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THE KERNEL METHOD FOR DISCRETE QUEUES

JOHAN VAN LEEUWAARDEN

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1. Introduction

Discrete queues can be described in terms of lattice paths, and are therefore analyzable through combinatorial methods. One popular such method is the so-called *kernel method*. This method refers to a way of solving a functional equation for a multivariate generation function by using couplings of the variables. These couplings arise naturally as those functions for which the kernel (a part of the functional equation) vanishes. We shall demonstrate the kernel method for a discrete queue that can be described the Markov chain $\{Q_n, n \geq 0\}$, where

$$Q_{n+1} = \max\{0, Q_n + X_n\}, \quad n = 0, 1, \ldots, \tag{1}$$

with $(X_n)_{n\in\mathbb{N}}$ a sequence of i.i.d. discrete random variables. Recursion (1) is exactly Lindley's recursion for the waiting time in the $GI/G/1$ queue. To see this, set $X_n = B_n - C_n$ with B_n the service time of customer n and C_n the interarrival time between customers n and $n+1$. However, we shall exploit the additional feature that X_n is integer valued, and hence, the random walks to be considered are on the set of integers.

Let X denote a generic random variable with $X \stackrel{d}{=} X_1$ and

$$X \in \{-s, -s+1, \ldots, -1, 0, 1, \ldots, d\} \tag{2}$$

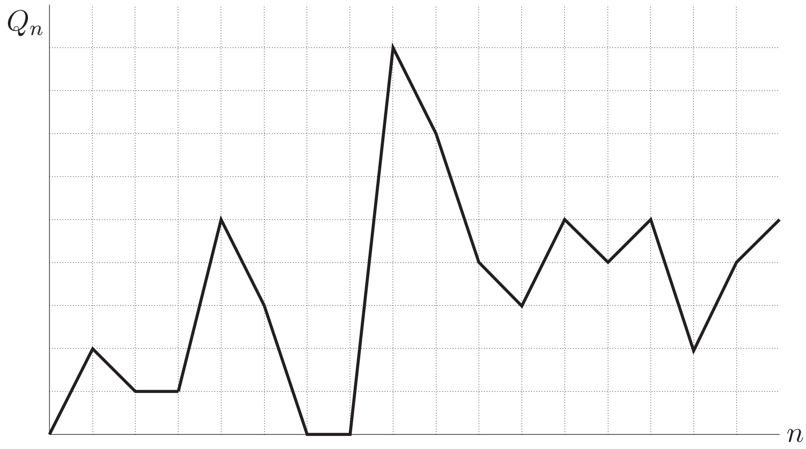

FIGURE 1. Sample path of the queueing process.

with $\mathbb{P}(X = -s) > 1$, s some positive integer and d possibly infinite. This means that we can write the probability generating function (pgf) of X as

$$\mathbb{E}(z^X) = A(z)z^{-s},$$

where $A(z)$ is the pgf of a nonnegative integer-valued random variable A, i.e.,

$$Q_{n+1} = \max\{0, Q_n + A_n - s\}, \quad n = 0, 1, \ldots, \tag{3}$$

where $(A_n)_{n\in\mathbb{N}}$ consist of i.i.d. discrete random variables with $A \stackrel{d}{=} A_1$.

2. NONNEGATIVE PATHS

Let us first consider paths $\{Q_0^*, Q_1^*, \ldots, Q_n^*\}$ that never go below the horizontal axis, in the sense that the maximum operator in (3) is not needed. Let $\mathbf{1}_{\{\cdot\}}$ be the indicator function and $(\cdot)^+ = \max\{0, \cdot\}$.

$$\begin{aligned}\mathbb{E}(z^{Q^*_{n+1}}) &= \mathbb{E}(z^{(Q^*_n+X_n)^+}\mathbf{1}_{\{Q^*_n+X_n\geq 0\}})\\ &= \mathbb{E}(z^{Q^*_n+X_n}) - \mathbb{E}(z^{Q^*_n+X_n}\mathbf{1}_{\{Q^*_n+X_n<0\}}),\end{aligned} \tag{4}$$

where $\mathbb{E}(z^{Q^*_n+X_n}) = \mathbb{E}(z^{Q^*_n})\mathbb{E}(z^{X_n}) = \mathbb{E}(z^{Q^*_n})A(z)z^{-s}$ and

$$\begin{aligned}\mathbb{E}(z^{Q^*_n+X_n}\mathbf{1}_{\{Q^*_n+X_n<0\}}) &= \sum_{k=-s}^{-1}\mathbb{P}(Q^*_n + A_n - s = k)z^k\\ &= \sum_{r=0}^{s-1}\mathbb{P}(Q^*_n + A_n = r)z^{r-s}.\end{aligned} \tag{5}$$

We thus obtain

$$\mathbb{E}(z^{Q^*_{n+1}}) = \mathbb{E}(z^{Q^*_n})A(z)z^{-s} - \sum_{r=0}^{s-1}\mathbb{P}(Q^*_n + A_n = r)z^{r-s}. \tag{6}$$

Next introduce the bivariate generating function

$$F^*(u,z) = \sum_{n\geq 0} u^n \mathbb{E}(z^{Q_n^*}).$$

We get

$$F^*(u,z) = \mathbb{E}(z^{Q_0^*}) + uA(z)z^{-s}F^*(u,z) - u\sum_{r=0}^{s-1} z^{r-s}F_r^*(u) \tag{7}$$

with $F_r^*(u) = \sum_{n=0}^{\infty} \mathbb{P}(Q_n^* + A_n = r)u^n$. Upon some rewriting we arrive at

$$F^*(u,z) = \frac{N^*(u,z)}{z^s - uA(z)}, \tag{8}$$

where

$$N^*(u,z) = z^s\mathbb{E}(z^{Q_0^*}) - u\sum_{r=0}^{s-1} z^r F_r^*(u). \tag{9}$$

2.1. **Starting from an empty queue.** According to Rouché's theorem, the denominator $z^s - uA(z)$ of (8) has exactly s zeros within the unit disk denoted by $z_0(u), z_1(u), \ldots, z_{s-1}(u)$. The function $F^*(u,z)$ is analytic in the polydisk $|u| < 1$, $|z| < 1$. Therefore, the zeros in $|u| < 1$ of the denominator in (8) should also be the zeros of the numerator.

If we set $Q_0^* = 0$, i.e. $\mathbb{E}(z^{Q_0^*}) = 1$, the numerator $N^*(u,z)$ is a polynomial in z of degree s, with leading monomial z^s, so that the polynomial factorizes as

$$N^*(u,z) = \prod_{k=0}^{s-1}(z - z_k(u)). \tag{10}$$

We thus find that

$$F^*(u,z) = \frac{\prod_{k=0}^{s-1}(z - z_k(u))}{z^s - uA(z)}. \tag{11}$$

2.2. **Busy periods.** We now consider those non-negative paths that end exactly at level zero at step n. The generating function is given by $E(u) = F^*(u,0) = \sum_{n=0}^{\infty} u^n \mathbb{P}(Q_n^* = 0)$, i.e.

$$E(u) = \frac{(-1)^{s-1}}{uA(0)} \prod_{k=0}^{s-1} z_k(u). \tag{12}$$

Consider the case $s = 1$ and let BP denote the number of customers served in a busy period. It is immediate that

$$\mathbb{E}(u^{\mathrm{BP}}) = uA(0)E(u) = z_0(u).$$

Example 1. ($M/M/1$ queue) Customers arrive according to a Poisson process with rate λ and have exponential service requirements with mean $1/\mu$. We have

$$A(z) = p + pqz + pq^2z^2 + \ldots = \frac{p}{1-qz} \quad \text{with} \quad p = \frac{\mu}{\lambda+\mu}, \quad q = \frac{\lambda}{\lambda+\mu}$$

and hence z_0 is the solution in $|z| < 1$ of

$$z - uA(z) = 0 \quad \Leftrightarrow \quad qz^2 - z + pu = 0.$$

We thus find that

$$\mathbb{E}(u^{\mathrm{BP}}) = z_0(u) = \frac{1-\sqrt{1-4pqu}}{2q} = \frac{\lambda+\mu-\sqrt{(\lambda+\mu)^2-4\lambda\mu u}}{2\lambda}. \tag{13}$$

Explicit inversion is possible since

$$\frac{1-\sqrt{1-4pqu}}{2q} = p\sum_{n=1}^{\infty}\frac{1}{n}\binom{2n-2}{n-1}(pq)^{n-1}u^n$$

and hence (let $[u^n]f(u)$ denote the coefficient of u^n in $f(u)$)

$$\mathbb{P}(\mathrm{BP}=n) = [u^n]\mathbb{E}(u^{\mathrm{BP}}) = \frac{1}{n}\binom{2n-2}{n-1}p^n q^{n-1}.$$

Example 2. ($M/G/1$ queue) Customers arrive according to a Poisson process with rate λ and B denotes a generic service requirement with $\lambda\mathbb{E}B<1$. The probability of having n arrivals during a service time is

$$\alpha_n = \int_0^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}d\mathbb{P}(B<t)$$

and so

$$\sum_{n=0}^{\infty}\alpha_n z^n = \mathbb{E}(e^{-\lambda(1-z)B}) = A(z).$$

Hence, $\mathbb{E}(z^{\mathrm{BP}})$ in the $M/G/1$ queue is the solution, inside the unit circle, of

$$z - u\mathbb{E}(e^{-\lambda(1-z)B}) = 0. \tag{14}$$

The reader can check that $\mathbb{E}(e^{-\omega B}) = \frac{\mu}{\mu+\omega}$ again leads to (13).

