; F)$ such that $B/B[\{u, r_1\}] = W$.

Lemma

Let $G = (V, E)$ be a graph and let $r_1, r_2 \in V$. Let $\overline{G}$ be obtained from G by identifying r_1, r_2. Then

$$M(\overline{G}; F) \leq M(G; F) + 1.$$

- Add a new vertex u to G and this to r_1 and r_2. Let G' be the graph obtained.
- Then $M(\overline{G}; F) = \max\{\text{corank}(A) \mid A = (a_{i,j}) \in S(G'; F), a_{u,u} = 0\}$.
- We know $\max\{\text{corank}(A) \mid A = (a_{i,j}) \in S(G'; F), a_{u,u} = 0\} \leq M(G'; F) \leq M(G; F) + 1$.
- Hence $M(\overline{G}; F) \leq M(G; F) + 1$.

Lemma

Let (G_1, G_2) *be a 2-separation of* G*, let* $\{r_1, r_2\} = V(G_1) \cap V(G_2)$*, and let* $\overline{G_1}$ *and* $\overline{G_2}$ *be obtained from* G_1 *and* G_2*, respectively, by identifying* r_1 *and* r_2*. Then*

$$M(G; F) \geq M(\overline{G_1}; F) + M(\overline{G_2}; F) - 2.$$

- Let $\overline{G}$ be obtained from G by identifying r_1 and r_2.
- Then $M(\overline{G}; F) \leq M(G; F) + 1$.
- The theorem on 1-separations shows that $M(\overline{G_1}; F) + M(\overline{G_2}; F) - 1 \leq M(\overline{G}; F)$.
- Hence $M(\overline{G_1}; F) + M(\overline{G_2}; F) - 2 \leq M(G; F)$.

Must now show that at least one of the following holds:

- $M(G;F) \leq M(G_1;F) + M(G_2;F) - 2$,
- $M(G;F) \leq M(G_1 + e;F) + M(G_2 + e;F) - 2$,
- $M(G;F) \leq M(\overline{G_1};F) + M(\overline{G_2};F) - 2$,
- $M(G;F) \leq M(G_1 - r_1;F) + M(G_2 - r_1;F) - 2$,
- $M(G;F) \leq M(G_1 - r_2;F) + M(G_2 - r_2;F) - 2$, and
- $M(G;F) \leq M(G_1 - \{r_1, r_2\};F) + M(G_2 - \{r_1, r_2\};F) - 2$.

Let $A \in S(G; F)$ with $\operatorname{corank}(A) = M(G; F)$.

Lemma

There is an $S \subseteq V(G) - \{r_1, r_2\}$ *such that* $Z = (z_{i,j}) = A/A[S]$ *satisfies* $Z[V(G) - (\{r_1, r_2\} \cup S)] = 0$.

We know that $\operatorname{corank}(Z) = \operatorname{corank}(A)$.

Definition

The graph $G(Z)$ of Z has as vertex-set $V(G) - (\{r_1, r_2\} \cup S)$ and as edge-set $\{ij \mid i \neq j, z_{i,j} \neq 0\}$.

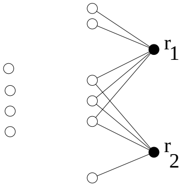


Figure: An example of $G(Z)$.

We now apply a case study. Suppose there is a vertex w in $G(Z)$, $w \neq r_1, r_2$, adjacent to only r_1.

- $\operatorname{corank}(A(r_1)) = \operatorname{corank}(Z(r_1)) = \operatorname{corank}(Z) + 1 = \operatorname{corank}(A) + 1$.
- $A(r_1) \in S(G - r_1; F)$.
- The theorem on 1-separation shows that
  - $\operatorname{corank}(A(r_1)) \leq M(G_1 - r_1; F) + M(G_2 - r_1; F) - 1$, or
  - $\operatorname{corank}(A(r_1)) \leq M(G_1 - \{r_1, r_2\}; F) + M(G_2 - \{r_1, r_2\}; F) - 2$.
- Thus
  - $M(G; F) \leq M(G_1 - r_1; F) + M(G_2 - r_1; F) - 2$, or
  - $M(G; F) \leq M(G_1 - \{r_1, r_2\}; F) + M(G_2 - \{r_1, r_2\}; F) - 2$.

We now assume that there is a vertex w in $G(Z)$ adjacent to both r_1 and r_2.

- Add a new vertex u to G and connect this to r_1 and r_2; let G' be the new graph.
- Extend A to a matrix $A' = (a'_{i,j}) \in S(G'; F)$ such that $a'_{u,u} = 0$ and $\text{corank}(A') = \text{corank}(A) + 1$.
- Identify r_1 and r_2. Let the new graph be $\overline{G}$ and new vertex be r.
- We know $M(\overline{G}; F) \geq \text{corank}(A') = \text{corank}(A) + 1 = M(G; F)$.
- Theorem on 1-separations shows that
  - $M(\overline{G}; F) \leq M(\overline{G_1}; F) + M(\overline{G_2}; F) - 1$, or
  - $M(G; F) \leq M(\overline{G_1} - r; F) + M(\overline{G_2} - r; F) - 1$.
- So
  - $M(G; F) \leq M(\overline{G_1}; F) + M(\overline{G_2}; F) - 2$, or
  - $M(G; F) \leq M(G_1 - \{r_1, r_2\}; F) + M(G_2 - \{r_1, r_2\}; F) - 2$.

Remains the case in which each vertex w in $G(Z)$ is nonadjacent to r_1 and r_2.

A corollary

Proposition

If G_2 is a path of length 2 connecting r_1 and r_2, then

$$M(G;F) = \max\{M(G_1 + e; F), M(\overline{G_1}; F)\}$$

Let P be an induced path in G with ends p_1, p_2. Let G' be obtained from G by removing P and identifying p_1 and p_2. Let G'' be obtained from G by removing P and adding an edge between p_1 and p_2.

Proposition

$M(G;F) = \max\{M(G';F), M(G'';F)\}$.

$M_+(G)$

Definition

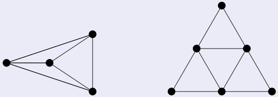
$M_+(G)$ is the largest possible corank over all positive semidefinite matrices in $S(G)$.

Theorem (Fiedler)

For any tree T, $M_+(T) \leq 1$.

Theorem (vdHolst, independently Kotlov)

A 2-connected graph G *has* $M_+(G) \leq 2$ *if and only if* G *has no subgraph homeomorphic to*



Theorem

Let (G_1, G_2) *be a* 1*-separation of* G. *Then*

$$M_+(G) = M_+(G_1) + M_+(G_2) - 1$$

Theorem

Let (G_1, G_2) *be a* 2*-separation of* G. *Then*

$$M_+(G) = \max\{M_+(G_1) + M_+(G_2) - 2, M_+(G_1 + e) + M_+(G_2 + e) - 2\}.$$