Streams from a rewriting perspective

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- Focus on computer supported techniques: enter stream specification in appropriate tool, after pressing a button a proof of the desired property is generated fully automatically by the tool

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- How to prove that stream specifications are *well-defined* or *productive*?
- *Term rewriting* is our central mechanism
- Focus on uniquely defined streams by requiring *orthogonality*
- Focus on computer supported techniques: enter stream specification in appropriate tool, after pressing a button a proof of the desired property is generated fully automatically by the tool
- We omit several technical details

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Example:

One equation $\text{ones} = 1 : \text{ones}$

$$\text{ones} \to 1 : \text{ones} \to 1 : 1 : \text{ones} \to 1 : 1 : 1 : \text{ones} \to \cdots$$

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The Fibonacci stream

$$\mathsf{Fib} = 0 : 1 : 0 : 0 : 1 : 0 : 1 : 0 : 0 : 1 : 0 : 0 : 1 : 0 : 1 : 0 : 0 : 1 : \cdots$$

is uniquely defined by

$$\begin{array}{rcl} \mathsf{Fib} & = & f(\mathsf{Fib}) \\ f(0 : \sigma) & = & 0 : 1 : f(\sigma) \\ f(1 : \sigma) & = & 0 : f(\sigma) \end{array}$$

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or equivalently by

$$\begin{array}{rcl} \mathsf{Fib} & = & 0:c \\ c & = & 1:f(c) \end{array} \qquad \begin{array}{rcl} f(0:\sigma) & = & 0:1:f(\sigma) \\ f(1:\sigma) & = & 0:f(\sigma) \end{array}$$

Visualization of streams

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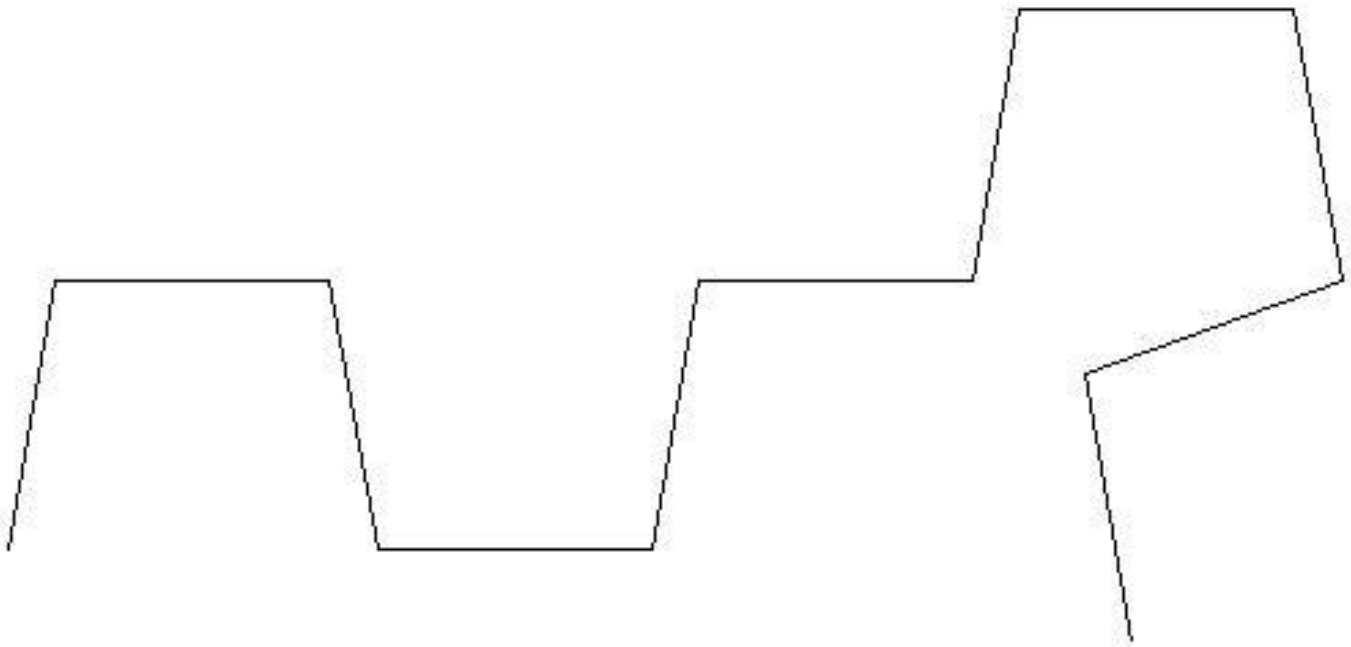
Drawing algorithm:
fix angles α, β, fix initial direction
traverse first N elements of the stream
for every 0 the direction goes α to the right
for every 1 the direction goes β to the left
after every symbol read a unit step is done