Example 3. ($M/D/1$ queue) When the service requirements are deterministic and equal to b, we have that $\mathbb{E}(e^{-\omega B}) = e^{-\omega b}$ and (14) becomes (with $\rho=\lambda b$)

$$z(u) = ue^{\rho(z(u)-1)}. \tag{15}$$

This is a famous and well studied equation, as the pgf of the total progeny in a Poisson branching process satisfies (evidently) the same equation. Moreover, (15) can be written in terms of the Lambert W function (see De Bruijn (1981), p. 23), which is defined as $W(x)e^{W(x)}=x$. That is, $z(u) = -\rho^{-1}W(-u\rho e^{-\rho})$. Because

$$W(x) = -\sum_{n=1}^{\infty}\frac{n^{n-1}}{n!}(-x)^n, \tag{16}$$

we obtain that

$$\mathbb{P}(\mathrm{BP}=n) = [u^n]\left(-\frac{1}{\rho}W(-u\rho e^{-\rho})\right) = \frac{(\rho n)^{n-1}}{n!}e^{-\rho n}.$$

3. The queueing process

We see that

$$\begin{aligned}\mathbb{E}(z^{Q_{n+1}}) &= \mathbb{E}(z^{(Q_n+X_n)^+})\\ &= \mathbb{E}(z^{(Q_n+X_n)^+}\mathbf{1}_{\{Q_n+X_n\geq 0\}}) + \mathbb{E}(z^{(Q_n+X_n)^+}\mathbf{1}_{\{Q_n+X_n<0\}})\\ &= \mathbb{E}(z^{Q_n+X_n}) - \mathbb{E}(z^{Q_n+X_n}\mathbf{1}_{\{Q_n+X_n<0\}}) + \mathbb{P}(Q_n+X_n<0), \end{aligned} \tag{17}$$

where $\mathbb{E}(z^{Q_n+X_n}) = \mathbb{E}(z^{Q_n})A(z)z^{-s}$,

$$\mathbb{E}(z^{Q_n+X_n}\mathbf{1}_{\{Q_n+X_n<0\}}) = \sum_{r=0}^{s-1}\mathbb{P}(Q_n+A_n=r)z^{r-s} \tag{18}$$

and $\mathbb{P}(Q_n+X_n<0) = \sum_{r=0}^{s-1}\mathbb{P}(Q_n+A_n=r)$. We thus obtain

$$\mathbb{E}(z^{Q_{n+1}}) = \mathbb{E}(z^{Q_n})A(z)z^{-s} + \sum_{r=0}^{s-1}\mathbb{P}(Q_n+A_n=r)(1-z^{r-s}). \tag{19}$$

For the bivariate generating function

$$F(u,z) = \sum_{n\geq 0} u^n\mathbb{E}(z^{Q_n})$$

we find that

$$F(u,z) = \mathbb{E}(z^{Q_0}) + uF(u,z)A(z)z^{-s} + u\sum_{r=0}^{s-1}(1-z^{r-s})F_r(u) \tag{20}$$

with $F_r(u) = \sum_{n=0}^{\infty}\mathbb{P}(Q_n+A_n=r)u^n$. Upon some rewriting we arrive at

$$F(u,z) = \frac{N(u,z)}{z^s-uA(z)}, \tag{21}$$

where

$$N(u,z) = z^s\mathbb{E}(z^{Q_0}) + u\sum_{r=0}^{s-1}(z^s-z^r)F_r(u). \tag{22}$$

If we, as before, set $Q_0=0$, the numerator $N(u,z)$ is a polynomial in z of degree s, with coefficients depending on u. Hence, for $Q_0=0$, we have that

$$N(u,z) = \gamma(u)\prod_{k=0}^{s-1}(z-z_k(u)), \tag{23}$$

where $\gamma(u)$ follows from $N(u,1)=1$. We thus find that

$$F(u,z) = \frac{1}{z^s-uA(z)}\prod_{k=0}^{s-1}\frac{z-z_k(u)}{1-z_k(u)}. \tag{24}$$

Denote by Q the limit of Q_n as n goes to infinity (which exists if $\rho = A'(1)/s < 1$), and let $F(z) = \mathbb{E}(z^Q)$.

$$F(z) = \lim_{u\to 1}(1-u)F(u,z)$$

$$= \lim_{u\to 1} \frac{1-u}{1-z_0(u)} \frac{z-z_0(u)}{z^s - uA(z)} \prod_{k=1}^{s-1} \frac{z-z_k(u)}{1-z_k(u)}$$

$$= \frac{(s-A'(1))(z-1)}{z^s - A(z)} \prod_{k=1}^{s-1} \frac{z-z_k}{1-z_k}, \tag{25}$$

with $z_k \equiv z_k(1)$ and where we have used that $z_0'(1) = 1/(s - A'(1))$.

Example 4. ($M/G/1$ queue) For $s = 1$ and $A(z) = \mathbb{E}(e^{-\lambda(1-z)B})$ the expression (25) reduces to

$$F(z) = \frac{(1-A'(1))(1-z)}{A(z)-z} = \frac{(1-\rho)(1-z)}{\mathbb{E}(e^{-\lambda(1-z)B}) - z}, \tag{26}$$

and represents the pgf of Q_d, the steady-state queue length (without the customer in service) just after a departure. PASTA and distributional Little's law then say that $\mathbb{E}(z^{Q_d}) = \mathbb{E}(e^{-\lambda(1-z)W})$, with W the stationary waiting time, and hence

$$\mathbb{E}(e^{-\omega W}) = \frac{(1-\rho)\omega}{\omega - \lambda + \lambda\mathbb{E}(e^{-\omega B})}. \tag{27}$$

This formula is known as the Pollaczek-Khintchine formula.

Example 5. ($M/D/1$ queue) A.K. Erlang's 1909 paper introducing the $M/D/1$ queue is generally considered to be the starting point of queueing theory. For Poisson arrivals with rate λ, deterministic service requirements b, and first-come-first-served, Erlang's result on the stationary waiting time W reads (assuming $\lambda b < 1$ for stability)

$$\mathbb{P}(W < t) = (1-\lambda b)e^{\lambda t} \sum_{j=0}^{\lfloor t/b \rfloor} (-\lambda e^{-\lambda b})^j \frac{(t-jb)^j}{j!}, \quad t \geq 0. \tag{28}$$

For a direct inversion of (27) for $\mathbb{E}(e^{-\omega B}) = e^{-\omega b}$ see Van Leeuwaarden, Löpker & Janssen (2008).

4. Existence of the roots

From now on we shall focus on the steady-state analysis of the queue, which comes down to studying the stochastic equation

$$Q \stackrel{d}{=} (Q + A - s)^+, \tag{29}$$

whose solution is given by (25). What shall now investigate more closely the roots of $z^s = A(z)$ in $|z| < 1$ (denoted by $z_1, \ldots, z_{s-1}$). But before we do so, we first give a direct derivation of (25).

From (25) we get

$$\mathbb{P}(Q = 0) = \sum_{j=0}^{s} \mathbb{P}(Q + A = j), \tag{30}$$

$$\mathbb{P}(Q = k) = \mathbb{P}(Q + A = s + k), \quad k \geq 1. \tag{31}$$

Multiplying by z^k and summing over all values of k yields

$$
\begin{aligned}
F(z) &= z^{-s}\sum_{k=1}^{\infty}\mathbb{P}(Q+A=s+k)z^{k+s}+\sum_{j=0}^{s}\mathbb{P}(Q+A=j) \\
&= z^{-s}\Big[F(z)A(z)-\sum_{j=0}^{s}\mathbb{P}(Q+A=j)z^j\Big]+\sum_{j=0}^{s}\mathbb{P}(Q+A=j) \\
&= z^{-s}F(z)A(z)+\sum_{j=0}^{s}\mathbb{P}(Q+A=j)(1-z^{j-s}) \qquad (32)
\end{aligned}
$$

and hence

$$
F(z)=\frac{\sum_{j=0}^{s-1}\mathbb{P}(Q+A=j)(z^s-z^j)}{z^s-A(z)}. \qquad (33)
$$

This description of $F(z)$ still contains s unknowns. Here, the kernel $z^s - A(z)$ comes into play. From $F(1) = 1$ we get

$$
\sum_{j=0}^{s-1}\mathbb{P}(Q+A=j)(s-j)=s-A'(1). \qquad (34)
$$

Write

$$
\sum_{j=0}^{s-1}\mathbb{P}(Q+A=j)(z^s-z^j)=\gamma(z-1)\prod_{k=1}^{s-1}(z-z_k), \qquad (35)
$$

where the constant γ can be determined from differentiating both sides of (35) with respect to z, and using (34). This gives

$$
\gamma=\frac{s-A'(1)}{\prod_{k=1}^{s-1}(1-z_k)}, \qquad (36)
$$

and so (25) follows.

4.1. **Classical approach.** In the vast majority of queueing problems to which Rouché's theorem is applied, the analytic function of interest is given by $z^s - A(z)$, where $s \in \mathbb{N}$ and $A(z)$ is the pgf of a nonnegative discrete random variable A. Denoting $\mathbb{P}(A = j)$ by a_j, we have that

$$
A(z)=\sum_{j=0}^{\infty}a_jz^j, \qquad (37)
$$

which is known to be analytic in the open disk $\{z \in \mathbb{C} : |z| < 1\}$ and continuous up to the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. For continuous-time bulk service queues, $M/G/1$ and $G/M/1$-type queues, the $A(z)$ is typically of the form $A(z) = \tilde{B}(\lambda(1 - z))$, where $\tilde{B}(s)$ is the Laplace-Stieltjes transform of a continuous random variable and λ is some positive real constant.

Let us first state Rouché's theorem (see e.g. Titchmarsh (1939))

Theorem 6. (Rouché) *Let the bounded region D have as its boundary a simple closed contour C. Let $f(z)$ and $g(z)$ be analytic both in D and on C. Assume that $|f(z)| < |g(z)|$ on C. Then $f(z) - g(z)$ has in D the same number of zeros as $g(z)$, all zeros counted according to their multiplicity.*

When $A(z)$ has a radius of convergence larger than 1, we can prove the following result concerning the number of zeros on and within the unit circle of $z^s - A(z)$ by using Rouché's theorem:

Lemma 1. *Let $A(z)$ be a* pgf *that is analytic in $|z| \le 1+\nu$, $\nu > 0$. Assume that $A'(1) < s$, $s \in \mathbb{N}$. Then the function $z^s - A(z)$ has exactly s zeros in $|z| \le 1$.*

Proof Define the functions $f(z) := A(z)$, $g(z) := z^s$. Because $f(1) = g(1)$ and $f'(1) = A'(1) < s = g'(1)$, we have, for sufficiently small $\epsilon > 0$,

$$f(1+\epsilon) < g(1+\epsilon). \tag{38}$$

Consider all z with $|z| = 1+\epsilon$, where $0 < \epsilon < \nu$. By the triangle inequality and (38) we have that

$$|f(z)| \le \sum_{j=0}^{\infty} a_j |z|^j \;=\; f(1+\epsilon) < g(1+\epsilon) = |g(z)|, \tag{39}$$

and hence $|f(z)| < |g(z)|$. Because both $f(z)$ and $g(z)$ are analytic for $|z| \le 1+\epsilon$, Rouché's theorem tells us that $g(z)$ and $f(z) - g(z)$ have the same number of zeros in $|z| \le 1+\epsilon$. Letting ϵ tend to zero yields the proof. □

The application of Lemma 1 is limited to the class of functions $A(z)$ with a radius of convergence larger than 1. In case $A(z)$ has radius of convergence 1, the results of the next section can be applied.

