

Examination Automated Reasoning 2IW15

June 29, 2011, 9.00 - 12.00

Including solutions

This examination consists of 5 problems each having the same weight. The final result for this course will be the average of the result for the practical assignment and the result for this examination, as long as both results are at least 5.

Problem 1.

Consider the CNF consisting of the following eight clauses

- | | |
|--------------------------|--------------------------|
| (1) $p \vee w$ | (5) $u \vee \neg w$ |
| (2) $\neg r \vee v$ | (6) $\neg p \vee \neg v$ |
| (3) $\neg p \vee r$ | (7) $p \vee \neg s$ |
| (4) $\neg t \vee \neg u$ | (8) $s \vee t$ |

- a. Determine whether this CNF is satisfiable by using the four rules UnitPropagate, Decide, Fail and Backtrack. Make clear at every step what is the corresponding list M of literals and which clause was used.

Solution:

p^d (Decide)
 $p^d \neg v$ (UnitPropagate, 6)
 $p^d \neg v r$ (UnitPropagate, 3)
 $\neg p$ (Backtrack, 2)
 $\neg p w$ (UnitPropagate, 1)
 $\neg p w u$ (UnitPropagate, 5)
 $\neg p w u \neg t$ (UnitPropagate, 4)
 $\neg p w u \neg t s$ (UnitPropagate, 8)
(Fail, 7)

- b. Such a propositional CNF in which every clause contains at most two literals is called a 2-CNF. Explain why the classical Davis Putnam procedure restricted to 2-CNF is polynomial in the number of variables. (Hint: show that the total number of clauses in a 2-CNF is polynomial)

Solution:

The total number of clauses having at most two literals on n variables is $O(n^2)$ since there are $2n$ literals and every such clause is identified by at most two such literals.

When doing resolution on a 2-CNF, the result is again a 2-CNF, since the result of a resolution step contains at most two literals: one from each of the originating clauses.

By the classical Davis Putnam procedure at most n rounds are executed, where a round consists of doing all resolution steps with respect to a single variable. Since in every round the set of clauses is a 2-CNF, the size of this set

is $O(n^2)$, by which doing all resolution steps with respect to a single variable requires a polynomial number of steps. Since the total number of rounds is at most n , the complexity of the classical Davis Putnam procedure is polynomial in n .

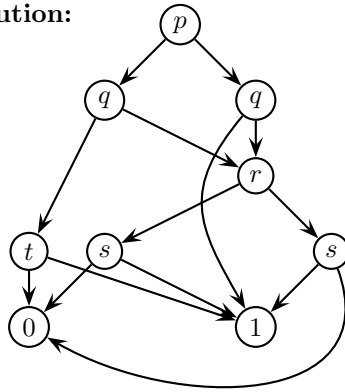
Problem 2.

Let the boolean function f on the boolean variables p, q, r, s, t be defined by:

- If q is true, then $f(p, q, r, s, t)$ is true if and only if at least one of the variables p and t is false.
- If q is false, then $f(p, q, r, s, t)$ is true if and only if r and s have different values.

Compute the ROBDD of the function f with respect to the order $p < q < r < s < t$.

Solution:



Problem 3.

a. Prove by semantic tableaux that

$$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

is a tautology.

Solution:

First negate the formula, and then try to prove unsatisfiability.

Without branching the following can be derived:

$$\neg(((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r))$$

$$(p \rightarrow q) \wedge (q \rightarrow r)$$

$$\neg(p \rightarrow r)$$

$$p \rightarrow q$$

$$q \rightarrow r$$

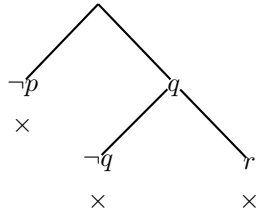
$$p$$

$$\neg r$$

Now branching is required, take $p \rightarrow q$ yielding a branch $\neg p$ and a branch q .

The branch $\neg p$ is closed.

For the branch q we need another instance of branching: take $q \rightarrow r$ yielding a branch $\neg q$ and a branch r . As both branches are closed, all paths are closed, and the negation of the given formula has been proven to be unsatisfiable. In a picture:



Hence the original formula is a tautology.

- b. Write down the Tseitin's transformation of $(p \leftrightarrow (q \vee r)) \vee (r \rightarrow \neg p)$.

