Solution to Exercise 8.18 of [1]

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Preliminaries

The *Convergent Linear Recursive Specification Principle (CL-RSP)* states that every convergent linear process equation (LPE) has at most one solution.

The mapping $even : Nat \rightarrow Bool$ is defined by the following equations, with $n : Nat$:

$$\begin{aligned} even(0) &= T \\ even(n+1) &= \neg even(n) \end{aligned}$$

Exercise

Consider the process declarations

$$X(n : Nat) = a(even(n)) \cdot X(n+3) \tag{1}$$

and

$$\begin{aligned} Y &= a(T) \cdot Z \\ Z &= a(F) \cdot Y \end{aligned}$$

We have to prove, for all $n : Nat$ that can be divided by 6:

$$X(n) = Y \tag{2}$$

Solution

If we want to apply the CL-RSP principle, then Y has to be an LPE. We can linearise Y and Z to the LPE P, that is defined as follows:

$$P(b : Bool) = a(b) \cdot P(\neg b)$$

Then $Y = P(T)$ holds. Because n can be divided by 6, and hence by 2, we have $even(n) = T$. By transitivity, (2) is equivalent to:

$$X(n) = P(even(n)) \tag{3}$$

If we have proven (3) for all $n : Nat$, then clearly we have proven (2) for all n that can be divided by 6.

Proof of (3)

We know that X is a solution for X in (1). When we have another solution Q for X in (1), we may conclude $X(n) = Q(n)$ for all $n : Nat$, from CL-RSP. Q is a solution for X in (1) if can substitute Q for X, i.e. we have to prove:

$$Q(n) = a(even(n)) \cdot Q(n+3) \tag{4}$$

Then we have proven (3), if we have proven (4) and:

$$Q(n) = P(even(n)) \tag{5}$$

Note that we couldn't just take P as a solution for X in (1), because P and X have different domains.

Inspired by (5), we define Q as follows:

$$Q = \lambda n{:}Nat.P(even(n))$$

Then we can easily prove (5):

$$\begin{array}{ll} & Q(n) \\ = & \quad \{ \text{ definition of } Q \ \} \\ & \lambda n{:}Nat.P(even(n))\ n \\ = & \quad \{ \ \beta\text{-reduction } \} \\ & P(even(n)) \end{array}$$

Now we only have to prove (4), which we do as follows:

$$\begin{array}{ll} & Q(n) \\ = & \quad \{ \text{ theorem (5) } \} \\ & P(even(n)) \\ = & \quad \{ \text{ definition of } P \ \} \\ & a(even(n)) \cdot P(\neg even(n)) \\ = & \quad \{ \ \neg\neg b = b, \text{ for all } b : Bool \ \} \\ & a(even(n)) \cdot P(\neg\neg\neg even(n)) \\ = & \quad \{ \text{ definition of } even \text{ (3 times) } \} \\ & a(even(n)) \cdot P(even(n+3)) \\ = & \quad \{ \text{ theorem (5) } \} \\ & a(even(n)) \cdot Q(n+3) \end{array}$$

Hence we have proven (3). □

References

[1] W.J. Fokkink, J.F. Groote, M.A. Reniers. *Modelling Distributed Systems*. Unpublished.