4.2. **Modified approach.** We first prove a result on the number and location of zeros of $z^s - A(z)$ on the unit circle. Thereto, we define the period p of a series $\sum_{-\infty}^{\infty} b_j z^j$ as the largest integer for which its holds that $b_j = 0$ whenever j is not divisible by p.

Lemma 2. *Let $A(z)$ be a* pgf *of some nonnegative discrete random variable with $A(0) > 0$ and $A'(1) < s$, where s is a positive integer. If $z^s - A(z)$ has period p, then $z^s - A(z)$ has exactly p simple zeros on the unit circle given by the p-th roots of unity $\tau_k = \exp(2\pi ik/p)$, $k = 0, 1, \ldots, p-1$.*

Proof Obviously, any zero ξ of $z^s - A(z)$ with $|\xi| = 1$ is simple, since $|A'(\xi)| \le A'(|\xi|) = A'(1) < s$ and, thus, $s\xi^{s-1} - A'(\xi) \ne 0$. Furthermore, for any z with $|z| = 1$, $|A(z)| = A(1)$ iff $z^k = 1$ whenever $a_k > 0$. This easily follows from the fact that $|a_0 + a_k z^k| < a_0 + a_k$ if $z^k \ne 1$. So, for z with $|z| = 1$ and $A(z) - z^s = 0$ it holds that $z^k = 1$ for all $a_k > 0$, and $z^s = 1$. This implies that $z^p = 1$, which completes the proof. □

Note that the requirement $a_0 = A(0) > 0$ involves no essential limitation: If a_0 was zero we would replace the distribution $\{a_i\}_{i\ge 0}$ by $\{a'_i\}_{i\ge 0}$ where $a'_i = a_{i+m}$, a_m being the first non-zero entry of $\{a_i\}_{i\ge 0}$, and a corresponding decrease in s according to $s' = s - m$.

We are now in a position to give the main result:

Theorem 7. *Let $A(z)$ be a* pgf *of some nonnegative discrete random variable with $A(0) > 0$ and $A'(1) < s$, where s is a positive integer. Also, let $z^s - A(z)$ have period p. Then the function $z^s - A(z)$ has p zeros on the unit circle given by $\tau_k = \exp(2\pi ik/p)$, $k = 0, 1, \ldots, p-1$ and exactly $s - p$ zeros in $|z| < 1$.*

Proof Lemma 2 tells us that $R(z) = z^s - A(z)$ has p equidistant zeros *on* the unit circle, and so it remains to prove that this function has exactly $s - p$ zeros *within* the unit circle. Thereto, define, for $N \in \mathbb{N}$, the truncated pgf

$$A_N(z) = \sum_{j=0}^{N-1} a_j z^j + \sum_{j=N}^{\infty} a_j z^N, \tag{40}$$

where N is a multiple of p. Then $R_N(z) = z^s - A_N(z)$ has obviously s zeros in $z \in D = \{z \in \mathbb{C} \mid |z| \leq 1\}$, since $A_N(z)$ is a polynomial satisfying $A'_N(1) < s$, and Lemma 1 thus applies. By Lemma 2 we know that $R_N(z)$ has p simple and equidistant zeros on the unit circle. We further have that

$$|A(z) - A_N(z)| \leq 2\sum_{j=N}^{\infty} a_j, \qquad |z| \leq 1, \tag{41}$$

$$|A'(z) - A'_N(z)| \leq 2\sum_{j=N}^{\infty} j a_j, \qquad |z| \leq 1. \tag{42}$$

Thus, $A_N(z)$ and $A'_N(z)$ *converge uniformly* to $A(z)$ and $A'(z)$ on $z \in D$, respectively. Moreover, if $G : D \to \mathbb{C}$ is continuous, then $G(A_N(z))$ is uniformly convergent to $G(A(z))$ on $z \in D$.

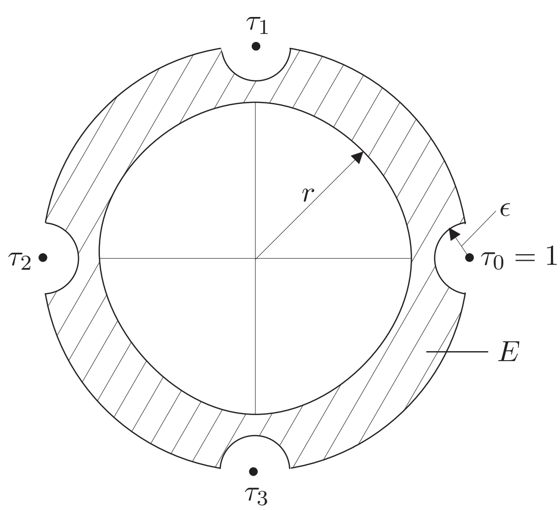

Figure 2. Graphical representation of the compact set E.

For each z on the unit circle $C = \{z \in \mathbb{C} \mid |z| = 1\}$ there exists a $D(z, \eta) := \{\xi \in D \mid 0 < |\xi - z| < \eta\}$, such that $R(\xi) \neq 0$ for $\xi \in D(z, \eta)$. Since C is compact, it can be covered by finitely many $D(z, \eta)$'s. Hence, there is a $0 < r < 1$ such that $R(z)$ has has no zeros in $r \leq |z| < 1$.

Now we prove that for large N the function $R_N(z)$, as the function $R(z)$, has no zeros in $r \leq |z| < 1$. Thereto, we show that there is an $\epsilon > 0$ and $M \in \mathbb{N}$ such that $R_N(z) \neq 0$ for all $N \geq M$ and $0 < |z - \tau_k| < \epsilon$, $k = 0, 1, \ldots, p-1$. Because $R'(z)$ is continuous and $R'_N(z)$ converges uniformly to $R'(z)$ on $z \in D$, there is an $\epsilon > 0$ and $M \in \mathbb{N}$ such that (for $k = 0, 1, \ldots, p-1$)

$$|R'_N(z) - R'(\tau_k)| < \delta < |R'(\tau_k)|, \qquad 0 < |z - \tau_k| < \epsilon, \quad N \geq M. \tag{43}$$

Furthermore, we have (for $k = 0, 1, \ldots, p-1$)

$$|R_N(z) - R'(\tau_k)(z - \tau_k)| = \Big| \int_{[\tau_k, z]} (R'_N(s) - R'(\tau_k)) \mathrm{d}s \Big|, \tag{44}$$

where the integration is carried out along the straight line that connects τ_k and z. Hence, for $0 < |z - \tau_k| < \epsilon$ and $N \geq M$, we obtain (for $k = 0, 1, \ldots, p-1$)

$$\Big| \int_{[\tau_k, z]} (R'_N(s) - R'(\tau_k)) \mathrm{d}s \Big| \leq |z - \tau_k| \max_{[\tau_k, z]} |R'_N(s) - R'(\tau_k)| < |z - \tau_k| \delta. \tag{45}$$

So, it follows that for $0 < |z - \tau_k| < \epsilon$ and $N \geq M$ (for $k = 0, 1, \ldots, p-1$)

$$\begin{aligned} |R_N(z)| &= |R_N(z) - R'(\tau_k)(z - \tau_k) + R'(\tau_k)(z - \tau_k)| & (46) \\ &\geq |R'(\tau_k)||z - \tau_k| - |R_N(z) - R'(\tau_k)(z - \tau_k)| & (47) \\ &> (|R'(\tau_k)| - \delta)|z - \tau_k| > 0. & (48) \end{aligned}$$

Since $R_N(z)$ converges uniformly to $R(z)$ and $R(z) \neq 0$ on the compact set (see Figure 2)

$$E = \{z \in \mathbb{C} \mid r \leq |z| \leq 1\} \setminus \bigcup_{k=0}^{p-1} D(\tau_k, \epsilon), \tag{49}$$

there exists an $K \in \mathbb{N}$ such that $R_N(z) \neq 0$ for all $N \geq K$, where $r \leq z < 1$. Hence, for all $N \geq K$ the number of zeros of $R_N(z)$ with $|z| < r$ is equal to $s - p$. This number can be expressed by the argument principle (see e.g. Whittaker and Watson [**?**]) as follows

$$s - p = \frac{1}{2\pi i} \oint_{|z|=r} \frac{R'_N(z)}{R_N(z)} \mathrm{d}z. \tag{50}$$

The integrand converges uniformly to $R'(z)/R(z)$, and thus

$$\lim_{N \to \infty} \frac{1}{2\pi i} \oint_{|z|=r} \frac{R'_N(z)}{R_N(z)} \mathrm{d}z = \frac{1}{2\pi i} \oint_{|z|=r} \frac{R'(z)}{R(z)} \mathrm{d}z = s - p. \tag{51}$$

Hence, the number of zeros of $R(z)$ with $|z| < r$ is also $s - p$. This completes the proof. □

Example 8. Due to Theorem 7, the $A(z)$ with a radius of convergence of 1 do not have to be excluded from the analysis of the zeros of $z^s - A(z)$. This further means that these pgf's can be incorporated in the general formulation of the solution to the queueing models of interest. The $A(z)$ that have radius of convergence 1 are typically those associated with heavy-tailed random variables. Some examples are given below.

(i) The discrete Pareto distribution (e.g. Johnson et al. [?]), defined by

$$a_j = c\frac{1}{j^{p+1}}, \qquad j = 1, 2, \dots, \tag{52}$$

with

$$c = \Big(\sum_{j=1}^{\infty} a_j\Big)^{-1} = \zeta(p+1)^{-1}, \tag{53}$$

$\zeta(\cdot)$ the Riemann zeta function and $p > 1$. For $k < p$, the k^{th} moment μ_k of the discrete Pareto distribution is given by

$$\mu_k = \frac{\zeta(p-k+1)}{\zeta(p+1)}, \tag{54}$$

whereas for $k \geq p$ the moments are infinite. The discrete Pareto distribution is also known as the Zipf or Riemann zeta distribution

(ii) The discrete standard lognormal distribution, defined by

$$a_j = ce^{-\frac{(\log j)^2}{2}}, \qquad j = 1, 2, \dots, \tag{55}$$

where c is a normalization constant.

(iii) The discrete distribution, related to the continuous Weibull distribution, defined by

$$a_j = cp^{-\sqrt{j}}, \qquad j = 0, 1, \dots, \tag{56}$$

where $p > 1$ and c is a normalization constant.

