Process Algebra (2IMF10)

Recursion

Bas Luttik

MF 6.072
s.p.luttik@tue.nl
http://www.win.tue.nl/~luttik

Lecture 7

Term model (simplified)

Let E be a recursive specification over $\mathrm{BSP}(A)$ and V_R.

Definition
The term algebra for $\mathrm{BSP}(A)+E$ is the algebra

$$\mathbb{P}(\mathrm{BSP}(A)+E) = (\mathcal{C}(\mathrm{BSP}(A)+E), +, (a.)_{a\in A}, 0, 1, (X)_{X\in V_R}) \ .$$

Proposition
Bisimilarity is a congruence on $\mathbb{P}(\mathrm{BSP}(A)+E)$.

Theorem
The equational theory $\mathrm{BSP}(A)+E$ *is a sound axiomatisation of* $\mathbb{P}(\mathrm{BSP}(A)+E)/\underline{\leftrightarrow}$.

Is $\mathrm{BSP}(A)+E$ also ground-complete for $\mathbb{P}(\mathrm{BSP}(A)+E)/\underline{\leftrightarrow}$?

Equivalence of recursion variables

Example
Consider the recursive specification

$$\left\{ \begin{array}{l} X = a.X \ , \\ Y = a.a.Y \end{array} \right\}$$

Then $X \underline{\leftrightarrow} Y$.

But this equation cannot be derived from $\mathrm{BSP}(A) + E$. (Prove!)

Conclusion: we need additional methods to reason about the equivalence of recursion variables.

[Remark: we shall not discuss a full-fledged ground-complete axiomatisation of the algebra $\mathbb{P}(\mathrm{BSP}(A)+E)/\underline{\leftrightarrow}$, but we'll come close.]

Recursion: algebraic approach

Let E be a rec. spec. over signature Σ and set of variables V_R.

Furthermore, let $\mathbf{A}$ be a Σ-algebra and ι the associated interpretation.

Definition
A solution of E in $\mathbf{A}$ is an extension κ of ι with interpretations of the recursion variables in V_R as elements of $\mathbf{A}$ such that $\mathbf{A}, \kappa \models X = t_X$ for every equation $X = t_X$ in E. If κ is a solution of E and $X \in V_R$, then we shall call $\kappa(X)$ a *solution* of X in E.

NB: a solution of E in an algebra of transition systems modulo bisimilarity is an assignment of closed terms to recursion variables such that the recursion equations are true up to bisimilarity.

Solutions: examples

1. The recursive specification $E_1 = \{X = a.1\}$ has a solution both in $\mathbb{P}(\mathrm{BSP}(A))/\underline{\leftrightarrow}$ and in $\mathbb{P}(\mathrm{BSP}(A)+E_1)/\underline{\leftrightarrow}$.
2. The recursive specification $E_2 = \{X = a.X\}$ has a solution in $\mathbb{P}(\mathrm{BSP}(A)+E_2)/\underline{\leftrightarrow}$, but not in $\mathbb{P}(\mathrm{BSP}(A))/\underline{\leftrightarrow}$.
3. The recursive specification $E_3 = \{X = X\}$ has many solutions, both in $\mathbb{P}(\mathrm{BSP}(A))/\underline{\leftrightarrow}$ and in $\mathbb{P}(\mathrm{BSP}(A)+E_3)/\underline{\leftrightarrow}$.

Example: equivalence of rec. vars.

Consider the recursive specification

$$\left\{ \begin{array}{l} X = a.X \ , \\ Y = a.a.Y \end{array} \right\} .$$

Note that we can argue that every solution of X is a solution of Y too:

$$X = a.X = a.a.X \ .$$

Hence, any solution κ of E in some algebra $\mathbf{A}$ satisfies

$$\kappa(X) = \kappa(a.a.X) = \mathbf{a}.\mathbf{a}.\kappa(X) \qquad (\mathbf{a}. \text{ the interpretation of } a. \text{ in } \mathbf{A}) \ ,$$

so $\kappa(X)$ is a solution of Y in E.