Example turtle figure

Taking the first 11 elements 10011010001 of a stream, starting horizontally and choose $\alpha = \beta = 80^o$ yields

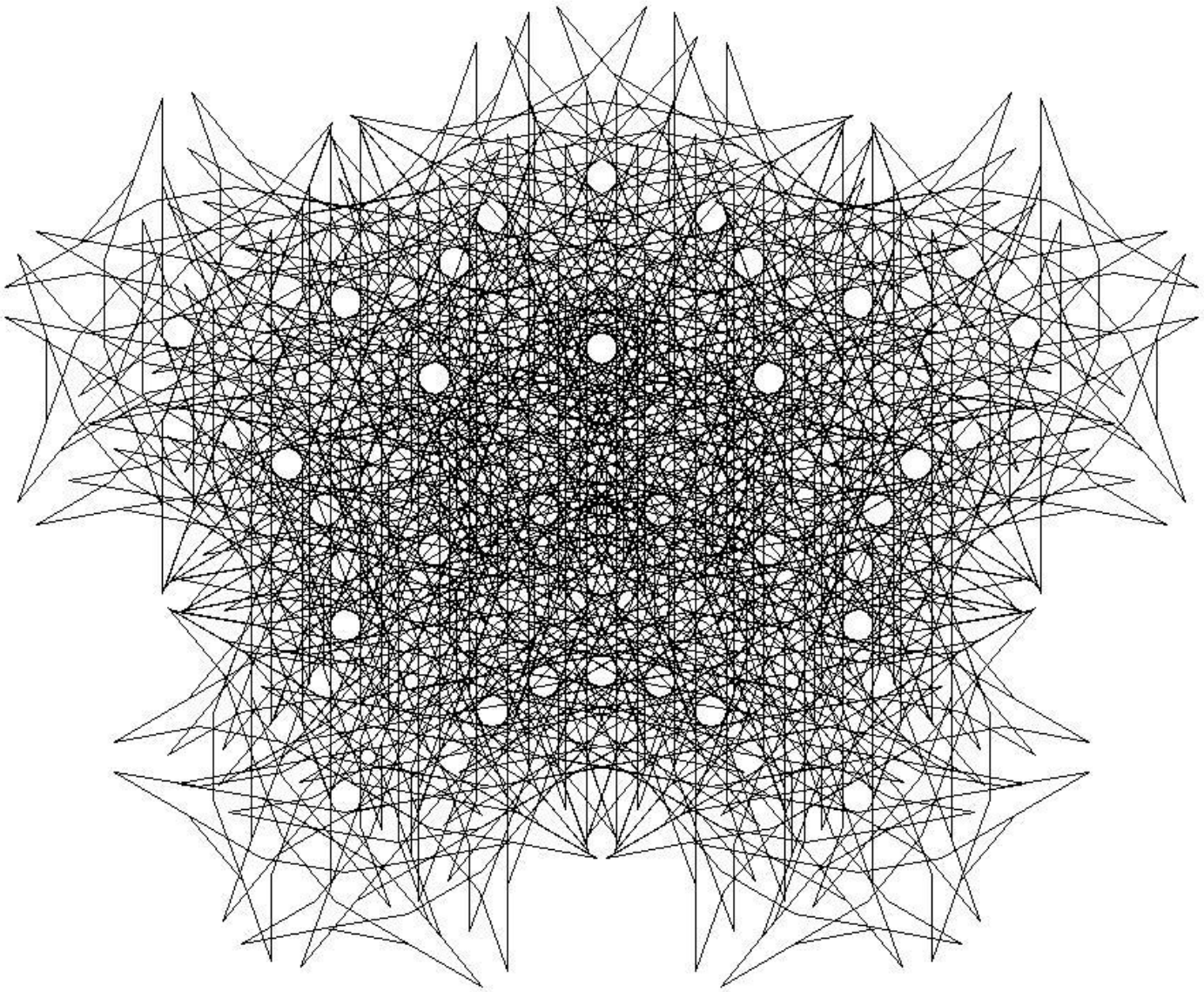


Fibonacci stream

$N =$ 18.000

$\alpha = 20^o$

$\beta = 160^o$

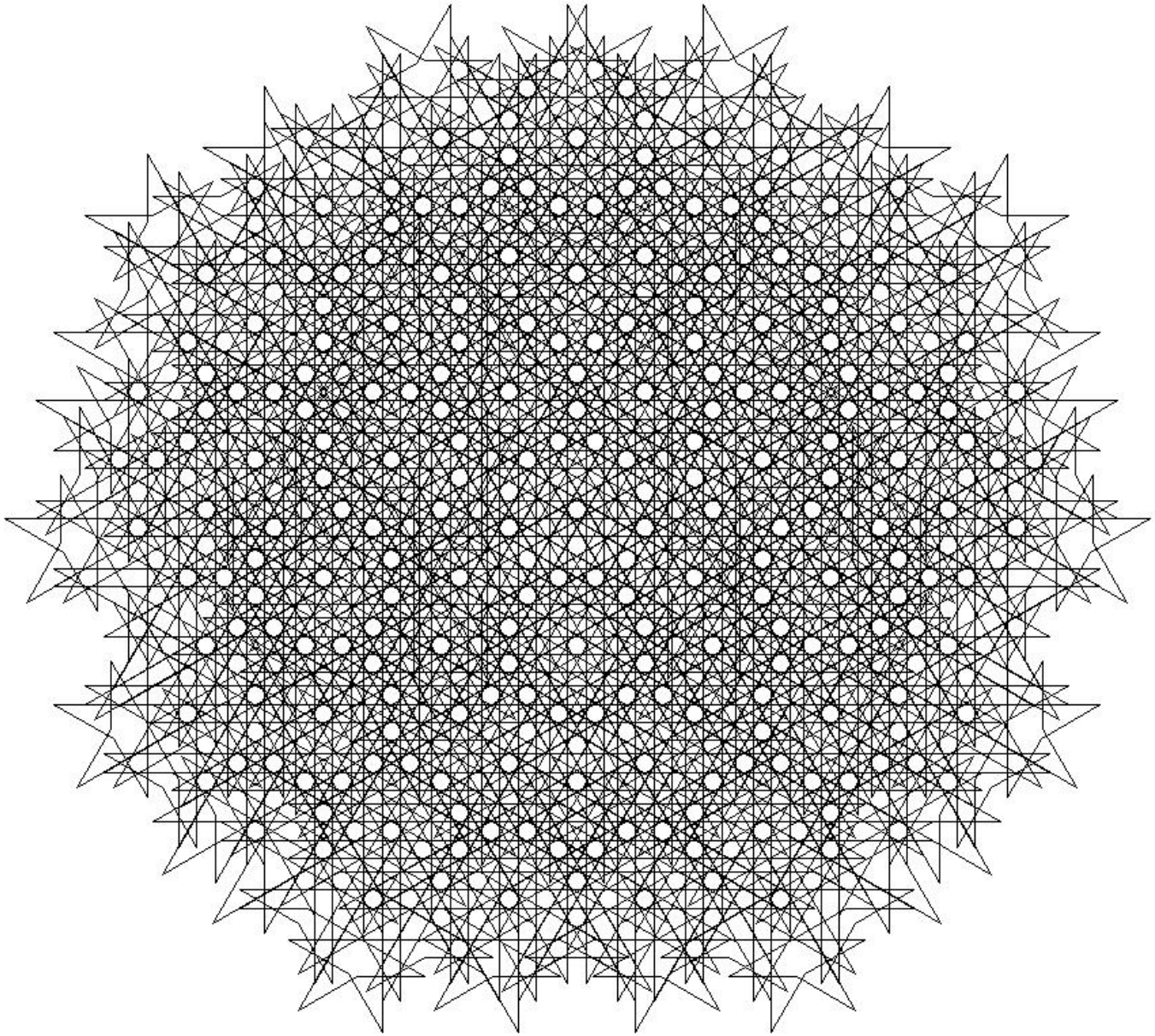


Fibonacci stream

$\alpha = 30^o$

$\beta = 150^o$

200.000 steps



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- *well-definedness*: there exists exactly one stream satisfying the given equations
- *productivity*: when using the equations as rewrite rules, all elements of the stream can be computed