(iv) The Haight's zeta distribution, defined by

$$a_j = \frac{1}{(2j-1)^p} - \frac{1}{(2j+1)^p}, \qquad j = 1, 2, \dots, \quad p > 1. \tag{57}$$

5. Finding the roots

We now pay further attention to the roots of $z^s = A(z)$. We first present an explicit expression for each of the roots as a Fourier series. Next, we elaborate on finding the roots using a fixed point iteration. We also point out how the conditions needed for the Fourier series representation and the fixed point iteration are related.

5.1. **Fourier series representation.** The roots of $z^s = A(z)$ lie on, what is called in [20], the generalized Szegö curve, defined by

$$\mathcal{S}_{A,s} := \{z \in \mathbb{C} \ : \ |z| \leq 1, \ |A(z)| = |z|^s\}. \tag{58}$$

For the notions used below from complex function theory we refer to [15, 28]. We impose the following condition:

Condition 1. *$\mathcal{S}_{A,s}$ is a Jordan curve with 0 in its interior, and $A(z)$ is zero-free on and inside $\mathcal{S}_{A,s}$.*

Recall that $a_0 > 0$ so that we have $|A(z)| > |z|^s$ for z in the interior of $\mathcal{S}_{A,s}$. Condition 1 is geometric in nature, and can be visually checked using some standard software package. A useful geometric formulation equivalent with Condition 1 is as follows:

Lemma 3. *Condition 1 is satisfied if and only if there is a Jordan curve J with $\mathcal{S}_{A,s}$ in its interior such that $A(z)$ is zero-free on and inside J while $|A(z)| < |z|^s$ on J.*

The proof that Condition 1 implies the existence of a J as in Lemma 3 uses continuity of A on $\mathcal{S}_{A,s}$ and some basic considerations of Jordan curve theory. The proof of the reverse implication can be based on the considerations in the proof of Lemma 4. For brevity we omit the details.

To present an equivalent form of Condition 1 of more analytic nature, we use the following result:

Lemma 4. *Condition 1 is satisfied if and only if the coefficients $[z^{l-1}]A^{l/s}(z)$ decay exponentially in l.*

Proof. Assume that Condition 1 holds. Letting J as in Lemma 3 we see that we can define an analytic root $A^{1/s}(z)$ for z on and inside J that is positive at $z = 0$. We thus have by Cauchy's theorem

$$[z^{l-1}]A^{l/s}(z) = \frac{1}{2\pi i} \int_{z \in J} \frac{A^{l/s}(z)}{z^l}\, dz, \quad l = 1, 2, \ldots . \tag{59}$$

Since $|A(z)| < |z|^s$ for $z \in J$, it follows that

$$[z^{l-1}]A^{l/s}(z)| \le \frac{1}{2\pi} \text{length}(J) \Big(\max_{z \in J} \Big| \frac{A(z)}{z^s} \Big|^{1/s} \Big)^l, \tag{60}$$

and this decays exponentially, as required.

Now assume that $[z^{l-1}]A^{l/s}(z)$ decays exponentially. We shall sketch the proof that Condition 1 is valid; full details can be found in [20], proof of Lemma 4.1. We consider for w in a neighborhood of 0 the equation

$$zA^{-1/s}(z) = w, \tag{61}$$

where we have taken in a neighborhood of $z = 0$ the root $A^{-1/s}$ of A that is positive at $z = 0$ (recall $a_0 > 0$).

Theorem 9. (Lagrange inversion, see e.g. [32], p. 133) *For $f(z)$ analytic on and inside a contour $\mathcal{J}$ surrounding the origin, and for w satisfying*

$$|wf(z)| < |z|,$$

for every z on $\mathcal{J}$, the equation

$$z/f(z) = w,$$

has a unique solution $z = \tilde{z}(w)$ inside $\mathcal{J}$ and

$$\tilde{z}(w) = \sum_{l=1}^{\infty} \frac{w^l}{l!} \Big[\Big(\frac{d}{dz}\Big)^{l-1} [f^l(z)] \Big]_{z=0} .$$

By the Lagrange inversion theorem, the solution $\tilde{z}(w)$ of (61) has the power series representation

$$\tilde{z}(w) = \sum_{l=1}^{\infty} c_l w^l, \tag{62}$$

for w in a neighborhood of 0 in which

$$c_l = \frac{1}{l!}\Big(\frac{d}{dz}\Big)^{l-1}\Big(\frac{z}{zA^{-1/s}(z)}\Big)^l\Big|_{z=0} = \frac{1}{l}[z^{l-1}]A^{l/s}(z). \tag{63}$$

By assumption, we have that $c_l \to 0$ exponentially, whence the power series in (62) for $\tilde{z}(w)$ has a radius of convergence $R > 1$. It follows then from basic considerations in analytic function theory that $A^{-1/s}$ extends analytically to the open set $\{\sum_{l=1}^{\infty} c_l w_k^l \mid |w| < R\}$ and that $\tilde{z}(w)$ extends according to (62) on the set $|w| < R$. The Szegö set $\mathcal{S}_{A,s}$ in (58) occurs as

$$\mathcal{S}_{A,s} = \{\tilde{z}(e^{i\alpha}) : \alpha \in [0, 2\pi]\}, \tag{64}$$

and it can be shown that the parametrization

$$\alpha \in [0, 2\pi] \to \tilde{z}(e^{i\alpha}) = \sum_{l=1}^{\infty} c_l e^{il\alpha} \in \mathcal{S}_{A,s} \tag{65}$$

has no double points while a homotopy between $\{0\}$ and $\mathcal{S}_{A,s}$ is obtained according to

$$r \in [0, 1] \to \{\tilde{z}(re^{i\alpha}) : \alpha \in [0, 2\pi]\}. \tag{66}$$

From the latter facts it follows that $\mathcal{S}_{A,s}$ is a Jordan curve with 0 in its interior, and this completes the sketch of the proof of the converse statement. □

We now turn to the representation of the s roots of $z^s = A(z)$ in $|z| \le 1$. These roots all lie inside the Jordan curve J in Lemma 3 and are given by

$$z_k = w_k A^{1/s}(z_k), \quad k = 0, 1, \ \dots\ , s-1, \tag{67}$$

where $w_k = e^{2\pi ki/s}$. Hence, from (65) we have

$$z_k = \sum_{l=1}^{\infty} c_l w_k^l, \quad k = 0, 1, \ \dots\ , s-1, \tag{68}$$

where c_l are explicitly given in (63).

The Condition 1 and its equivalent forms as given per Lemmas 3 and 4 are equally useful in deciding whether a given A satisfies it. We present now some instances where Condition 1 is satisfied.

i. $A(z)$ is zero-free in $|z| \le 1$. An appropriate Jordan curve J is found as $|z| = 1 + \delta$ with sufficiently small $\delta > 0$. Indeed, the assumptions on A imply that there is a $\delta > 0$ such that $0 < |A(z)| < |z|^s$ for $1 < |z| \le 1 + \delta$.
ii. $A(z)$ is zero-free in $|z| < 1$. There may occur now a finite number of zeros of A on $|z| = 1$, necessitating a modification of the Jordan curve J in (i). We indent this J around the zeros such that the zeros are outside the new J while $|A(z)| < |z|^s$ for all z on the new J. As one sees, this technique may also work in cases where there

are zeros of A strictly inside $|z| = 1$. A class of examples follows from Kakeya's theorem [22] as follows:

- when $a_0 > a_1 > \dots$, we have that $A(z)$ is zero-free in $|z| \le 1$,
- when $a_0 \ge a_1 \ge \dots$, we have that $A(z)$ is zero-free in $|z| < 1$.

iii. The c_l in (63) are all non-negative. It follows from Pringsheim's theorem [30] and the fact that $\tilde{z}(w)$ is well-defined for $w \in [0, 1+\delta]$ with some $\delta > 0$, that the radius of convergence of the power series in (62) exceeds 1. Thus Lemma 4 applies and it follows that Condition 1 is satisfied.

Below we give two examples where one can compute the $c_l = [z^{l-1}]A^{l/s}(z)$ explicitly, so that the criterion in Lemma 4 can be verified.

Example 10. Consider the Poisson case, $a_j = e^{-\lambda}\lambda^j/j!$, $j = 0, 1, ...,$ and $A(z) = \exp(\lambda(z-1))$ with $0 \le \theta := \lambda/s < 1$. In this case, Condition 1 is always satisfied. Furthermore, there holds that

$$c_l = e^{-l\theta}\,\frac{(l\theta)^{l-1}}{l!}. \tag{69}$$

In Figure 3 we have pictured $\mathcal{S}_{A,s}$ for $\theta = 0.1, 0.5, 1.0$. The dots on the curves indicate the roots z_k for the case $s = 20$, obtained by calculating the sum in (68) up to $l = 50$.

Example 11. Consider the binomial case, $a_j = \binom{n}{j} q^j (1-q)^{n-j}$, $j = 0, ..., n$, and $A(z) = (p + qz)^n$ where p, $q \ge 0$, $p + q = 1$ and $A'(1) = nq < s$. We compute in this case

$$c_l = \frac{1}{l}\,p^{l\beta - l + 1}\,q^{l-1}\binom{l\beta}{l-1}, \quad l = 1, 2, \ ... \tag{70}$$

where $\beta := n/s$. In [20] the c_l are shown to have exponential decay for $\beta \ge 1$ (which covers in fact all practically relevant instances). It is further shown that for $0 \le \beta < 1$ the c_l have exponential decay if and only if

$$p^{\beta-1} q (1-\beta)^{1-\beta} \beta^{\beta} < 1. \tag{71}$$

For $\beta = 1/2$, $s = 20$, constraint (71) requires q to be less than $2(\sqrt{2} - 1)$. In Figure 4 we plotted the $\mathcal{S}_{A,s}$ for $q = 0.82 < 2(\sqrt{2} - 1)$, and the dots indicate the roots z_k obtained by calculating the sum in (68) up to $l = 50$. When q is increased, such that $q > 2(\sqrt{2} - 1)$, $\mathcal{S}_{A,s}$ turns from a smooth Jordan curve containing zero into two separate closed curves (see [20]), and (68) no longer holds.

For the Poisson and binomial distribution we have (69) and (70), respectively, to determine the c_l. In general, the values of the c_l can be determined using the following property:

Property 1. *For $A(z) = \sum_{j=0}^{\infty} a_j z^j$ and $\alpha \in \mathbb{R}$, and $A^{\alpha}(z) = \sum_{j=0}^{\infty} b_j z^j$, the coefficients b_j follow from the coefficients a_j according to $b_0 = a_0^{\alpha}$ and*

$$b_{j+1} = \alpha a_0^{\alpha-1} a_{j+1} + \frac{1}{(j+1)a_0}\sum_{n=0}^{j-1}[\alpha(n+1) - (j-n)]a_{n+1}b_{j-n}, \quad j = 0, 1, \dots. \tag{72}$$

The proof of Property 1 consists of computing the b_j's successively by equating coefficients in $A(z)(A^{\alpha})'(z) = \alpha A'(z) A^{\alpha}(z)$.

