Critical window for connectivity in the Configuration Model

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joint work with Remco van der Hofstad
The Configuration Model

CMₙ(\(d\)) is a random graph made of

- A set of \([n]\) vertices
The Configuration Model (CM) is a random graph made of

- A set of \([n]\) vertices
- A degree sequence \(d_n\).

Conditioning on simplicity, the output is uniform among all the graphs with degree sequence \(d_n\).
The Configuration Model

The Configuration Model $\text{CM}_n(d)$ is random graph made of

- A set of $[n]$ vertices
- A degree sequence $d_n$.

Each vertex is given as many half edges as its degree. All the half edges are paired at random.
Conditioning on simplicity the output is uniform among all the graphs with degree sequence $d_n$. 
Description of the model

1 Pictures courtesy of Dr. Julia Komjathy
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\[ v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8 \]

\[ \text{Pictures courtesy of Dr. Julia Komjathy} \]
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![Graph diagram showing nodes and connections]

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\begin{center}
\includegraphics{graph.png}
\end{center}

\footnotesize
\textsuperscript{1}Pictures courtesy of Dr. Julia Komjathy
Description of the model
Connectivity of the Erdős-Rényi Random Graphs

**Theorem (Connectivity Threshold for ERRG (Erdős, Rényi, 1959))**

Let $G(n, p)$ be an ERRG, let $\lambda$ be the expected degree of each vertex, then if $\lim (\lambda - \log n) = t$

$$\lim_{n \to \infty} \mathbb{P}(G(n, p) \text{ is connected}) = e^{-e^{-t}}$$
Critical Window for connectivity

We define $D_n$ as the degree of a randomly chosen vertex.

**Conditions**

We define a sequence $\mathcal{CM}_n(d)$ to be in the critical window for connectivity when the following conditions are satisfied:

1. There exists a limiting degree variable $D$ such that $D_n \xrightarrow{d} D$;
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3. $\lim_{n \to \infty} \frac{n_1}{\sqrt{n}} = q_1 \in [0, \infty)$;
4. $\lim_{n \to \infty} \frac{n_2}{n} = p_2 \in [0, 1)$;
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3. $\lim_{n \to \infty} n_1 / \sqrt{n} = q_1 \in [0, \infty)$;
4. $\lim_{n \to \infty} n_2 / n = p_2 \in [0, 1)$;
5. $\lim_{n \to \infty} \mathbb{E}[D_n] = d < \infty$.

In the Critical Window for connectivity $CM_n(d)$ has non-trivial asymptotical probability of being connected.
Connectivity in the Configuration Model

**Theorem (Łuczak, 1992)**

Let us consider $\text{CM}_n(d)$, with minimum degree 2, then, under finite variance assumption on $D$,

$$\lim_{n \to \infty} \mathbb{P}_\lambda(\text{CM}_n(d) \text{ is connected} \mid \text{CM}_n(d) \text{ is simple}) = \left(\frac{d - 2p_2}{d}\right)^{\frac{1}{2}} \exp\left(-\frac{p_2^2 + dp_2}{d^2}\right)$$
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$$\lim_{n \to \infty} \mathbb{P}(\text{CM}_n(d) \text{ is connected}) = \left( \frac{d - 2p_2}{d} \right)^{1/2} \exp \left( - \frac{q_1^2}{2(d - 2p_2)} \right)$$
Structure of the proof

We want to study the properties of the Configuration Model within the critical window. In particular to obtain our results is crucial:

1. To show that the number of lines and cycles outside the giant component converges to a Poisson random variable;
We want to study the properties of the Configuration Model within the critical window.

In particular to obtain our results is crucial:

1. To show that the number of lines and cycles outside the giant component converges to a Poisson random variable;

2. To prove that whp all vertices of degree at least 3 are in the giant component.
Lines and Cycles

Figure: $k$-line: 2 vertices of degree 1, $k - 2$ of degree 2

Figure: $k$-cycle: $k$ vertices of degree 2
Lines and Cycles

Figure: \(k\)-line: 2 vertices of degree 1, \(k - 2\) of degree 2

Figure: \(k\)-cycle: \(k\) vertices of degree 2

Self loops and multi edges among vertices of degree 2 count as 1 or 2-cycles

Figure: 1-cycle

Figure: 2-cycle
Poisson convergence of number of lines and cycles

We find the asymptotic distribution of the number of cycles and lines

Number of k-lines $= L_k \xrightarrow{d} \text{Poi} \left( \frac{q_1^2 (2p_2)^{k-2}}{2d^{k-1}} \right)$,

Number of k-cycles $= C_k \xrightarrow{d} \text{Poi} \left( \frac{(2p_2)^k}{2kd^k} \right)$.

If we have finite degree variance also self loops and multiedges are asymptotically Poisson, independent from lines and from cycles of length $\geq 3$. 
Smaller components

Theorem (Łuczak, 1992)

Let us consider $\text{CM}_n(d)$, with minimum degree 2, then

$$\frac{|C_{\text{max}}|}{n} \xrightarrow{P} 1.$$
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Let us consider $\mathbb{CM}_n(d)$, with minimum degree 2, then

$$\frac{|C_{\text{max}}|}{n} \xrightarrow{P} 1.$$ 


Consider $\mathbb{CM}_n(d)$ in the critical window for connectivity. Then

$$n - |C_{\text{max}}| \xrightarrow{d} X,$$

where $X$ is a random variable with finite expectation.
Exploration of the graph
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Going down in the exploration

At each step we keep track of the variable $S_t$ which counts the active half-edges. We exhaust the connected component when $S_t = 0$.

We bound the probability of events which reduce $S_t$: 

- $P(\text{finding a vertex of degree 1}) \approx \frac{n - 1}{2}$
- $P(\text{creating a loop}) \approx \frac{S_t}{n}$

So it is very unlikely to lose active half edges unless we have many.

If the exploration starts from or hits at any point a vertex of degree at least 3 it is very unlikely to die out before exploring $\frac{n}{2}$ vertices.
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So it is very unlikely to lose active half edges unless we have many. If the exploration starts from or hits at any point a vertex of degree at least 3 it is very unlikely to die out before exploring $n/2$ vertices.
Conclusions

- The connectivity critical window in the configuration model is almost fully explored, except when $n_2/n \to 1$, in that regime the component size is highly volatile.
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- Since we are explicitly ruling out isolated vertices the connectivity threshold for the Configuration Model is far below the one of the ERRG. Also connectivity is much more fragile with respect to percolation.

- We can now asymptotically enumerate the number of connected simple graphs in the connectivity critical window.

- When \( n_1 \gg \sqrt{n} \) to have positive probability to produce a connected graph we need unbounded average degree.
The End
Thanks for the attention!