Term rewriting

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Example

$$\begin{array}{rcl} 0 + x & \to & x \\ s(x) + y & \to & s(x + y) \\ 0 * x & \to & 0 \\ s(x) * y & \to & y + (x * y) \end{array}$$

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Every symbol in Σ has a fixed *arity*: the number of arguments it requires

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If R is a TRS, then a *rewrite step* $t \to_R t'$ is the replacement of t by t', where t' is obtained from t by replacing a subterm of t of the shape $\ell\sigma$ by $r\sigma$, for some rule $\ell \to r$ of R

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In this way we can compute $2 + 2 = 4$:

$$\begin{aligned} s(s(0)) + s(s(0)) &\to_R s(\underbrace{s(0) + s(s(0))}) \\ &\to_R s(s(\underbrace{0 + s(s(0))})) \\ &\to_R s(s(s(s(0)))) \end{aligned}$$

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- on every ground term containing a defined symbol a rule is applicable: the TRS is *exhaustive*
- the TRS is *terminating*, so starting in a ground term, rewriting will always end in a constructor ground term
- the TRS is *confluent*, so this resulting constructor ground term is independent of possible choice in the computation

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- Data elements are constructor normal forms of sort d, for instance $\mathbf{N}$ composed from s and 0
- There are no finite constructor ground terms of sort s
- The intended stream is an infinite constructor ground term of sort s, and can be seen as the limit of an infinite reduction

For example, the specification $\mathsf{ones} = 1 : \mathsf{ones}$ coincides with the TRS $\mathsf{ones} \to 1 : \mathsf{ones}$, yielding an infinite reduction

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So the TRS is never terminating

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So the process of computing a stream by rewriting is *not* about infinite terms

Example

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Thue-Morse stream Morse: $D = \{0, 1\}$

$$\begin{array}{rcl} \mathsf{Morse} & \longrightarrow & 0 : c \\ c & \longrightarrow & 1 : f(c) \\ f(0 : xs) & \longrightarrow & 0 : 1 : f(xs) \\ f(1 : xs) & \longrightarrow & 1 : 0 : f(xs) \end{array}$$

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$$\mathsf{Morse} \to 0 : c \to 0 : 1 : f(c) \to 0 : 1 : f(1 : f(c)) \to$$
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\mathrm{merge}(x : xs, y : ys) & \rightarrow & F(c(x, y), x : xs, y : ys) \\
F(0, x : xs, y : ys) & \rightarrow & x : \mathrm{merge}(xs, ys) \\
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$$
\begin{array}{rcll}
c(0, 0) & \to & 0 & \\
c(s(x), s(y)) & \to & c(x, y) & [c(x, y) \to^* 0 \text{ if } x = y] \\
c(0, s(x)) & \to & s(0) & [c(x, y) \to^* s(0) \text{ if } x < y] \\
c(s(x), 0) & \to & s(s(0)) & [c(x, y) \to^* s(s(0)) \text{ if } x > y]
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instead of

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- There **N** was assumed to exist, with $+, *, <, >, =$ with their usual meaning, while we encode this by term rewriting as part of our specification
- There non-trivial human effort was required to prove that this specification is *well-defined* = has a unique solution

Differences, continued:

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Before discussing details of productivity first we present a rewriting technique for proving well-definedness

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Idea:
If we can observe the n-th element $\mathrm{head}(\mathrm{tail}^{n-1}(\sigma))$ of a stream σ for every n, and this result is unique, then the whole stream is well-defined

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If we can observe the n-th element $\mathrm{head}(\mathrm{tail}^{n-1}(\sigma))$ of a stream σ for every n, and this result is unique, then the whole stream is well-defined

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The Observational Variant

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Theorem (Z09)

If R is a stream specification for which $\mathrm{Obs}(R)$ is terminating, then R is well-defined

Example

$$\begin{array}{rcl} \mathsf{Ham} & \rightarrow & s(0) : \mathsf{merge}(m_2(\mathsf{Ham}), m_3(\mathsf{Ham})) \\ m_2(x : xs) & \rightarrow & d(x) : m_2(xs) \end{array}$$