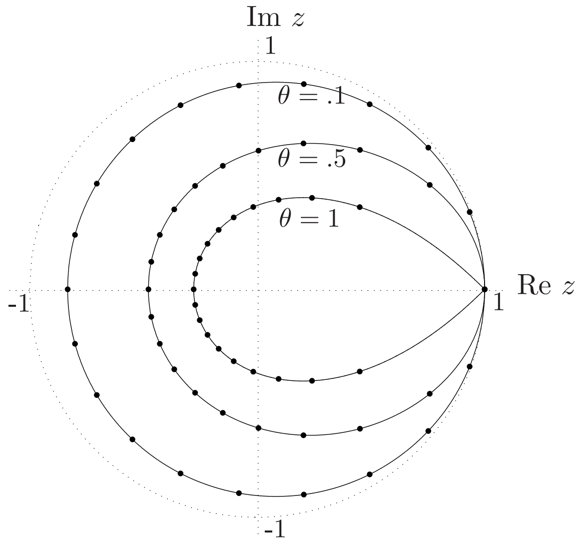

FIGURE 3. $\mathcal{S}_{A,s}$ for Poisson case, $\theta = .1, .5, 1$. The dots indicate $z_0, \ldots, z_{19}$ for $s = 20$, obtained by calculating the sum in (68) up to $l = 50$.

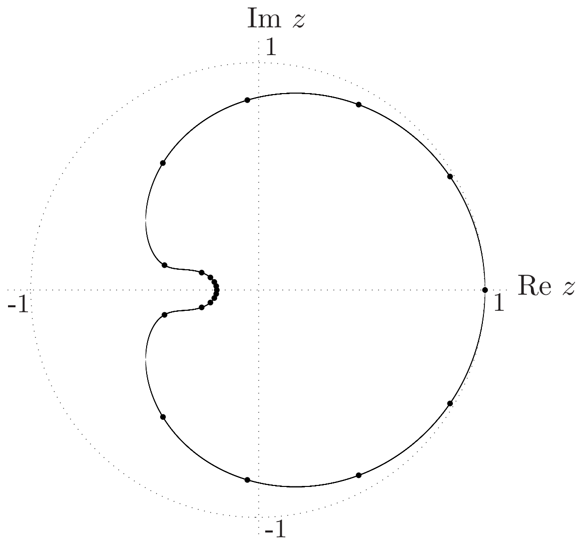

FIGURE 4. $\mathcal{S}_{A,s}$ for binomial case, $\beta = 0.5$, $q = .82$. The dots indicate $z_0, \ldots, z_{19}$ for $s = 20$, obtained by calculating the sum in (68) up to $l = 50$.

In [7] it is shown that the condition that A is infinitely divisible, or the somewhat weaker condition that $A(z)$ has no zeros inside the unit circle, are sufficient for the roots of $z^s = A(z)$ on and within the unit circle to be distinct. However, examples exist of $A(z)$ having zeros inside the unit circle and at the same time having distinct roots (see e.g. Example 11). It is therefore that in both [7] and [18] the urge of finding a necessary condition for distinctness is expressed. In this respect, we have the following result:

Lemma 5. *When Condition 1 is satisfied, the roots of $z^s = A(z)$ on and within the unit circle are distinct.*

Proof. The roots lie inside J, and satisfy (67). Since $|A(z)|^{1/s} < |z|$ for all $z \in J$, it follows from Rouché's theorem that for each w_k, the function $z - w_k A^{1/s}(z)$ has as many zeros inside J as z. □

Although Condition 1 is not necessary for the roots to be distinct (as appears to be the case in Example 11 with $\beta = 1/2$ and $q = 0.83$), it covers a far larger class of distributions of A than those for which $A(z)$ has no zeros within the unit circle.

5.2. **Fixed point iteration.** We now discuss a way to determine the roots by applying successive substitution to a fixed-point equation. This idea originates from the work of Harris et al. [18] on root-finding for the continuous-time $G/E_k/1$ queue, and was presented more formally by Adan & Zhao [2] who distinguished a class of continuous random variables for which the method works. We further investigate the method for discrete random variables A. We present necessary conditions for the method to work and compare these to the conditions needed for the Fourier series representation of the roots introduced in the previous section.

When $A(z)$ is assumed to have no zeros for $|z| \leq 1$, we know that the s roots of $z^s = A(z)$ in $|z| \leq 1$ satisfy

$$z = wG(z), \tag{73}$$

with $G(z) = A^{1/s}(z)$ and $w^s = 1$. For each feasible w, Equation (73) can be shown as in Lemma 5 to have one unique root in $|z| \leq 1$. One could try to solve the equations by successive substitutions (see [2, 18]) as

$$z_k^{(n+1)} = w_k G(z_k^{(n)}), \quad k = 0, 1, ..., s-1, \tag{74}$$

with starting values $z_k^{(0)} = 0$.

Lemma 6. *When for $|z| \leq 1$, $A(z)$ is zero-free and $|G'(z)| < 1$, the fixed point equations (74) converge to the desired roots.*

Proof. For $|z| \leq 1$, $|w| \leq 1$,

$$|wG(z)| \leq G(|z|) \leq G(1) = 1, \tag{75}$$

so $wG(z)$ maps $|z| \leq 1$ into itself. For $|\tilde{z}|, |\hat{z}| \leq 1$ we have that

$$|wG(\tilde{z}) - wG(\hat{z})| \leq |\tilde{z} - \hat{z}| \max_{0 \leq t \leq 1} |G'(\hat{z} + t(\tilde{z} - \hat{z}))|. \tag{76}$$

Hence, from (76) and $|G'(z)| < 1$ for all $|z| \leq 1$, we conclude that $wG(z)$ is a contraction on $|z| \leq 1$. □

For the Poisson distribution with $\lambda < s$, it is readily seen that $A(z) \neq 0$ and $|G'(z)| < 1$ for $|z| \leq 1$, so that the iteration (74) works. We want to consider, however, also distributions for which $A(z)$ has zeros within the unit circle (see e.g. Example 11). We restrict here naturally to $A(z)$ that allow a root $G(z) = A^{1/s}(z)$ that is analytic around $\mathcal{S}_{A,s}$ and positive at 0. Hence we introduce the following condition:

Condition 2. *Condition 1 should be satisfied and for all points $z \in \mathcal{S}_{A,s}$ there should hold that $|G'(z)| < 1$.*

According to the maximum principle we have that Condition 2 implies that $|G'(z)| < 1$ holds for all points inside $\mathcal{S}_{A,s}$ as well. Condition 2 thus ensures that for $\alpha \in [0, 2\pi]$ the point z_k is an attractor for the iteration (74).

Note that Condition 2 is what is minimally needed to ensure (74) to converge locally. However, under Condition 2 the iterates are by no means guaranteed to stay in $\mathcal{S}_{A,s}$ and its interior. This is already seen for the binomial case with $\beta < 1$, s even, and the iteration (74) for $k = s/2$, i.e

$$z_{s/2}^{(n+1)} = -1(p + qz_{s/2}^{(n)})^{\beta}. \tag{77}$$

For this iteration, the $z_{s/2}^{(n)}$, $n = 0, 1, \ldots$, are alternatingly inside and outside $\mathcal{S}_{A,s}$. The iteration, though, converges to the correct point when q is not too large. It is difficult, in general, to give guarantees for convergence; nevertheless, convergence seems to occur in most cases where Condition 2 holds.

6. Infinite series and Spitzer's identity

Iterating the relation (1) yields

$$\begin{aligned} Q_{n+1} &= \max\{0, Q_n + X_n\} \\ &= \max\{0, X_n + \max\{0, Q_{n-1} + X_{n-1}\}\} \\ &= \max\{0, X_n, X_n + X_{n-1} + Q_{n-1}\} \\ &= \max\{0, X_n, X_n + X_{n-1} + \max\{0, Q_{n-2} + X_{n-2}\}\} \\ &= \max\{0, X_n, X_n + X_{n-1}, X_n + X_{n-1} + X_{n-2} + Q_{n-2}\} \\ &\vdots \\ &= \max\{0, X_n, X_n + X_{n-1}, \ldots, X_n + X_{n-1} + \cdots + X_0\} \qquad (78) \end{aligned}$$

where the last step uses the fact that $Q_0 = 0$. Hence,

$$\begin{aligned} \mathbb{P}(Q_{n+1} \geq x) &= \mathbb{P}(\max\{0, X_n, X_n + X_{n-1}, \ldots, X_n + X_{n-1} + \cdots + X_0\} \geq x) \\ &= \mathbb{P}(\max\{0, X_1, X_1 + X_2, \ldots, X_1 + X_2 + \cdots + X_{n+1}\} \geq x) \qquad (79) \end{aligned}$$

where the last equality follows from duality.

Let $S_n = X_1 + \cdots + X_n$ and $M_n = \max\{0, S_1, \ldots, S_n\}$, $n \geq 1$. From (79) we see that

$$\mathbb{E}(Q_{n+1}) = \mathbb{E}(M_{n+1}). \qquad (80)$$

Lemma 7.

$$\mathbb{E}M_n = \sum_{l=1}^{n} \frac{1}{l}\mathbb{E}(S_l^+).$$

Proof.

$$M_n = \mathbf{1}_{\{S_n>0\}}M_n + \mathbf{1}_{\{S_n\leq 0\}}M_n. \qquad (81)$$

Use

$$\begin{aligned} \mathbb{E}(\mathbf{1}_{\{S_n>0\}}M_n) &= \mathbb{E}(\mathbf{1}_{\{S_n>0\}}(X_1 + \max\{0, X_2, \ldots, X_2 + \cdots + X_n\})) \\ &= \mathbb{E}(\mathbf{1}_{\{S_n>0\}}X_1) + \mathbb{E}(\mathbf{1}_{\{S_n>0\}} \max\{0, X_2, \ldots, X_2 + \cdots + X_n\}) \\ &= \mathbb{E}(\mathbf{1}_{\{S_n>0\}}X_1) + \mathbb{E}(\mathbf{1}_{\{S_n>0\}}M_{n-1}) \qquad (82) \end{aligned}$$

