Predicate Logic
Lecture 3 (Chapters 8-9)

September 14, 2016

Unary predicate (example)

Consider the statement $4m < 5$.
Whether the statement is true or false depends on the value of $m$:

<table>
<thead>
<tr>
<th>$m$</th>
<th>on $\mathbb{Z}$</th>
<th>on $\mathbb{R}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-4$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
</tr>
<tr>
<td>$-3$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
</tr>
<tr>
<td>$0$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
</tr>
<tr>
<td>$1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2$</td>
<td>$\bullet$</td>
<td></td>
</tr>
<tr>
<td>$3$</td>
<td>$\bullet$</td>
<td></td>
</tr>
<tr>
<td>$4$</td>
<td>$\bullet$</td>
<td></td>
</tr>
</tbody>
</table>

$4m < 5$ is a unary predicate

Equivalence of predicates

Predicates $P(x)$ and $Q(x)$ are equivalent if

\[ \text{for all } x \text{ we have that } P(x) \text{ evaluates to true if, and only if, } Q(x) \text{ evaluates to true.} \]

Example

\[
4m < 5 \quad \text{val} \quad m < \frac{5}{4} \quad \text{(on } \mathbb{R} \text{ and } \mathbb{Z})
\]

\[
\text{val} \quad m \leq 1 \quad \text{(on } \mathbb{Z}, \text{ but not on } \mathbb{R}).
\]

Universal quantification

We express the statement

“for all $m \in \mathbb{Z}$ it holds that $4m < 5$”

as

\[
\forall m [m \in \mathbb{Z} : 4m < 5]
\]

Universal quantifier $\forall m$ turns the predicate $4m < 5$ into a proposition.
(This proposition is false; take, e.g., $m = 2, 3, \ldots$)
Universal quantification

In general, if $P(x)$ and $Q(x)$ are predicates, then we write

$$\forall_x [ P(x) : Q(x) ]$$

for

"all $x$ satisfying $P$ satisfy $Q.$"

Existential quantification

We express the statement

"there exists $m \in \mathbb{Z}$ such that $4m < 5$"

as

$$\exists_m [ m \in \mathbb{Z} : 4m < 5]$$

Existential quantifier $\exists_m$ turns the predicate $4m < 5$ into a proposition.
(This proposition is true; take $m = 1, 0, -1, \ldots$)

Existential quantification

In general, if $P(x)$ and $Q(x)$ are predicates, then we write

$$\exists_x [ P(x) : Q(x) ]$$

for

"there exists $x$ satisfying $P$ that also satisfies $Q.$"

Domain of Quantification

What are the truth values associated with the following propositions?

- $\exists_x [ x \in \mathbb{R} : x^2 = 1 ] \overset{val}{=} True$
- $\exists_x [ x \in \mathbb{R} \land x > 0 : x^2 = 1 ] \overset{val}{=} True$
- $\exists_x [ x \in \mathbb{R} \land x > 2 : x^2 = 1 ] \overset{val}{=} False$
- $\forall_x [ x \in \mathbb{R} : x^2 > 1 ] \overset{val}{=} False$
- $\forall_x [ x \in \mathbb{R} \land x > 0 : x^2 > 1 ] \overset{val}{=} False$
- $\forall_x [ x \in \mathbb{R} \land x > 2 : x^2 > 1 ] \overset{val}{=} True$
One- or zero-element domains

Empty domain

\( \forall x [\text{False} : P] \equiv \text{True} \)

\( \exists x [\text{False} : P] \equiv \text{False} \)

“All gnomes are green!”

One-element domain

\( \forall x [x = n : P] \equiv P[n \text{ for } x] \)

\( \exists x [x = n : P] \equiv P[n \text{ for } x] \)

Example:

\[ \forall m [m = 3 : 4 \cdot m < 5] \equiv 4 \cdot 3 < 5. \]

Domain weakening

NB: \( P \land Q \equiv P \) (all \( x \) satisfying \( P \land Q \) also satisfy \( P \)).