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Standard termination tools like AProVE succeed in proving termination of $\mathsf{Obs}(R)$ fully automatically for the Hamming example

APROVE

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Web Interface

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References

Contributors

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Program Type: Term Rewrite System (WST) Change

Please enter your program here: Load Example Show Help for Language TRS (in new window)

```
(VAR x y xs ys)
(RULES
hd(ham) -> s(0)
tl(ham) -> mer(m2(ham),m3(ham))
hd(m2(xs)) -> d(hd(xs))
tl(m2(xs)) -> m2(tl(xs))
d(0) -> 0
d(s(x)) -> s(s(d(x)))
hd(m3(xs)) -> t(hd(xs))
tl(m3(xs)) -> m3(tl(xs))
t(0) -> 0
t(s(x)) -> s(s(s(t(x))))
hd(mer(xs,ys)) -> hd(F(c(hd(xs),hd(ys)),xs,ys))
tl(mer(xs,ys)) -> tl(F(c(hd(xs),hd(ys)),xs,ys))
hd(F(0,xs,ys)) -> hd(xs)
tl(F(0,xs,ys)) -> mer(tl(xs),tl(ys))
hd(F(s(0),xs,ys)) -> hd(xs)
tl(F(s(0),xs,ys)) -> mer(tl(xs),ys)
hd(F(s(s(x)),xs,ys)) -> hd(ys)
tl(F(s(s(x)),xs,ys)) -> mer(xs,tl(ys))
c(0,0) -> 0
c(s(x),s(y)) -> c(x,y)
c(0,s(x)) -> s(0)
c(s(x),0) -> s(s(0))
)
```

Productivity

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We restrict to *orthogonal* systems (non-overlapping and left-linear), for which unicity of the result is obtained for free as soon as result can be computed

Definition

A stream specification R is productive for a ground term t if for every $n \in \mathbf{N}$ there exists a reduction

$$t \to_R^* \underbrace{d_1 : d_2 : \cdots : d_n :}_{n \text{ elements produced}} t'$$

Productivity

Note that this definition does not refer to infinitary rewriting

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Example:

$$\begin{array}{rcl} c & \to & f(c) \\ f(x : xs) & \to & 0 : c \end{array}$$

is well-defined, but not productive

Pebble flow

[EGHIK07] and [EGH08] identify decidable criteria

- in [EGHIK07]: weakly guarded
- in [EGH08]: flat / friendly nesting

for which productivity can be decided

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produces two elements 0, 1, after consuming one element x

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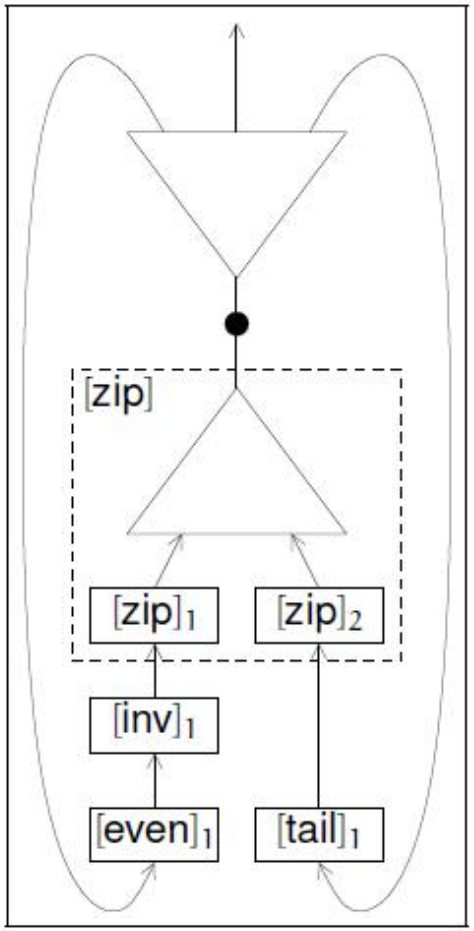
A *pebble flow net* mimicks the production / consumption of elements starting by the given constant

