Geometric Algorithms

Lecture: Arrangements and Duality
Question: In a set of $n$ points, are there 3 points on a line?
**Question:** What is the smallest area triangle with vertices from a given set of $n$ points?
Question: For a set of $n$ points compute for each point the angular sequences of the other points?
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**Faster algorithms:** use duality and arrangements
Question: Alice and Bob want to share a Pizza Hawaii. Can they cut it by one straight cut such that each slice has half of the pineapple and half of the ham?

Faster algorithms: use duality and arrangements
Duality

\[ \ell : y = mx + b \]

\[ p = (px, py) \]

**Note:**

Geometric Algorithms  Lecture: Arrangements and Duality
### Duality

**Primal Plane**

\[ \ell: y = mx + b \]

- \( p = (p_x, p_y) \)

**Dual Plane**

\[ p^*: y = p_x x - p_y \]

- \( \ell^* = (m, -b) \)

**Notes:**

- Point \( p = (p_x, p_y) \) maps to line \( p^*: y = p_x x - p_y \)
- Line \( \ell: y = mx + b \) maps to point \( \ell^* = (mx, -b) \)
Duality

**primal plane**

\[ \ell : y = mx + b \]

**dual plane**

\[ p^* : y = p_x x - p_y \]

\[ p = (p_x, p_y) \quad \rightarrow \quad \ell^* = (m, -b) \]

*Note:* self inverse \((p^*)^* = p\), \((\ell^*)^* = \ell\)
**Duality**

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Point \( p = (p_x, p_y) \) \(
\mapsto\) line \( p^* : y = p_x x - p_y \)

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\mapsto\) point \( \ell^* = (mx, -b) \)

*Note:* does not handle vertical lines
Duality

**primal plane**

\[ \ell : y = mx + b \]

\[ mp_x + b - p_y \]

\[ p = (p_x, p_y) \]

**dual plane**

\[ p^* : y = px x - p_y \]

\[ pxm - p_y + b \]

\[ \ell^* = (m, -b) \]

duality preserves vertical distances
Duality

primal plane

\( \ell : y = mx + b \)

\[ mp_x + b - p_y \]

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dual plane

\( p^* : y = p_xx - p_y \)

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duality preserves vertical distances

\[ \Rightarrow \] incidence preserving: \( p \in \ell \) if and only if \( \ell^* \in p^* \)
Duality

primal plane

\[ \ell : y = mx + b \]

\[ p = (px, py) \]

\[ mp_x + b - py \]

duality preserves vertical distances

\[ \Rightarrow \text{incidence preserving: } p \in \ell \text{ if and only if } \ell^* \in p^* \]

\[ \Rightarrow \text{order preserving: } p \text{ lies above } \ell \text{ if and only if } \ell^* \text{ lies above } p^* \]

dual plane

\[ p^* : y = px x - py \]

\[ \ell^* \]

\[ pxm - py + b \]

\[ \ell^* = (m, -b) \]
Duality

can be applied to other objects, e.g. segments

primal plane
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dual of a segment is a double wedge
Duality

A geometric interpretation:

- parabola $\mathcal{U} : y = x^2 / 2$
- point $p = (p_x, p_y)$ on $\mathcal{U}$
- derivative of $\mathcal{U}$ at $p$ is $p_x$, i.e., $p^*$ has same slope as tangent line
- tangent line intersects $y$-axis at $(0, -p_x^2 / 2)$
- $\Rightarrow p^*$ is tangent line at $p$
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Why use Duality?

It gives a new perspective!

E.g. Half-plane intersection

- upper envelope of lines $L$
- assume $\ell$ appears as $pq$
- $p$ and $q$ on or above all lines in $L$
- $p^*$ and $q^*$ are on or below all points in $L^*$
- $\ell^*$ point on lower hull of $L^*$
- $\Rightarrow$ Compute half-plane intersections in $O(n \log n)$ time
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E.g. 3 points on a line dualize to 3 lines intersecting in a point

\[
\begin{align*}
\text{primal plane} & \quad \text{dual plane} \\
\ell & \quad \ell^* \\
p_1 & \quad p_1^* \\
p_2 & \quad p_2^* \\
p_3 & \quad p_3^* \\
p_4 & \quad p_4^*
\end{align*}
\]
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**E.g.** 3 points on a line dualize to 3 lines intersecting in a point

next we use arrangements
Arrangement $\mathcal{A}(L)$: subdivision induced by a set of lines $L$.

