Theorems used by proofs generated by TORPA

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This document lists all theorems used by proofs generated by TORPA version 1.6, prepared for the Termination Competition 2006. They are identified by letters in square brackets; the output of TORPA 1.6 uses the same identification.

[A] Monotone Algebras

Theorem. *Let R, S, R' and S' be SRSs satisfying*

- $R \cup S = R' \cup S'$ *and* $R \cap S = R' \cap S' = \emptyset$, *and*
- $\mathsf{SN}(R'/S')$ *and* $\mathsf{SN}((R \cap S')/(S \cap S'))$.

Then $\mathsf{SN}(R/S)$.

Proof: [3], Theorem 1. □

Theorem. *Let A be a non-empty set and let $>$ be a well-founded order on A. Let $f_a : A \to A$ be strictly monotone for every $a \in \Sigma$, i.e., $f_a(x) > f_a(y)$ for every $x, y \in A$ satisfying $x > y$.*

Let R and S be disjoint SRSs over Σ such that $f_\ell(x) > f_r(x)$ for all $x \in A$ and $\ell \to r \in R$, and $f_\ell(x) \geq f_r(x)$ for all $x \in A$ and $\ell \to r \in S$.

Then $\mathsf{SN}(R/S)$.

Proof: [3], Theorem 4. □

These theorems are applied as follows: if $\mathsf{SN}(R/S)$ has to be proved then an interpretation is chosen for which $f_\ell(x) \geq f_r(x)$ for all $x \in A$ and $\ell \to r \in R \cup S$. Then R' is defined to consist of the rules $\ell \to r$ of $R \cup S$ satisfying $f_\ell(x) > f_r(x)$ for all $x \in A$, and $S' = (R \cup S) \setminus R'$.

Then $\mathsf{SN}(R'/S')$ holds by the second theorem, and by the first theorem the remaining proof obligation is $\mathsf{SN}((R \cap S')/(S \cap S'))$.

[B] Recursive Path Order

For an order $>$ on the finite set Σ the order $>_{rpo}$ has the following defining property: $s >_{rpo} t$ if and only if s can be written as $s = as'$ for $a \in \Sigma$, and either

- $s' = t$ or $s' >_{rpo} t$, or
- t can be written as $t = bt'$ for $b \in \Sigma$, and either
  - $a > b$ and $s >_{rpo} t'$, or
  - $a = b$ and $s' >_{rpo} t'$.

Theorem. *If $\ell >_{rpo} r$ for all rules $\ell \to r$ of an SRS R, then R is terminating.*

Proof: [1], Theorem 6.4.3. □

[C] Reverse

For a string s write s^{rev} for its reverse. For an SRS R write

$$R^{\mathsf{rev}} = \{ \ \ell^{\mathsf{rev}} \to r^{\mathsf{rev}} \mid \ell \to r \ \in \ R \ \}.$$

Theorem. *Let R and S be disjoint SRSs. Then $\mathsf{SN}(R/S)$ if and only if $\mathsf{SN}(R^{\mathsf{rev}}/S^{\mathsf{rev}})$.*

Proof: This follows from the observation that if $s \to_R t$ for any SRS R then $s^{\mathsf{rev}} \to_{R^{\mathsf{rev}}} t^{\mathsf{rev}}$. □

This is Lemma 2 in [3].

[D] RFC-match-bounds

For an SRS R over an alphabet Σ we define the infinite SRS $\mathsf{match}(R)$ over $\Sigma \times \mathbf{N}$ to consist of all rules $(a_1, n_1) \cdots (a_p, n_p) \to (b_1, m_1) \cdots (b_q, m_q)$ for which $a_1 \cdots a_p \to b_1 \cdots b_q \in R$ and $m_i = 1 + \min_{j=1,\ldots,p} n_j$ for all $i = 1, \ldots, q$.