Pebble flow



$$
\begin{aligned}
[\mathsf{zip}] &= \mathsf{gate}([\mathsf{zip}]_1, [\mathsf{zip}]_2) \\
[\mathsf{zip}]_1 &= -+[\mathsf{zip}]_2 = -++[\mathsf{zip}]_1 = \overline{-++} \\
[\mathsf{zip}]_2 &= +[\mathsf{zip}]_1 = +-+[\mathsf{zip}]_2 = \overline{+-+} \\
[\mathsf{inv}] &= \mathsf{gate}([\mathsf{inv}]_1) \\
[\mathsf{inv}]_1 &= -+[\mathsf{inv}]_1 = \overline{-+} \\
[\mathsf{even}] &= \mathsf{gate}([\mathsf{even}]_1) \\
[\mathsf{even}]_1 &= -+[\mathsf{odd}]_1 = -+-[\mathsf{even}]_1 = \overline{-+-} \\
[\mathsf{odd}]_1 &= -[\mathsf{even}]_1 = --+[\mathsf{odd}]_1 = \overline{--+} \\
[\mathsf{tail}] &= \mathsf{gate}([\mathsf{tail}]_1) \\
[\mathsf{tail}]_1 &= -\overline{-+}
\end{aligned}
$$

$$M \to 0 : \mathsf{zip}(\mathsf{inv}(\mathsf{even}(M)), \mathsf{tail}(M))$$

Pebble flow

The system is productive for the given constant if this produces elements forever, that is, in this process there are never more elements consumed than have been produced

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Has been implemented

Web interface of pebble flow tool

Online Version

Enter your stream specification, and press "GO" to check for productivity.

```
Signature(
  M : stream(bit),
  0,1 : bit,
  tail : stream(x) -> stream(x),
  zip : stream(x) -> stream(x) -> stream(x),
  inv : stream(bit) -> stream(bit)
)

M = 0:zip(inv(M),tail(M))

zip(a:s,t) = a:zip(t,s)
tail(a:s) = s
inv(0:s) = 1:inv(s)
inv(1:s) = 0:inv(s)
```

--- GO ---

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- Restricts to *data oblivious* productivity $\approx$ independent of data values, so fails to prove productivity of

$$\begin{array}{rcl} M & \to & f(c) \\ c & \to & 1 : c \\ f(0 : xs) & \to & f(xs) \\ f(1 : xs) & \to & 1 : f(xs) \end{array}$$

Other techniques to prove productivity

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Fruitful lemma:

Lemma

A stream specification R is productive for all (finite) ground terms if and only if every ground term t rewrites to a term of the shape $d : t'$

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A stream specification R is productive for all (finite) ground terms if and only if every ground term t rewrites to a term of the shape $d : t'$

So proving productivity for all ground terms may be easier than for a single constant

[ZR09]: A stream specification R is productive for all (finite) ground terms if and only if $R \cup \{x : xs \to \text{overflow}\}$ is *balanced outermost terminating*

Example

$$
\begin{array}{rcl}
M & \to & f(c) \\
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f(0 : xs) & \to & f(xs) \\
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x : xs & \to & \text{overflow}
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is outermost terminating, as is proved automatically by several tools

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Unfortunately, for many productive examples plain outermost termination does not hold, while there are no techniques for proving balanced outermost termination automatically

Lemma (ZR10)

Let R' be obtained from R by replacing $\ell \to r$ by $\ell \to r'$, where $r \to_R^* r'$, and R' is productive
Then R is productive

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Lemma (ZR10)

Let R be terminating with respect to the context-sensitive rewrite relation is which rewriting in the right argument of ":" is disallowed
Then R is productive

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Has been implemented by R; a web interface is available

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Successful for wide range of examples, including

- Hamming numbers, Thue Morse stream, Fibonacci stream, ...
- examples that do not satisfy data oblivious productivity
- more general infinite data structures, like binary trees

Summary of results

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We conclude this presentation by some more examples

