Rewriting on string diagrams and categorification

Benjamin DUPONT

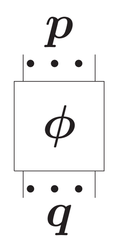
Master Student, Institut Camille Jordan, UMR 5208
Université Claude Bernard. Lyon, France

Objective

- This works presents a symbolic computation based on rewriting to study algebraic structures that appear in higher dimensional representation theory.
- This algebraic structures are higher dimensional representations occuring in the context of categorification of algebras, see [3], [4].
- The main objective is to compute by rewriting methods some bases of the spaces of 2-cells in these categories.

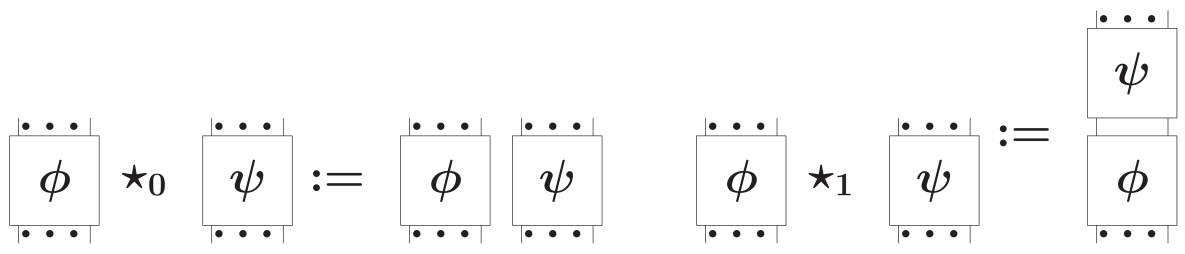
Linear $(2, 2)$-categories

- A *linear $(2, 2)$-category* is a 2-category with only one 0-cell (in the cases we consider), 1-cells and 2-cells. Besides, there is a linear structure on the spaces of 2-cells, see [1].
- These categories are presented by rewriting systems called *linear $(3, 2)$-polygraphs*. In those rewriting systems, the generating 2-cells have the form of a circuit as follows :

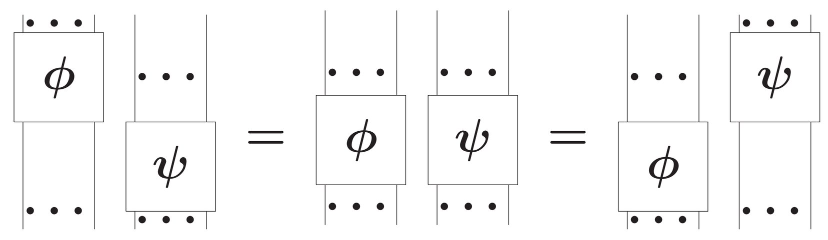


where p and q are two 1-cells of the category.
- These generators can be composed in two ways

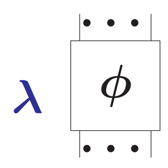
Horizontally Vertically



- All these compositions are made modulo the exchange law of the 2-category :



- One can also make linear combinations of these circuits with scalars in a ground field $\mathbb{K}$. An element of the form



where ϕ is a 2-cell obtained with the previous compositions of generating 2-cells and $\lambda \in \mathbb{K}$ is called *a monomial* in the linear $(2, 2)$-category.

Rewriting in linear $(2, 2)$-categories

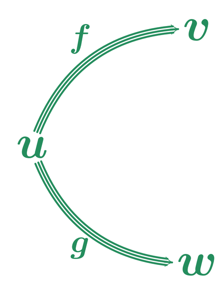
- A *rewriting step* of Σ is a 3-cell of the form

$$\lambda m_1 \star_1 (m_2 \star_0 s_2(\alpha) \star_0 m_3) \star_1 m_4 + u \Rrightarrow \lambda m_1 \star_1 (m_2 \star_0 t_2(\alpha) \star_0 m_3) \star_1 m_4 + u$$

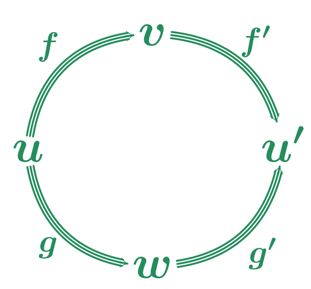
where $s_2(\alpha)$ and $t_2(\alpha)$ are two parallel 2-cells such that the monomial $\lambda m_1 \star_1 (m_2 \star_0 s_2(\alpha) \star_0 m_3) \star_1 m_4$ does not appear in the monomial decomposition of u.
- A *rewriting sequence* of Σ is a finite or infinite sequence :

$$u_0 \Rrightarrow \cdots \Rrightarrow u_n \Rrightarrow \cdots$$

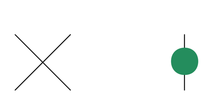
of rewriting steps of Σ.
- A *branching* of Σ is



- A branching is *confluent* if it can be completed by rewriting sequences f' and g' as follows :



- A *local branching* of Σ is a pair of rewriting steps of Σ with the same 2-source.
- A linear $(3, 2)$-polygraph is :
  - *confluent (resp. locally confluent)* if all its (resp. local) branchings are confluent.
  - *terminating* if it has no infinite rewriting sequence.
  - *left monomial* is every source of a 3-cell in Σ is a monomial.
- **Example.** Here, an example of linear $(3, 2)$-polygraph with one 0-cell, one 1-cell, two generating 2-cells



and two 3-cells :