- consists of \textit{faces}, \textit{edges} and \textit{vertices} (some unbounded)
- also arrangements of other geometric objects, e.g., segments, circles, higher-dimensional objects
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Combinatorial Complexity:

- \( \leq n(n - 1)/2 \) vertices
- \( \leq n^2 \) edges
- \( \leq n^2/2 + n/2 + 1 \) faces: add lines incrementally
  \[ 1 + \sum_{i=1}^{n} i = n(n+1)/2 + 1 \]
- equality holds in simple arrangements
**Arrangements of Lines**

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Overall \( O(n^2) \) complexity
Goal: Compute $\mathcal{A}(L)$ in bounding box in DCEL representation

- plane sweep for line segment intersection:
  $O((n+k)\log n) = O(n^2 \log n)$
- faster: incremental construction
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Constructing Arrangements

**Goal:** Compute $A(L)$ in bounding box in DCEL representation

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Algorithm \textsc{ConstructArrangement}(L)

\textit{Input.} Set $L$ of $n$ lines.
\textit{Output.} DCEL for $A(L)$ in $B(L)$.

1. Compute bounding box $B(L)$.
2. Construct DCEL for subdivision induced by $B(L)$.
3. for $i \leftarrow 1$ to $n$
4. \hspace{1em} do insert $\ell_i$. 

\textbf{Introduction} \\
\textbf{Duality} \\
\textbf{Arrangements} \\
\textbf{Incremental Construction} \\
\textbf{Motion Planning} \\
\textbf{k-Levels}
**Algorithm** `CONSTRUCT_ARRANGEMENT(L)`

*Input.* A set $L$ of $n$ lines in the plane.

*Output.* DCEL for subdivision induced by $B(L)$ and the part of $A(L)$ inside $B(L)$, where $B(L)$ is a suitable bounding box.

1. Compute a bounding box $B(L)$ that contains all vertices of $A(L)$ in its interior.
2. Construct DCEL for the subdivision induced by $B(L)$.
3. for $i ← 1$ to $n$
4. do  Find the edge $e$ on $B(L)$ that contains the leftmost intersection point of $\ell_i$ and $A_i$.
5.  $f ←$ the bounded face incident to $e$
6. while $f$ is not the unbounded face, that is, the face outside $B(L)$
7.    do  Split $f$, and set $f$ to be the next intersected face.
Face split:

\[ f \rightarrow \ell_i \]
**Algorithm** \textsc{ConstructArrangement}(L)

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Incremental Construction

Runtime analysis:

1. $O(n^2)$

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Geometric Algorithms  
Lecture: Arrangements and Duality
Incremental Construction

Runtime analysis:

1. $O(n^2)$
2. constant

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Zone Theorem

The zone of a line $\ell$ in an arrangement $\mathcal{A}(L)$ is the set of faces of $\mathcal{A}(L)$ whose closure intersects $\ell$. 
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**Theorem:** The complexity of the zone of a line in an arrangement of $m$ lines is $O(m)$. 
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**Proof:**
- We can assume $\ell$ horizontal and no other line horizontal.
- We count number of *left-bounding* edges.
- We show by induction on $m$ that this at most $5m$:
  - $m = 1$: trivially true.
  - $m > 1$: only at most 3 new edges if $\ell$ is unique.
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  - $m = 1$: trivially true
  - $m > 1$: only at most 3 new edges if $\ell_1$ is unique, at most 5 if $\ell_1$ is not unique.

$$5(m - 1) + 5 = 5m$$
Incremental Construction

Run time analysis:

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3 Points on a Line

Algorithm:
run incremental construction algorithm for dual problem
stop when 3 lines pass through a point

Run time:
$O(n^2)$
Algorithm:
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Run time: $O(n^2)$
Example: Motion Planning

Where can the rod move by translation (no rotations) while avoiding obstacles?

- pick a **reference point**: lower end-point of rod
- shrink rod to a point, expand obstacles accordingly: locus of **semi-free placements**
- reachable configurations: cell of initial configuration in arrangement of line segments
Example: Motion Planning

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What about a moving disc among discs?
The level of a point in an arrangement of lines is the number of lines strictly above it.
k-levels in Arrangements

The **level** of a point in an arrangement of lines is the number of lines strictly above it.

**Open problem:** What is the complexity of k-levels?
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**Dual problem:** What is the complexity k-sets in a point set?
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**Known bounds:**
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