Since X_i, S_n has the same joint distribution for all i,

$$\begin{aligned} \mathbb{E}(S_n\mathbf{1}_{\{S_n>0\}}) &= \mathbb{E}(\sum_{i=1}^{n} X_i\mathbf{1}_{\{S_n>0\}}) \\ &= n\mathbb{E}(X_1\mathbf{1}_{\{S_n>0\}}) \qquad (83) \end{aligned}$$

and hence

$$\mathbb{E}(\mathbf{1}_{\{S_n>0\}}X_1) = \frac{1}{n}\mathbb{E}(S_n\mathbf{1}_{\{S_n>0\}}) = \frac{1}{n}\mathbb{E}(S_n^+). \qquad (84)$$

In addition, since $S_n \leq 0$ implies that $M_n = M_{n-1}$ we have that

$$\mathbf{1}_{\{S_n\leq 0\}}M_n = \mathbf{1}_{\{S_n\leq 0\}}M_{n-1}. \qquad (85)$$

Combined with the preceding this gives

$$\begin{aligned}\mathbb{E}(M_n) &= \mathbb{E}(\mathbf{1}_{\{S_n>0\}}M_n) + \mathbb{E}(\mathbf{1}_{\{S_n\le 0\}}M_n)\\ &= \frac{1}{n}\mathbb{E}(S_n^+) + \mathbb{E}(\mathbf{1}_{\{S_n>0\}}M_{n-1}) + \mathbb{E}(\mathbf{1}_{\{S_n\le 0\}}M_{n-1})\\ &= \mathbb{E}(M_{n-1}) + \frac{1}{n}\mathbb{E}(S_n^+). \end{aligned} \tag{86}$$

Hence,

$$\begin{aligned}\mathbb{E}(M_n) &= \frac{1}{n}\mathbb{E}(S_n^+) + \frac{1}{n-1}\mathbb{E}(S_{n-1}^+) + \mathbb{E}(M_{n-2})\\ &= \sum_{l=2}^{n}\frac{1}{l}\mathbb{E}(S_l^+) + \mathbb{E}(M_1), \end{aligned} \tag{87}$$

which proves the result since $M_1 = S_1^+$. □

More general results can be obtained (remember that $Q_0 = 0$):

Theorem 12. (Spitzer's identity)

$$\begin{aligned}F(u,z) &= \sum_{n\ge 0}\mathbb{E}(z^{Q_n})u^n\\ &= \frac{1}{z^s - uA(z)}\prod_{k=0}^{s-1}\frac{z - z_k(u)}{1 - z_k(u)}\\ &= \exp\Big\{\sum_{l=1}^{\infty}\frac{1}{l}\mathbb{E}(z^{S_l^+})u^l\Big\} \end{aligned} \tag{88}$$

For that stationary queue length distribution it follows from Abel's theorem that

$$F(z) = \lim_{u\uparrow 1}(1-u)\sum_{n=0}^{\infty}u^n\mathbb{E}z^{Q_n} = \exp\Big\{\sum_{l=1}^{\infty}l^{-1}(\mathbb{E}z^{S_l^+} - 1)\Big\}. \tag{89}$$

Moments of the stationary queue length follow from taking derivatives of (124), e.g.,

$$F'(1) = \sum_{l=1}^{\infty}\frac{1}{l}\mathbb{E}(S_l^+). \tag{90}$$

6.1. **Cumulants.** Spitzer's identity holds for both discrete and continuous increments X_l. In fact, with M the all-time maximum, $\mathbb{E}X_l < 0$, and X having moments of all orders, we have

$$\mathbb{E}(e^{\omega M}) = \exp\Big\{\sum_{l=1}^{\infty}\frac{1}{l}\mathbb{E}(e^{\omega S_l^+} - 1)\Big\}, \quad \mathrm{Re}\ \omega \le 0. \tag{91}$$

The k-th cumulant of a random variable Y is defined as the k-th derivative of $\log \mathbb{E}e^{\omega Y}$ evaluated at $\omega = 0$. We then see that

$$\log \mathbb{E}(e^{\omega M}) = \sum_{n=1}^{\infty}\frac{1}{l}\mathbb{E}(e^{\omega S_l^+} - 1)$$

$$= \sum_{l=1}^{\infty} \frac{1}{l} \int_0^\infty (\omega x + \tfrac{1}{2}\omega^2 x^2 + \ldots) f_{S_l^+}(x) \mathrm{d}x, \tag{92}$$

with $f_{S_l^+}$ the density function of S_l^+, and so the k-th cumulant of M equals

$$\frac{\mathrm{d}^k}{(\mathrm{d}\omega)^k} \log \mathbb{E}(e^{\omega M}) \Big|_{\omega=0} = \sum_{l=1}^{\infty} \frac{1}{l} \mathbb{E}((S_l^+)^k), \quad k = 1, 2, \ldots. \tag{93}$$

Recall that the first cumulant is the mean, the second cumulant is the variance, the third cumulant is the central moment $\mathbb{E}(M - \mathbb{E}M)^3$, and the fourth cumulant is $\mathbb{E}(M - \mathbb{E}M)^4 - 3\mathbb{E}(M - \mathbb{E}M)^2$.

7. Wiener-Hopf technique

The Wiener-Hopf technique stems from mathematical physics, and found its way to the field of applied probability Perhaps the most famous application of the Wiener-Hopf technique is in the context of random walks, see e.g. Cohen [10] due to the fact that the Wiener-Hopf technique is a powerful tool for the analysis of Markov processes whose evolution equation contains the $\max\{0, \cdot\}$ operator

Let us first describe the role of the $\max\{0, \cdot\}$ operator. From recursion (3) we have

$$\begin{aligned} \mathbb{E}(z^{Q_{n+1}}) &= \mathbb{E}(\mathbf{1}\{Q_n + A_n \le s\}) + \mathbb{E}(z^{Q_n + A_n - s}\mathbf{1}\{Q_n + A_n > s\}) \\ &= \mathbb{P}(Q_n + A_n \le s) + \mathbb{E}(z^{Q_n + A_n - s}) - \mathbb{E}(z^{Q_n + A_n - s}\mathbf{1}\{Q_n + A_n \le s\}). \end{aligned} \tag{94}$$

Letting $n \to \infty$ and observing that Q_n and A_n are independent then yields

$$\xi_+(z)(1 - z^{-s}A(z)) = \xi_-(z), \tag{95}$$

where $\xi_+(z) = F(z)$ and $\xi_-(z) = \mathbb{P}(Q + A \le s) - \mathbb{E}(z^{Q+A-s}\mathbf{1}\{Q + A \le s\})$. Observe that ξ_+ (respectively ξ_-) is analytic and bounded in $|z| < 1$ (respectively $|z| > 1$), and both ξ_+, ξ_- are continuous up to $|z| = 1$.

In order to find an explicit expression for $\xi_+(z)$ we need to factorize the function $1 - z^{-s}A(z)$. In more general terms, we need to factorize a function $1 - Y(z)$, where $Y(z)$ is the pgf of a random variable Y for which it holds that $\mathbb{E}Y < 0$. Such a factorization is known as the *Wiener-Hopf factorization*. The Wiener-Hopf factorization identity then reads

Theorem 13. (Wiener-Hopf factorization identity) *The following decomposition exists:*

$$1 - Y(z) = \phi_+(z)\phi_-(z), \quad |z| = 1, \tag{96}$$

where ϕ_+ (respectively ϕ_-) is analytic and bounded in $|z| < 1$ (respectively $|z| > 1$), and both ϕ_+, ϕ_- are continuous up to $|z| = 1$.

Hence, once we know the functions ϕ_+, ϕ_- we can write (95) as

$$\xi_+(z)\phi_+(z) = \frac{\xi_-(z)}{\phi_-(z)}, \tag{97}$$

where the left-hand side (respectively right-hand side) of (97) represents a function that is analytic and bounded in $|z| < 1$ (respectively $|z| > 1$), and both sides of (97) are functions continuous up to $|z| = 1$. Therefore, their analytic continuation contains no singularities in the entire complex plane. Liouville's theorem then says

Theorem 14. (Liouville) *Let $f(z)$ be analytic for all values of z and let $|f(z)| < K$ for all values of z, where K is a constant (so that $|f(z)|$ is bounded as $|z| \to \infty$). Then $f(z)$ is seen to be constant.*

Whence upon using Liouville's theorem the left-hand side of (97) is constant, and since $\xi_+(1) = 1$, we obtain

$$\xi_+(z) = \frac{\phi_+(1)}{\phi_+(z)}. \tag{98}$$

With the machinery described above, we can prove earlier mentioned results, where we rely on two different factorizations of the function $1 - z^{-s}A(z)$.

Proof of (124)
Start from the basic identity

$$1 - z = \exp\{\ln(1-z)\} = \exp\Big\{ -\sum_{l=1}^{\infty} \frac{1}{l} z^l \Big\}, \quad |z| \leq 1,\ z \neq 1. \tag{99}$$

Hence, we can write (for $|z| = 1$)

$$1 - z^{-s}A(z) = \exp\Big\{ -\sum_{l=1}^{\infty} \frac{1}{l} (z^{-s}A(z))^l \Big\} = \phi_+(z)\phi_-(z), \tag{100}$$

where

$$\phi_+(z) = \exp\Big\{ -\sum_{l=1}^{\infty} \frac{1}{l} \mathbb{E}(z^{S_l} \mathbf{1}\{S_l > 0\}) \Big\}, \tag{101}$$

$$\phi_-(z) = \exp\Big\{ -\sum_{l=1}^{\infty} \frac{1}{l} \mathbb{E}(z^{S_l} \mathbf{1}\{S_l \leq 0\}) \Big\}. \tag{102}$$

Observe that

$$\phi_+(1) = \exp\Big\{ -\sum_{l=1}^{\infty} \frac{1}{l} \mathbb{P}(S_l > 0) \Big\}, \tag{103}$$

which by (98) completes the proof. □

Alternative proof of (25)
We construct an explicit factorization of $1 - z^{-s}A(z)$ by choosing

$$\phi_+(z) = \frac{z^s - A(z)}{\prod_{k=0}^{s-1}(z - z_k)}, \quad \phi_-(z) = \frac{\prod_{k=0}^{s-1}(z - z_k)}{z^s}. \tag{104}$$

With

$$\phi_+(1) = \lim_{z \to 1} \frac{z^s - A(z)}{(z-1)\prod_{k=1}^{s-1}(z - z_k)} = \frac{s - \mu_A}{\prod_{k=1}^{s-1}(1 - z_k)}, \tag{105}$$

this completes the proof. □

8. From roots to infinite series

It is possible to derive expressions in terms of infinite series from expressions in terms of the roots.