For an SRS R over an alphabet Σ we define the SRS $R_\#$ over $\Sigma \cup \{\#\}$ by

$$R_\# \ = \ R \ \cup \ \{ \ \ell_1 \# \to r \mid \ell \to r \in R \ \wedge \ \ell = \ell_1 \ell_2 \ \wedge \ \ell_1 \neq \epsilon \neq \ell_2 \ \}.$$

Theorem. *Let R be an SRS and let $N \in \mathbf{N}$ such that for all rhs's $b_1 \cdots b_q$ of R and all $k \in \mathbf{N}$ and all reductions*

$$(b_1, 0) \cdots (b_q, 0)(\#, 0)^k \ \to^*_{\mathsf{match}(R_\#)} \ (c_1, n_1) \cdots (c_r, n_r)$$

it holds that $n_i \leq N$ for all $i = 1, \ldots, r$. Then R is terminating.

Proof: [3], Theorem 14. □

In TORPA termination of an SRS is proved by RFC-match-bounds by the construction of a finite automaton M over the alphabet $(\Sigma \cup \{\#\}) \times \mathbf{N}$, where Σ is the alphabet of R, satisfying:

- for every rhs $b_1 \cdots b_q$ of R and every $k \in \mathbf{N}$ the automaton M accepts $(b_1, 0) \cdots (b_q, 0)(\#, 0)^k$, and
- M is closed under $\mathsf{match}(R_\#)$, i.e., if M accepts v and $v \rightarrow_{\mathsf{match}(R_\#)} u$ then M accepts u too.

Such an automaton is called *compatible*. The pair $(a, k) \in (\Sigma \cup \{\#\}) \times \mathbf{N}$ is shortly written as a_k, and the number k is called the *label* of this pair. It is easy to see that if a (finite) compatible automaton M has been found then for N being the biggest label occurring in M the condition of the theorem holds.

[E] Semantic Labelling

Fix a non-empty set A and maps $f_a : A \rightarrow A$ for all $a \in \Sigma$ for some alphabet Σ. Let f_s for $s \in \Sigma^*$ be defined as before. Let $\overline{\Sigma}$ be the alphabet consisting of the symbols a_x for $a \in \Sigma$ and $x \in A$. The *labelling function* $\mathsf{lab} : \Sigma^* \times A \rightarrow \overline{\Sigma}^*$ is defined inductively as follows:

$$\mathsf{lab}(\epsilon, x) = \epsilon \quad \text{for } x \in A,$$

$$\mathsf{lab}(sa, x) = \mathsf{lab}(s, f_a(x))a_x \quad \text{for } s \in \Sigma^*, a \in \Sigma, x \in A.$$

For an SRS R over Σ define

$$\mathsf{lab}(R) = \{ \mathsf{lab}(l, x) \rightarrow \mathsf{lab}(r, x) \mid l \rightarrow r \in R, x \in A \}.$$

Theorem. *Let R and S be two disjoint SRSs over an alphabet Σ. Let $>$ be a well-founded order on a non-empty set A. Let $f_a : A \rightarrow A$ be defined for all $a \in \Sigma$ such that*

- *$f_a(x) \geq f_a(y)$ for all $a \in \Sigma, x, y \in A$ satisfying $x > y$, and*
- *$f_\ell(x) \geq f_r(x)$ for all $\ell \rightarrow r \in R \cup S, x \in A$.*

Let Dec be the SRS over $\overline{\Sigma}$ consisting of the rules $a_x \rightarrow a_y$ for all $a \in \Sigma, x, y \in A$ satisfying $x > y$.

Then $\mathsf{SN}(R/S)$ if and only if $\mathsf{SN}(\mathsf{lab}(R)/(\mathsf{lab}(S) \cup \mathsf{Dec}))$.

Proof: [3], Theorem 15. □

In TORPA this is only applied for $A = \{0, 1\}$. In case the relation $>$ is empty the set A together with the functions f_a for $a \in \Sigma$ is called a *model* for the SRS, otherwise it is called a *quasi-model*. It is called a model since then for every rule $\ell \to r$ the interpretation f_ℓ of ℓ is equal to the interpretation f_r of r. Note that $\mathsf{Dec} = \emptyset$ in case of a model.

If TORPA applies [E] Semantic Labelling, then the 'if'-part of this theorem is used.

[F] Removal of Labels

Here the same theorem [E] Semantic Labelling is used, but then the 'only if'-part.