Paper folding

P is the boolean stream defined by

$$P = \mathrm{zip}(\mathrm{alt}, P)$$

$$\mathrm{alt} = 0 : 1 : \mathrm{alt}$$

$$\mathrm{zip}(x : xs, ys) = x : \mathrm{zip}(ys, xs)$$

in which x is a boolean variable and xs, ys are stream variables

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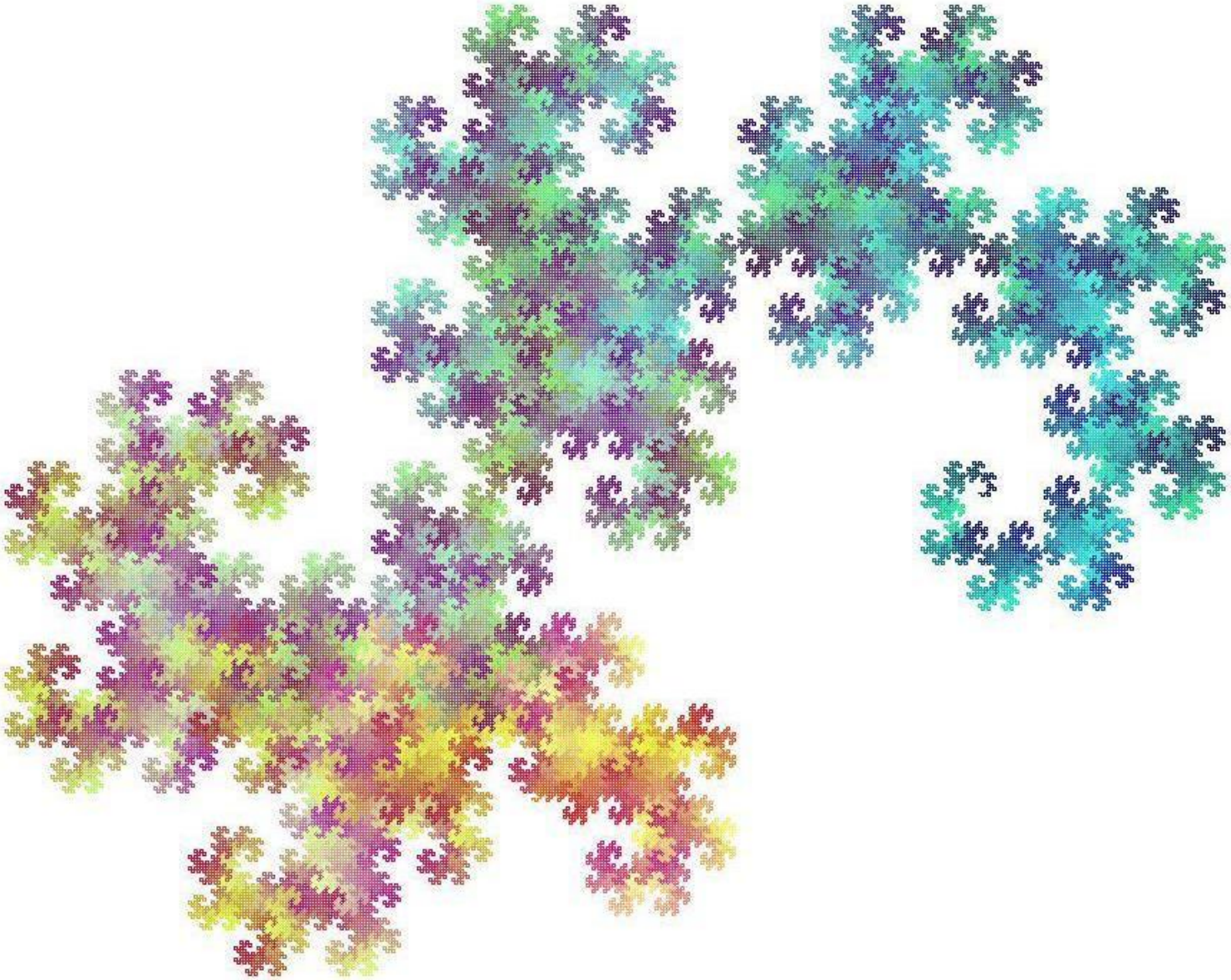
The turtle figure with $\alpha = \beta = 90^o$ is called the *dragon curve* or *dragon fractal*

The dragon fractal



The Kolakoski stream

Let f, g be defined by

$$f(0 : xs) = 0 : 0 : g(xs)$$

$$f(1 : xs) = 0 : g(xs)$$

$$g(0 : xs) = 1 : 1 : f(xs)$$

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The Kolakoski stream K is the unique fixpoint of g, and satisfies