Solution:

Let A be the name of the formula itself,

let B be the name of $p \leftrightarrow (q \vee r)$,

let C be the name of $r \rightarrow \neg p$, and

let D be the name of $q \vee r$. Then we obtain the Tseitin's transformation consisting of the following clauses:

$$\begin{aligned}
 & A, \\
 & \neg A \vee B \vee C, \quad A \vee \neg B, \quad A \vee \neg C, \\
 & B \vee p \vee D, \quad B \vee \neg p \vee \neg D, \quad \neg B \vee \neg p \vee D, \quad \neg B \vee p \vee \neg D, \\
 & \neg C \vee \neg r \vee \neg p, \quad C \vee r, \quad C \vee p, \\
 & \neg D \vee q \vee r, \quad D \vee \neg q, \quad D \vee \neg r.
 \end{aligned}$$

Problem 4.

- a. Let w, x, y, z be variables. Determine the results of all single resolution steps that can be executed starting from the two clauses $P(x, f(x, y)) \vee \neg Q(f(x, x))$ and $\neg P(f(z, z), w) \vee Q(f(z, w))$.

Solution:

The terms $Q(f(x, x))$ and $Q(f(z, w))$ unify with mgu σ satisfying $\sigma(x) = \sigma(z) = \sigma(w) = x$. Then resolution yields

$$P(x, f(x, y)) \vee \neg P(f(x, x), x).$$

The terms $P(x, f(x, y))$ and $P(f(z, z), w)$ unify with mgu σ satisfying $\sigma(x) = f(z, z), \sigma(y) = y, \sigma(z) = z$ and $\sigma(w) = f(f(z, z), y)$. Then resolution yields

$$Q(z, f(f(z, z), y)) \vee \neg Q(f(f(z, z), f(z, z))).$$

b. Apply skolemization to a prenex normal form of

$$(\exists x \forall y P(x, y)) \rightarrow \forall y (R(y) \wedge \neg Q(f(y)))$$

Solution:

First rename one of the y s:

$$(\exists x \forall y P(x, y)) \rightarrow \forall z (R(z) \wedge \neg Q(f(z))).$$

Elaborate \rightarrow :

$$(\neg \exists x \forall y P(x, y)) \vee \forall z (R(z) \wedge \neg Q(f(z))).$$

Push \neg inwards:

$$(\forall x \exists y \neg P(x, y)) \vee \forall z (R(z) \wedge \neg Q(f(z))).$$

Bring $\forall z$ to the front:

$$\forall x \exists y \forall z (\neg P(x, y) \vee (R(z) \wedge \neg Q(f(z)))).$$

Transform the body to CNF by distributivity:

$$\forall x \exists y \forall z (\neg P(x, y) \vee R(z)) \wedge (\neg P(x, y) \vee \neg Q(f(z))).$$

Now the formula is in prenex normal form. For applying skolemization y has to be replaced by $g(x)$ for the fresh symbol g , resulting in

$$\forall x \forall z (\neg P(x, g(x)) \vee R(z)) \wedge (\neg P(x, g(x)) \vee \neg Q(f(z))).$$

Problem 5.

The term rewriting system R is defined to consist of the rules

$$\begin{aligned} f(g(x), y) &\rightarrow f(x, y), \\ h(g(x)) &\rightarrow g(g(h(x))), \\ f(x, h(y)) &\rightarrow g(x). \end{aligned}$$

a. Prove that R is terminating.

Solution:

Choose monotone interpretations

$$[f](x, y) = x + y, \quad [g](x) = x + 1, \quad [h](x) = 3x.$$

Then for all α one obtains

$$[f(g(x), y), \alpha] = \alpha(x) + \alpha(y) + 1 > \alpha(x) + \alpha(y) = [f(x, y), \alpha],$$

$$[h(g(x)), \alpha] = 3(\alpha(x) + 1) > 3\alpha(x) + 2 = [g(g(h(x))), \alpha],$$

$$[f(x, h(y)), \alpha] = \alpha(x) + 3\alpha(y) > \alpha(x) + 1 = [g(x), \alpha].$$

For the last inequality it is essential to exclude 0 from the natural numbers. Alternatively one could choose $[f](x, y) = x + y + 2$.

Another proof can be given by the lexicographic path order, choosing any precedence satisfying $f > g$ and $h > g$.

b. Give all non-trivial critical pairs of R .

Solution:

$[g(g(x)), f(x, h(y))]$ and $[f(x, g(g(h(y))))], g(x)]$.

c. Find a term with more than two normal forms.

Solution:

Take the term $f(g(g(x)), h(x))$. It admits the following reductions to three distinct normal forms:

$$f(g(g(x)), h(x)) \rightarrow g(g(g(x))),$$

$$f(g(g(x)), h(x)) \rightarrow f(g(x), h(x)) \rightarrow g(g(x)),$$

$$f(g(g(x)), h(x)) \rightarrow f(g(x), h(x)) \rightarrow f(x, h(x)) \rightarrow g(x).$$