[G] Dependency Pairs

Write $\Sigma^\# = \Sigma \cup \{a^\# \mid a \in \Sigma_D\}$. The SRS $DP(R)$ over $\Sigma^\#$ is defined to consist of all rules of the shape

$$a^\# \ell' \;\to\; b^\# r''$$

for which $a\ell' = \ell$ and $r = r'br''$ for some rule $\ell \to r$ in R and $a, b \in \Sigma_D$. Rules of $DP(R)$ are called *dependency pairs*.

Theorem. *Let R be an SRS in which all lhs's are non-empty. Then* $\mathsf{SN}(R)$ *if and only if* $\mathsf{SN}(DP(R)/R)$.

Proof: [3], Theorem 6. □

[H] Looping

If $u \to_R^+ vuw$ then an infinite reduction of the following shape exists:

$$u \to_R^+ vuw \to_R^+ vvuww \to_R^+ vvvuwww \to_R^+ \cdots,$$

proving non-termination. Such a reduction is called *looping*. The way TORPA searches for looping reductions is described in [3], Section 9.

[I] Dummy Elimination

A *dummy symbol* is a symbol occurring in a rhs but in no lhs. Now for a dummy symbol a in an SRS R we define

$$\mathsf{DE}'_a(R) = \{\ell \to u \mid u = \mathsf{cap}'_a(r) \vee u \in \mathsf{dec}'_a(r) \text{ for a rule } \ell \to r \in R\},$$

where

$$
\begin{array}{rcll}
\mathsf{cap}'_a(\lambda) & = & \lambda & \\
\mathsf{cap}'_a(fs) & = & f\mathsf{cap}'_a(s) & \text{for all symbols } f \text{ with } f \neq a \text{ and all strings } s \\
\mathsf{cap}'_a(as) & = & a_\$ & \\
\mathsf{dec}'_a(\lambda) & = & \emptyset & \\
\mathsf{dec}'_a(fs) & = & \mathsf{dec}'_a(s) & \text{for all symbols } f \text{ with } f \neq a \text{ and all strings } s \\
\mathsf{dec}'_a(as) & = & \mathsf{dec}'_a(s) \cup \{{}_\$a(\mathsf{cap}'_a(s))\}. &
\end{array}
$$

Theorem. *Let R be an SRS having a dummy symbol a. Then R is terminating if and only if $\mathsf{DE}'_a(R)$ is terminating.*

Proof: [2], Theorem 11. □

[J] Reducing Right Hand Sides

Theorem. *Let R be a TRS for which a rhs is not in normal form, i.e., R contains a rule $\ell \to r$ and a rule of the shape $\ell' \to C[\ell\sigma]$. Assume that*

- *$\ell \to r$ is left-linear,*
- *$\ell \to r$ is non-erasing,*
- *$\mathsf{WCR}(\{\ell \to r\})$, and*
- *there is no overlap between ℓ and the lhs of any rule of $R \setminus \{\ell \to r\}$.*

Let R' be obtained from R by replacing the rule $\ell' \to C[\ell\sigma]$ by $\ell' \to C[r\sigma]$. Then R is terminating if and only if R' is terminating.

Proof: [2], Theorem 4. □

[K] Strip First/Last Symbol

Theorem. *Let R' be an SRS obtained from an SRS R by replacing a rule of the shape $a\ell \to ar$ by $\ell \to r$ (strip first symbol), or by replacing a rule of the shape $\ell a \to ra$ by $\ell \to r$ (strip last symbol). If R' is terminating then R is terminating.*

Proof: Obvious since every $\to_R$-step is an $\to_{R'}$-step too. □

References

[1] H. Zantema. Termination. In *Term Rewriting Systems, by Terese*, pages 181–259. Cambridge University Press, 2003.

[2] H. Zantema. Reducing right-hand sides for termination. In A. Middeldorp, V. van Oostrom, F. van Raamsdonk, and R. de Vrijer, editors, *Processes, Terms and Cycles: Steps on the Road to Infinity: Essays Dedicated to Jan Willem Klop on the Occasion of His 60th Birthday*, volume 3838 of *Lecture Notes in Computer Science*, pages 173–197, Berlin, 2005. Springer-Verlag.

[3] H. Zantema. Termination of string rewriting proved automatically. *Journal of Automated Reasoning*, 34:105–139, 2005.