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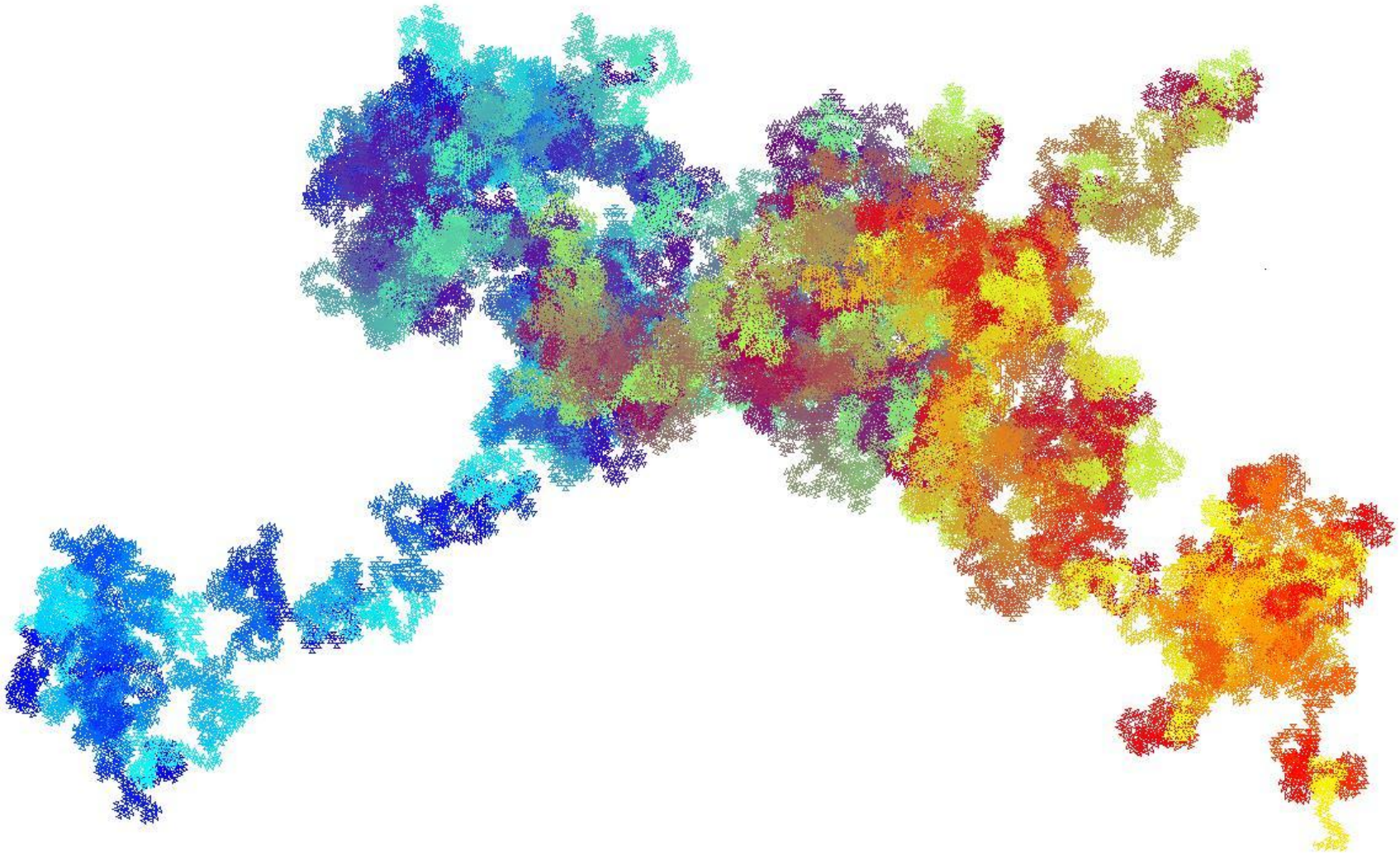
K behaves like a completely random boolean stream, and shows up many open problems

The Kolakoski stream



One more example...

$$Q = 1 : f(Q)$$
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Again a very simple stream specification for which productivity and well-definedness are easily proved

Taking the first 200.000 elements and choose $\alpha = 118^o$ yields the following turtle figure

For all of these pictures the initial part of the stream was constructed by rewriting: start by the corresponding constant, and apply rewrite rules until the required number of elements has been produced.

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Thank you for your attention