Example: equivalence of rec. vars.

Consider the recursive specification

$$\left\{ \begin{array}{l} X = a.X \ , \\ Y = a.a.Y \end{array} \right\} .$$

The reasoning on the previous slide allows us to conclude that every solution of X in whatever algebra (!) must also be a solution of Y in that algebra.

The converse, however, need not hold:

Exercise: construct a model of $\mathrm{BSP}(A)$ in which Y has a solution that is not also a solution of X.

Example: equivalence of rec. vars.

Consider the recursive specification

$$\left\{ \begin{array}{l} X = a.X \ , \\ Y = a.a.Y \end{array} \right\} .$$

Perhaps if we exclude some models (e.g., the answers to the exercise on the previous slide), then we may be able to say more about the equivalence of X and Y in the above recursive specification.

Note that, for models in which X and Y both have a unique solution, the reasoning on slide 14 would suffice to conclude that X and Y indeed denote the same process!

Definition

An occurrence of a recursion variable X in a closed term s is guarded if it occurs in the scope of an action prefix.

A term s is completely guarded if all occurrences of all recursion variables in s are guarded.

A recursive specification E is completely guarded if all right-hand sides of all equations in E are completely guarded.

Exercise 5.5.1

Determine whether, in the following terms, the occurrences of the recursion variables X and Y are guarded, unguarded, or both:

$$a.X \ , \quad Y + b.X \ , \quad b.(X + Y) \ , \quad a.Y + X \ .$$

Example

1. The recursive specification

$$E_1 = \{X_1 = a.X_1, \ Y_1 = a.X_1\}$$

is completely guarded.

2. The recursive specification

$$E_2 = \{X_1 = a.X_1, \ Y_1 = X_1\}$$

is *not* completely guarded.

But E_1 and E_2 have exactly the same solutions in every model!

Definition

A recursive specification E is guarded if there exists a completely guarded recursive specification F with $V_R(E) = V_R(F)$ and $\mathrm{BSP}(A)+E \vdash X = t$ for all $X = t \in F$.

Example

Although the recursive specification

$$E_2 = \{X_1 = a.X_1, \ Y_1 = X_1\}$$

is not completely guarded, it is guarded.

Definition

A recursive specification E is guarded if there exists a completely guarded recursive specification F with $V_R(E) = V_R(F)$ and $\mathrm{BSP}(A)+E \vdash X = t$ for all $X = t \in F$.

Exercise 5.5.2

Determine whether the following recursive specifications are guarded or unguarded:

1. $\{X = Y, \ Y = a.X\}$; and
2. $\{X = a.Y + Z, \ Y = b.Z + X, \ Z = c.X + Y\}$.

Recursive Specification Principle

Definition

Let Σ be a signature; we say that Σ-algebra $\mathbf{A}$ satisfies the Recursive Specification Principle (RSP) if every *guarded* recursive specification E over Σ and some set V_R of variables has *at most* one solution.

RSP as a proof principle

Example

Consider rec. spec. E consisting of the following equations:

$$X = a.X + b.X \ ,$$
$$Y = a.Y + b.Z \ , \text{ and}$$
$$Z = a.Z + b.Y \ .$$

We can prove that $X = Y$ in the context of E as follows:

Define two sequences of terms $\tilde{t} = t_X, t_Y, t_Z$ and $\tilde{u} = u_X, u_Y, u_Z$ by $t_X \equiv X$, $t_Y \equiv Y$, $t_Z \equiv Z$, and $u_X \equiv X$, $u_Y \equiv X$, $u_Z \equiv X$.

Then both $\tilde{t}$ and $\tilde{u}$ denote solutions of E (verify!).

Since E is guarded, by RSP, $\tilde{t} = \tilde{u}$, so $X \equiv u_Y = t_Y \equiv Y$.