Examples:

\[ \exists x [x \in \mathbb{R} \land x > 0 : x^2 = 1] \]

\[ \equiv \exists x [x \in \mathbb{R} : x > 0 \land x^2 = 1] \]

\[ \forall x [x \in \mathbb{Z} \land x > 1 : x^2 > 1] \]

\[ \equiv \forall x [x \in \mathbb{Z} : x > 1 \Rightarrow x^2 > 1] \]

Domain weakening

\[ \forall x [P \land Q : R] \equiv \forall x [P : R] \land \forall x [Q : R] \]

\[ \exists x [P \land Q : R] \equiv \exists x [P : R] \land \exists x [Q : R] \]

De Morgan for quantifiers

\( \forall \) is generalised \( \land \)

\( \exists \) is generalised \( \lor \)

see book

De Morgan

\[ \neg \exists x [P : Q] \equiv \forall x [P : \neg Q] \]

\[ \neg \forall x [P : Q] \equiv \exists x [P : \neg Q] \]

not all = there exists one for which not
not one = all not

So:

\[ \neg \forall = \exists \neg \quad \text{and} \quad \neg \exists = \forall \neg \]

We also have:

\[ \neg \forall \neg = \exists \exists = \exists \quad \text{and} \quad \neg \exists \neg = \forall \neg \neg = \forall \].
Substitution

The Substitution Rule is valid for (quantified) predicates.

Example:
We have the following valid equivalence:
$$\forall x [P \land Q : R] \equiv \forall x [P : Q \Rightarrow R]$$
and hence, by the Substitution Rule, if we substitute $Q$ for $P$, $\neg P$ for $Q$, and $R \Rightarrow S$ for $R$, then we get another valid equivalence:
$$\forall x [Q \land \neg P : R \Rightarrow S] \equiv \forall x [Q : \neg P \Rightarrow (R \Rightarrow S)].$$

Leibniz

Leibniz’s Rule is valid for (quantified) predicates.

Example
We have the following valid equivalence:
$$\exists y [y = 2 : x \geq y] \equiv x \geq 2,$$
and hence, by Leibniz’s Rule, we make another valid equivalence by applying it in a bigger context:
$$\forall x [x \in D : \exists y [y = 2 : x \geq y]] \equiv \forall x [x \in D : x \geq 2].$$

Calculation with quantifiers (example)

We show with a calculation that
$$\forall x [P \land R : S] \land \forall x [Q \land R : S] \equiv \neg \exists x [P \lor Q : \neg (R \Rightarrow S)];$$

$$\neg \exists x [P \lor Q : \neg (R \Rightarrow S)] \equiv \{ \text{De Morgan} \} \equiv \forall x [P \land Q : R \Rightarrow S];$$

$$\forall x [P \land Q : R \Rightarrow S] \equiv \{ \text{Double Negation} \} \equiv \forall x [P : R \Rightarrow S];$$

$$\forall x [P : R \Rightarrow S] \equiv \{ \text{Domain Splitting} \} \equiv \forall x [P \land R : S] \land \forall x [Q : R \Rightarrow S];$$

$$\forall x [P \land R : S] \land \forall x [Q : R \Rightarrow S] \equiv \{ \text{Domain Weakening (2×)} \} \equiv \forall x [P \land R : S] \land \forall x [Q \land R : S].$$

Binary predicates (example)

The statement $3m + n > 3$ is a binary predicate.

$$3m + n = 3$$

$$\begin{array}{c}
\text{3m + n = 3} \\
N \\
\end{array}$$

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$$\begin{array}{c}
\text{3m + n = 3} \\
\text{N} \\
\end{array}$$
Equivalence of binary predicates

Predicates $P(x, y)$ and $Q(x, y)$ are equivalent if for all $x, y$ we have that $P(x, y)$ evaluates to true if, and only if, $Q(x, y)$ evaluates to true.

Example

Let $x, y \in \mathbb{Z}$.

The predicates

\[ \neg(x = 0 \Rightarrow y > 2) \]

\[ = \quad \text{Implication} \]

\[ \neg(\neg(x = 0) \lor y > 2) \]

and

\[ x = 0 \land y \leq 2 \]

\[ = \quad \text{De Morgan} \]

\[ \neg(x = 0) \land \neg(y > 2) \]

are equivalent.

Binary predicates (example)

The statement $3m + n > 3$ is a binary predicate.

\[ \begin{array}{c}
3m + n = 3 \\
N \\
\hline
\end{array} \]

assignment $m=0, n=2$ (false prop.)