For the sake of clarity, consider Q (stationary queue length) and the case where A is Poisson distributed with rate λ, so that the roots satisfy $z^s = \mathrm{e}^{\lambda(z-1)}$, and e.g.,

$$\mathbb{P}(Q=0) = \mathrm{e}^{\lambda}(-1)^{s-1}(s-\lambda)\prod_{k=0}^{s-1}\frac{z_k}{1-z_k}. \tag{106}$$

We also have

$$\mathbb{P}(Q=0) = \exp\Big\{-\sum_{l=1}^{\infty}\frac{1}{l}\sum_{m=1}^{\infty}\mathrm{e}^{-l\lambda}\frac{(l\lambda)^{ls+m}}{(ls+m)!}\Big\}. \tag{107}$$

It can be shown that

$$\prod_{k=1}^{s-1} z_k = (-1)^{s-1}\mathrm{e}^{-\lambda}\exp\Big\{\sum_{l=1}^{\infty}\frac{1}{l}\mathrm{e}^{-l\lambda}\frac{(l\lambda)^{ls}}{(ls)!}\Big\}, \tag{108}$$

and

$$\frac{s-\lambda}{\prod_{k=1}^{s-1}(1-z_k)} = \exp\Big\{-\sum_{l=1}^{\infty}\frac{1}{l}\sum_{m=0}^{\infty}\mathrm{e}^{-l\lambda}\frac{(l\lambda)^{ls+m}}{(ls+m)!}\Big\}. \tag{109}$$

The way to do this is by using the exact expressions for the roots and Fourier sampling.

9. Proof of Spitzer's identity

Theorem 15. (Spitzer's identity) *Assume that $A(z)$ is a polynomial of degree $n \geq s$. Then*

$$\begin{aligned} F(u,z) &= \frac{1}{z^s - uA(z)}\prod_{k=0}^{s-1}\frac{z-z_k(u)}{1-z_k(u)} \\ &= \frac{1}{1-u}\prod_{k=s}^{n-1}\frac{1-z_k(u)}{z-z_k(u)} \\ &= \exp\Big\{\sum_{k=1}^{\infty}\frac{u^k}{k}\mathbb{E}(z^{S_k^+})\Big\} \end{aligned} \tag{110}$$

Proof. Observe that

$$z^s - uA(z) = -ua_n\prod_{k=0}^{n-1}(z-z_k(u)) \tag{111}$$

and for $z=1$ this gives

$$1-u = -a_n\prod_{k=0}^{n-1}(z-z_k(u)). \tag{112}$$

Hence,

$$F(u,z) = \frac{1}{z^s - uA(z)} \prod_{k=0}^{s-1} \frac{z - z_k(u)}{1 - z_k(u)} = \frac{1}{1-u} \prod_{k=s}^{n-1} \frac{1 - z_k(u)}{z - z_k(u)}. \tag{113}$$

Let $\nu_1(u), \ldots, \nu_{n-s}(u)$ denote the $n-s$ roots in $|\nu| < 1$ of

$$\nu^{-s} - uA(\nu^{-1}) = 0 \quad \text{or} \quad \eta(\nu) := \frac{1}{\nu^s A(\nu^{-1})} = u, \tag{114}$$

so that

$$F(u,z) = \frac{1}{1-u} \prod_{k=1}^{n-s} \frac{1 - \nu_k^{-1}(u)}{z - \nu_k^{-1}(u)}. \tag{115}$$

Next consider a positively oriented contour $\mathcal{C}$ that encircles 0 and is such that for sufficiently small u the roots $\nu_1(u), \ldots, \nu_{n-s}(u)$ lie inside $\mathcal{C}$, and the remaining s roots of (114) lie outside $\mathcal{C}$. The residues theorem, for a function $h(\nu)$ analytic in a domain containing $\mathcal{C}$, says that

$$\sum_{k=1}^{n-s} h(\nu_k) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\eta'(\nu)}{\eta(\nu) - u} h(\nu) d\nu. \tag{116}$$

Expanding the denominator in the integral for small u (thanks to $(\eta - u)^{-1} = \sum_{k=0}^{\infty} u^k/\eta^{k+1}$ valid for $|u| < |\eta|$) yields

$$\sum_{k=1}^{n-s} h(\nu_k) = \frac{1}{2\pi i} \sum_{k=0}^{\infty} u^k \oint_{\mathcal{C}} \frac{\eta'(\nu)}{\eta^{k+1}(\nu)} h(\nu) d\nu. \tag{117}$$

Integration by parts on a closed contour $\oint a(\nu) b'(\nu) d\nu = -\oint a'(\nu) b(\nu) d\nu$ gives for $a = h$ and $b' = \eta'/\eta^{k+1}$

$$\sum_{k=1}^{n-s} h(\nu_k) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\eta'(\nu)}{\eta(\nu)} h(\nu) d\nu + \frac{1}{2\pi i} \sum_{k=1}^{\infty} \frac{u^k}{k} \oint_{\mathcal{C}} \frac{h'(\nu)}{\eta^k(\nu)} d\nu. \tag{118}$$

Then write (115) as

$$F(u,z) = \exp\Big\{ -\ln(1-u) + \sum_{k=1}^{n-s} \ln\Big(\frac{1 - \nu_k^{-1}(u)}{z - \nu_k^{-1}(u)} \Big) \Big\}. \tag{119}$$

We thus want to apply (118) to the function $h(\nu) = \ln \frac{1-\nu^{-1}}{z-\nu^{-1}}$, for which

$$h'(\nu) = \frac{\nu z - 1}{\nu - 1} \Big(\frac{\nu z - 1 - z(\nu - 1)}{(\nu z - 1)^2} \Big) = -\frac{1}{1-\nu} + \frac{z}{1 - \nu z} = -\sum_{j=0}^{\infty} \nu^j + \sum_{j=0}^{\infty} z^{j+1} \nu^j. \tag{120}$$

Notice that

$$\begin{aligned} \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\nu^j}{\eta^k(\nu)} d\nu &= \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{(\nu^s A(\nu^{-1}))^k}{\nu^{-j}} d\nu \\ &= \mathbb{P}\Big(\sum_{i=1}^{k} (s - A_i) = -j - 1 \Big) \end{aligned}$$

$$= \mathbb{P}(S_k = j+1) \tag{121}$$

Substituting (120) into (118) and using (121) yields

$$\begin{aligned}\sum_{k=1}^{n-s} \ln\Big(\frac{1-\nu_k^{-1}(u)}{z-\nu_k^{-1}(u)}\Big) &= \sum_{k=1}^{\infty} \frac{u^k}{k} \sum_{j=0}^{\infty} \mathbb{P}(S_k = j+1)(z^{j+1}-1) \\ &= \sum_{k=1}^{\infty} \frac{u^k}{k} \Big(\sum_{j=1}^{\infty} \mathbb{P}(S_k = j) z^j - \mathbb{P}(S_k \geq 1) \Big). \end{aligned} \tag{122}$$

Combining (122), (119) and using $-\ln(1-u) = \sum_{k=1}^{\infty} u^k/k$ leads to

$$F(u,z) = \exp\Big\{ \sum_{k=1}^{\infty} \frac{u^k}{k} \Big(\mathbb{P}(S_k \leq 0) + \sum_{j=1}^{\infty} \mathbb{P}(S_k = j) z^j \Big) \Big\}, \tag{123}$$

which completes the proof. □

10. Tail distribution

Again we assume that $A(z)$ is a polynomial of degree $n \geq s$. It follows from Abel's theorem that

$$F(z) = \lim_{u \uparrow 1} (1-u) F(u,z) = \prod_{k=s}^{n-1} \frac{1-z_k}{z-z_k}. \tag{124}$$

Then, upon using partial fraction expansion,

$$F(z) = \sum_{k=s}^{n-1} \frac{r_k}{z-z_k} = -\sum_{k=s}^{n-1} \frac{r_k}{z_k} \sum_{j=0}^{\infty} \Big(\frac{z}{z_k}\Big)^j \tag{125}$$

with the residues defined as

$$r_k = \lim_{z \to z_k} (z-z_k) F(z) = \frac{\prod_{j=s}^{n-1} (1-z_j)}{\prod_{j=s, j\neq k}^{n-1} (1-z_j)}. \tag{126}$$

From (125) we see that

$$\mathbb{P}(X = j) = -\sum_{k=s}^{n-1} r_k \Big(\frac{1}{z_k}\Big)^{j+1} \tag{127}$$

Let z_s be the unique real root in $(1,\infty)$ for which $|z_s| < |z_k|$, $k = s+1, \ldots, n-1$. Then $\mathbb{P}(X = j) \approx -r_s z_s^{-j-1}$ for large j, and we arrive at the so-called *dominant pole approximation*

$$\mathbb{P}(X \geq k) \approx -\sum_{j=k}^{\infty} r_s \Big(\frac{1}{z_s}\Big)^{j+1} = -\frac{r_s}{z_s - 1} \Big(\frac{1}{z_s}\Big)^k \quad \text{for large } k. \tag{128}$$

11. Inversion

Generating functions are widely applied in combinatorics and probability, where the coefficients typically stand for the numbers of objects of certain size, or probability of a certain event. In these cases the coefficients are nonnegative. A random variable X is fully characterized by its distribution function $F_X(x) = \mathbb{P}(X \leq x)$. A variable is discrete if it is supported by a finite or denumerable set, and in most cases this set is $\mathbb{Z}$ or $\mathbb{Z}_{\geq 0}$. If X is discrete and supported by $\mathbb{Z}$, it probability generating function (PGF) is defined as

$$G(z) = \mathbb{E}(z^X) = \sum_{k=-\infty}^{\infty} \mathbb{P}(X = k) z^k. \tag{129}$$

One can apply analytic methods to the generating function in order to extract information about the coefficients $\mathbb{P}(X = k)$. In favorable cases, this may lead to an explicit expressions for $\mathbb{P}(X = k)$, while in more difficult cases, one can obtain a numerical or asymptotic description of $\mathbb{P}(X = k)$. In obtaining such approximations, the Cauchy coefficient formula turns out to be instrumental.

11.1. Cauchy's coefficient formula.

We now quickly summarize some basic notions regarding analytic functions, leading to the Cauchy's coefficient formula, which has important consequences for transforms of random variables. For more details see for instance [17].

Definition 1. *A function $f(z)$ defined over a region Ω is analytic at a point $z_0 \in \Omega$, if for z in some open disc centered at z_0 and contained in Ω, it is representable by a convergent power series*

$$f(z) = \sum_{n \geq 0} c_n (z - z_0)^n. \tag{130}$$

A function is analytic in a region Ω iff it is analytic at every point of Ω.