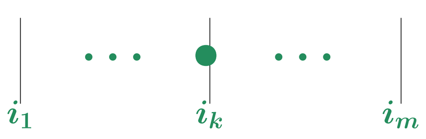
Rewriting results

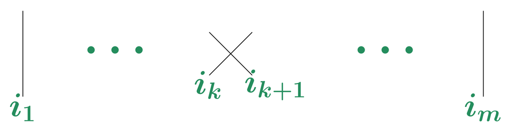
- In this setting, we have a version of classic rewriting results such as Noetherian's induction principle and Newmann's lemma. Thus, a terminating linear $(3, 2)$-polygraph is confluent if and only if all its critical branchings are confluent.
- **Proposition, [1].** *Let Σ be a confluent and terminating left-monomial linear $(3, 2)$-polygraph and $\mathcal{C}$ be the linear $(2, 2)$-category presented by Σ. Then, for any 1-cells u and v of $\mathcal{C}$ with same 0-source and 0-target, the set of monomials of Σ in normal form from u to v gives a basis of $\mathcal{C}(u, v)$.*

The simply-laced KLR algebras

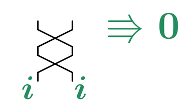
Let I be a set of vertices of a graph and $\mathcal{V} = \sum_{i \in I} \mathcal{V}_i \cdot i \in \mathbb{N}[I]$. Denote by $\mathrm{Seq}(\mathcal{V})$ the set of sequences of elements of I in which i appears exactly $\mathcal{V}_i$ times and $m := \sum_{i \in I} \mathcal{V}_i$. Fix $\cdot$ a bilinear pairing on I such that $i \cdot j \in \{0, -1\}$ for any i, j. The simply-laced KLR algebra is the algebra presented by :

- **generators** :

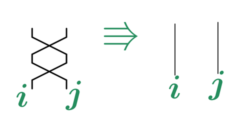
$x_{k,\mathbf{i}}$ =  for $1 \leq k \leq m, \mathbf{i} = i_1 \dots i_m \in \mathrm{Seq}(\mathcal{V})$

$\tau_{k,\mathbf{i}}$ =  for $1 \leq k \leq m-1, \mathbf{i} = i_1 \dots i_m \in \mathrm{Seq}(\mathcal{V})$

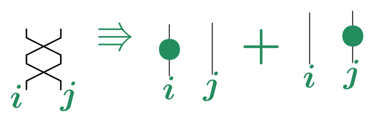
- **relations** :
  i) For any $i \in I$,



  ii) For any $i, j \in I$ such that $i \cdot j = 0$,



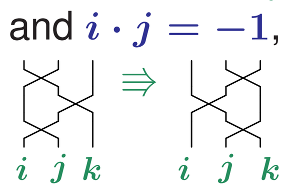
  iii) For any $i, j \in I$ such that $i \cdot j = -1$,



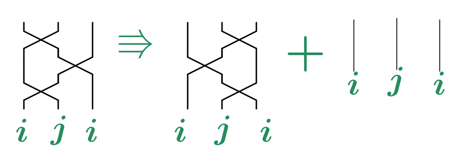
  iv) For any $i, j \in I$,

  v) For any $i \in I$,

  vi) For any $i, j, k \in I$, and unless $i = k$ and $i \cdot j = -1$,



  vii) For any $i, j \in I$ such that $i \cdot j = -1$,



Main results

We define the linear $(3, 2)$-polygraph KLR by :

- One 0-cell $\{*\}$
- The 1-cells are $\mathbf{i} \in \mathrm{Seq}(\mathcal{V})$ so that the generating 1-cells are $i \in I$
- The 2-cells between two 1-cells $\mathbf{i}$ and $\mathbf{j}$ are given by the braid-like diagrams which link $\mathbf{i}$ to $\mathbf{j}$, that is each vertex at the bottom is linked by a strand to a vertex of the top such that a strand doesn't intersect with itself.
- The 3-cells are given as above.

Theorem. *The linear $(3, 2)$-polygraphs KLR presents the simply-laced KLR algebras and are terminating and confluent.*

Corollary. *The simply-laced KLR algebras admit Poincaré-Birkhoff-Witt bases*, where these PBW bases can be described as the diagrams which contain a minimal number of crossings, all the Yang-Baxter oriented to the right and all the dots placed in the bottom of the strands.

Conclusions

- We found bases for the simply-laced KLR algebras which will be useful to prove a theorem of categorification of quantum groups.
- The 2-categories that categorify those groups has more generators than the KLR algebras (cups and caps) and more relations too. It is the next step to apply the same computation methods to find bases of those 2-categories in order to prove that the 2-category so defined does not have too huge spaces of 2-cells or extra relations that annihilates everything.

References

[1]-C. Alleaume, *Rewriting in higher dimensional linear categories and application to the affine Oriented Brauer category*, 2016, arXiv : 1603.02592, Journal of Pure and Applied Algebra (to be published).
[2]- Y. Guiraud, P. Malbos, *Higher dimensional categories with finite derivation type*, 2009, Theory and Applications of Categories Vol 22, p.420–478.
[3]-R. Rouquier, *2-Kac-Moody algebras*, 2008, arXiv :0812.5023.
[4]-M. Khovanov, A. Lauda *A diagrammatic approach to categorification of quantum groups III*, 2018, arXiv :0807.3250.