Binary predicates (example)

The statement $3m + n > 3$ is a binary predicate.

\[ \begin{array}{c}
3m + n = 3 \\
N \\
\hline
\end{array} \]

assignment $m=1, n=2$ (true prop.)
Renaming bound variables

Example:

$$\forall m \in \mathbb{N} : m + n > 6$$
(true if \( n > 6 \); false if \( n \leq 6 \))

$$\forall k \in \mathbb{N} : k + n > 6$$
(true if \( n > 6 \); false if \( n \leq 6 \))

$$\forall n \in \mathbb{N} : n + n > 6$$
(always false)

Bound variables may be renamed,
but the binding structure may not change!

Bound variable

$$\forall x [P : Q] \equiv \forall y [P[y \text{ for } x] : Q[y \text{ for } x]]$$
$$\exists x [P : Q] \equiv \exists y [P[y \text{ for } x] : Q[y \text{ for } x]]$$

provided that \( y \) does not occur in \( P \) or \( Q \)
(not even as part of \( \forall y \) or \( \exists y \))

Quantifying many-place predicates

We can turn the binary predicate \( 3m + n > 3 \) into a proposition by quantification:

$$\forall m [m \in \mathbb{R} : \exists n [n \in \mathbb{N} : 3m + n > 3]]$$

This proposition is true! Informal argument:
1. Consider an arbitrary \( m_0 \in \mathbb{R} \).
2. Does there exist \( n \in \mathbb{N} \) such that \( 3m_0 + n > 3 \)?
3. Yes! We can, for instance, take \( n_0 = [3 - 3m_0] + 1 \).

Explanation:

‘Not all vertical lines through the \( m \)-axis are entirely green.”
\( \forall m \in \mathbb{R} : \exists n [n \in \mathbb{N} : 3m + n > 3] \) \( \overset{\text{val}}{=} \) True

Explanation:
"On every vertical line through the \( m \)-axis at least one point is green."

\( \exists m \in \mathbb{R} : \forall n [n \in \mathbb{N} : 3m + n > 3] \) \( \overset{\text{val}}{=} \) True

Explanation:
"There exists a vertical line through the \( m \)-axis (e.g., the line \( m = 2 \)) such that all points are green."

NB: we only consider points whose \( n \)-coordinate is in \( \mathbb{N} \)

\( \exists m \in \mathbb{R} : \exists n [n \in \mathbb{N} : 3m + n > 3] \) \( \overset{\text{val}}{=} \) True

Explanation:
"There exists a vertical line through the \( m \)-axis with a green point."

(In fact, we have already seen that all vertical lines have this property.)

\( \forall n [n \in \mathbb{N} : \forall m [m \in \mathbb{R} : 3m + n > 3] \) \( \overset{\text{val}}{=} \) False

Explanation:
"Not all horizontal lines through the \( n \)-axis are entirely green."
\[ \forall n \in \mathbb{N} : \exists m \in \mathbb{R} : 3m + n > 3 \] \quad \text{val} \quad \text{True}

Explanation:
“All horizontal lines through the \( n \)-axis have a green point.”

\[ \exists n \in \mathbb{N} : \forall m \in \mathbb{R} : 3m + n > 3 \] \quad \text{val} \quad \text{False}

Explanation:
“There is no horizontal line through the \( n \)-axis that is entirely green.”

\[ \exists n \in \mathbb{N} : \exists m \in \mathbb{R} : 3m + n > 3 \] \quad \text{val} \quad \text{True}

Explanation:
“There exists a horizontal line through the \( n \)-axis with a green point.”

\[ \forall n \exists m \in \mathbb{R} : 3m + n > 3 \] instead of \[ \forall m \exists n \in \mathbb{N} : 3m + n > 3 \]

\[ \forall m,n \exists (m,n) \in \mathbb{R} \times \mathbb{N} : 3m + n > 3 \] instead of \[ \forall m \exists n \in \mathbb{N} : 3m + n > 3 \]

NB: we only contract multiple occurrences of the same quantifier (i.e., \( \forall m \exists n \) becomes \( \forall m,\exists n \), but \( \forall m \exists \exists n \) is not contracted).