Definition 2. *A function $f(z)$ is meromorphic at z_0 iff for z in a neighborhood of z_0 with $z \neq z_0$ it can be represented as $N(z)/D(z)$, with $N(z)$ and $D(z)$ being analytic at z_0. In that case, it admits near z_0 an expansion of the form*

$$f(z) = \sum_{n \geq -M} f_n (z - z_0)^n. \tag{131}$$

If $f_{-M} \neq 0$ and $M \geq 1$, then $f(z)$ is said to have a pole of order M at $z = z_0$. The coefficient f_{-1} is called the residue of $z = z_0$ and is written as $\mathrm{Res}[f(z); z = z_0]$.

Hence, for poles of order 1,

$$\mathrm{Res}[f(z); z = z_0] = \lim_{z \to z_0} (z - z_0) f(z). \tag{132}$$

Theorem 16 (Cauchy's residue theorem)**.** *Let $f(z)$ be meromorphic in the region Ω and let $\mathcal{C}$ be a positively oriented simple contour along which the function is analytic. Then*

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} f(z) dz = \sum_s \mathrm{Res}[f(z); z = s], \tag{133}$$

where the sum runs over all poles s of $f(z)$ enclosed by $\mathcal{C}$.

Theorem 17 (Cauchy's coefficient formula)**.** *Let $f(z)$ be analytic in a region Ω containing 0 and let $\mathcal{C}$ be a positively oriented simple contour along which the function is analytic. Then, the coefficient $[z^n]f(z)$ admits the integral representation*

$$f_n = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{z^{n+1}}\mathrm{d}z. \tag{134}$$

Proof. The residue theorem gives

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{z^{n+1}}dz = \mathrm{Res}[f(z)z^{-n-1}; z = 0] \tag{135}$$

and the latter is easily seen to equal the coefficient $[z^n]f(z)$. □

The coefficient formula presents a relation between the coefficient and the function itself, using adequately chosen contours of integration. It thus presents the opportunity to estimate the coefficients $[z^n]f(z)$ that appear in the expansion of $f(z)$ near 0 by using information of the function $f(z)$ away from 0.

11.2. **Singularities.** Singularities are points where a function ceases to be analytic.

Theorem 18 (Pringsheim's theorem)**.** *If $f(z)$ is representable at the origin by a power series expansion with nonnegative coefficients and radius of convergence R, then the point $z = R$ is a singularity of $f(z)$.*

From now on we assume $f(z)$ to have nonnegative coefficients.

Proposition 1 (Saddle-point bounds)**.** *Let $f(z)$ be analytic in the disc $|z| < R$ with $0 < R \leq \infty$. Let $f(z)$ have nonnegative coefficients at 0. Then, for any $r \in (0, R)$,*

$$[z^n]f(z) \leq \frac{f(r)}{r^n} \quad \Rightarrow \quad [z^n]f(z) \leq \inf_{r\in(0,R)} \frac{f(r)}{r^n}. \tag{136}$$

Proof. Follows from trivial bounds applied to the Cauchy coefficient formula. □

Notice that the value for z that provides the best bound in (136) follows from solving

$$r\frac{f'(r)}{f(r)} = n. \tag{137}$$

Let $h(r) = f(r)r^{-n}$. Then $h(0^+) = \infty$ and $h(r) \to +\infty$ as $r \to R^-$. The function $h(r)$ is upward convex for $r > 0$ so that the function $h(r)$ has a unique infimum. The convexity follows from

$$\frac{d^2}{dr^2}\frac{f(r)}{r^n} = \frac{r^2 f''(r) - 2nrf'(r) + n(n+1)f(r)}{r^{n+2}}, \tag{138}$$

which is positive for $r > 0$, since the numerator

$$\sum_{k\geq 0}(n+1-k)(n-k)f_k r^k, \quad f_k = [z^k]f(k) \tag{139}$$

has only nonnegative coefficients.

Theorem 19 (Geometric tail bounds)**.** *Let $G(z) = \mathbb{E}(z^X)$ be a pgf that is analytic for $|z| \le r$ with $r > 1$. Then*

$$\mathbb{P}(X = k) \le \frac{G(r)}{r^k}, \quad \mathbb{P}(X > k) \le \frac{G(r)}{r^k(r-1)}. \tag{140}$$

Proof. Trivial bounds applied to the Cauchy integral yield

$$\mathbb{P}(X = k) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{G(z)}{z^{k+1}} \mathrm{d}z \le \frac{G(r)}{r^k}. \tag{141}$$

The bound for the tail probability is derived from the integral representation

$$\mathbb{P}(X > k) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{G(z)}{z^{k+2}} \Big(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots \Big) \mathrm{d}z \le \frac{G(r)}{r^k}, \tag{142}$$

and again applying trivial bounds. □

Notice that these bounds always have a geometric decay in the value of k. This can also be established without resorting to complex analysis (see [31]). For $r \ge 1$,

$$\begin{aligned} -\log \mathbb{P}(X > k) &= -\log \sum_{j=k+1}^{\infty} \mathbb{P}(X = j) \ge -\log \sum_{j=k+1}^{\infty} r^{j-k-1} \mathbb{P}(X = j) \\ &\ge -\log \Big(r^{-k-1} \sum_{j=0}^{\infty} r^j \mathbb{P}(X = j) \Big) = (k+1)\log r - \log G(r). \end{aligned} \tag{143}$$

The tightest bound follows from the maximizer $\hat{r}$ of $(k+1)\log r - \log G(r)$, which is the solution to

$$\frac{\hat{r} G'(\hat{r})}{G(\hat{r})} = k + 1. \tag{144}$$

This then gives

$$\mathbb{P}(X > k) \le \frac{G(\hat{r})}{\hat{r}^k(\hat{r} - 1)}. \tag{145}$$

A similar result can be established for continuous random variables.

Theorem 20 (Exponential tail bounds)**.** *Let Y be a random variable whose moment generating function $\phi(s) = \mathbb{E}(\mathrm{e}^{sY})$ exists in an interval $[-a, b]$, where $-a < 0 < b$. Then*

$$\mathbb{P}(Y < -x) = O(\mathrm{e}^{-ax}), \quad \mathbb{P}(Y > x) = O(\mathrm{e}^{-bx}). \tag{146}$$

Proof. For any s such that $0 \le s < b$,

$$\mathbb{P}(Y > x) = \mathbb{P}(\mathrm{e}^{sY} > \mathrm{e}^{sx}) \le \phi(s)\mathrm{e}^{-sx}, \tag{147}$$

where the last step follows from Markov's inequality. It then suffices to choose $s = b$. The left-tail bounds can be established by symmetry. □

Theorems 19 and 20 give a first impression of the order of magnitude of rare event probabilities, and set the stage for more refined large deviation estimates.

11.3. **Numerical inversion.** We next discuss numerical transform inversion using the Cauchy integral. Let $\mathcal{C}_r$ denote a positively oriented circle about the origin of radius $r \in (0,1)$. Upon making the change of variables $z = r\mathrm{e}^{iu}$, we obtain

$$\begin{aligned}
\mathbb{P}(X=k) &= \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{G(z)}{z^{k+1}}dz \\
&= \frac{1}{2\pi r^k}\int_0^{2\pi} G(r\mathrm{e}^{iu})\mathrm{e}^{-iku}du \\
&= \frac{1}{2\pi r^k}\int_0^{2\pi} [\cos ku \ \mathrm{Re}(G(r\mathrm{e}^{iu})) + \sin ku \ \mathrm{Im}(G(r\mathrm{e}^{iu}))]du \qquad (148)
\end{aligned}$$

Calculating the integral in (148) approximately using the trapezoidal rule with step size π/k yields

$$\mathbb{P}(X=k) \approx \hat{p}_k = \frac{1}{2kr^k}\sum_{j=1}^{2k}(-1)^j \mathrm{Re}(G(re^{ij\pi/k})). \qquad (149)$$

Using the Poisson summation formula, Abate and Whitt [1] derive for $0 < r < 1,\ k \geq 1$ the error bound

$$|\mathbb{P}(X=k) - \hat{p}_k| \leq \frac{r^{2k}}{1-r^{2k}}. \qquad (150)$$

For practical purposes one can think of the error bound as r^{2k}, because $r^{2k}/(1-r^{2k}) \approx r^{2k}$ for r^{2k} small. To have accuracy up to the nth decimal, we let $r = 10^{-n/2k}$.

11.4. **General tail asymptotics.** The next theorem follows from Cauchy's coefficent formula, and pushing the contour of integration past singularities; see [17, p. 259].

Theorem 21 (Expansion of meromorphic functions)**.** *Let $f(z)$ be a meromorphic function at all points of the closed disc $|z| \leq R$, with poles at points $\alpha_1, \alpha_2, \ldots, \alpha_m$. Assume that $f(z)$ is analytic at all points of $|z| = R$ and at $z = 0$. Then there exist m polynomials $\{p_j(x)\}_{j=1}^m$ such that*

$$f_n = [z^n]f(z) = \sum_{j=1}^{m} p_j(n)\alpha_j^{-n} + O(R^{-n}), \qquad (151)$$

where the degree of p_j is equal to the order of the pole of f at α_j minus 1.

An important consequence of Theorem 21 is that the asymptotic behavior of all generating functions whose dominant singularities are poles can be easily analyzed. For instance, Theorem 21 can be applied to the pgf

$$F(z) = \frac{(s-A'(1))(z-1)}{z^s - A(z)}\prod_{k=1}^{s-1}\frac{z-z_k}{1-z_k}, \qquad (152)$$

which is a meromorphic function. For many choices of $A(z)$, including $A(z) = \exp(\lambda(z-1))$, it can be proved that the dominant singularity is the pole $z_s \in (1,\infty)$ of order one, leading to the asymptotic result

$$\mathbb{P}(X = k) \approx -r_s \Big(\frac{1}{z_s}\Big)^{k+1} \quad \text{for large } k \tag{153}$$

with

$$\begin{aligned} r_s &= \lim_{z \to z_s} (z - z_s) F(z) \\ &= \frac{(s - A'(1))(z_s - 1)}{s z_s^{s-1} - A'(z_s)} \prod_{k=1}^{s-1} \frac{z_s - z_k}{1 - z_k}. \end{aligned} \tag{154}$$

This gives

$$\mathbb{P}(X \geq k) \approx -\frac{r_s}{z_s - 1} \Big(\frac{1}{z_s}\Big)^{k} \quad \text{for large } k. \tag{155}$$

Note that this corresponds with (128), except that now the residue is calculated from the expressions for $F(z)$ that is in terms of the roots *inside* the unit circle. For an alternative derivation of (155), that does not use the notion of meromorphic functions, see Tijms [29, p. 365].

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