Topological Aspects of Hybrid Processes (a treatment using non-standard analysis)

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1 Introduction

Hybrid systems are systems in which both physical and computational behavior play an important role. In the study of such systems, using techniques from computer science, we often encounter problems of a topological nature. In this paper, we briefly discuss three of these problems, namely: the *continuity* of physical behavior, the occurrence of *Zeno-phenomena* and other limit-behavior due to the combination of computations and physical behavior, and the influence of *imprecise measurements*.

Before we are able to discuss the problems mentioned above, we need a way to state them formally. For this purpose, we use *topological transition systems*, i.e. labeled transition systems $\langle X, \Sigma, \rightarrow \rangle$ in which the state space X and signal space Σ are both equipped with a topology (see e.g. [8]). Most of the work on hybrid systems that deals with Zeno-behavior, for example, implicitly assumes such a topology. But the consequences of that topology for equivalences, and the like, are hardly ever made precise (see [5, 15, 7] for some exceptions).

To analyse these topological transition systems, we will use the methods of *non-standard analysis*, also known as the mathematics of infinitesimals. From a non-standard point of view, the presence of a topology means that the spaces X and Σ are lifted to spaces $X \subseteq {}^*X$ and $\Sigma \subseteq {}^*\Sigma$, on which a notion of approximate equivalence (denoted $\approx$) is defined. The mapping * is also defined for all relevant mathematical structures on X. For example, given the function $x \in \mathbb{R} \rightarrow \mathbb{R}$, we use ${}^*x \in {}^*(\mathbb{R} \rightarrow \mathbb{R})$ to denote the lifted version. The elements of X are called *standard elements*, while the elements in ${}^*X \setminus X$ are *non-standard elements*. The non-standard elements in, for example, the set of real numbers $\mathbb{R}$, are infinitesimals (i.e. numbers approximately equal to 0), near-standard elements (i.e. numbers that differ only by an infinitesimal from a standard element), and infinitely large numbers (i.e. numbers that are not approximately equal to any standard number). For earlier excursions by computer scientists into the non-standard domain, see for example [1, 11, 12].

In the coming sections, we are going to show some examples of models in which we use approximate equivalence to model continuous behavior and Zeno-phenomena. Furthermore, we propose variants of the familiar notion of bisimulation equivalence that reflect ways to preserve continuity, Zeno-phenomena and robustness against measurement errors.

2 Continuity

The first topological problem regarding the modeling of hybrid systems, that we will discuss, is that physical behavior is *continuous* in nature. Labeled transition systems often turn out to be an inadequate model for such behavior. Recently, they have been extended with behavioral systems, i.e. with sets of functions from time (the real line) to states and observations, to overcome this (see e.g. [10, 14, 6, 3]). Using non-standard analysis on topological transition systems, we find an alternative model for hybrid systems.

Intuitively, continuity of behavior means that a systems progress takes place in infinitesimally small steps. Using a non-standard topological transition system, this intuition is easily formalized by stating that:

Definition 1 (Continuity) *A transition relation* $\rightarrow \subseteq {}^*X \times {}^*\Sigma \times {}^*X$ *is continuous iff* $\langle x \rangle \xrightarrow{\sigma} \langle x' \rangle$ *implies* $x \approx x'$.

In physics, continuous behavior is often described using differential equations. The non-standard definition of differentiability (see e.g. [9]) tells us that the time derivative $\dot{\underline{x}}$ of a function $\underline{x} \in \mathbb{R} \rightarrow \mathbb{R}$ has the property that $\dot{\underline{x}}(t) \approx \frac{{}^*\underline{x}(t') - {}^*\underline{x}({}^*t)}{t' - {}^*t}$ for all standard $t \in \mathbb{R}$ and ${}^*t \approx t' \in {}^*\mathbb{R}$. Inspired by this definition, we can build a transition system that mimics the behavior of such a differential equation. This is reflected in the following conjecture.

Conjecture 1 *Let* $\rightarrow \subseteq {}^*X \times {}^*\Sigma \times {}^*X$ *be defined by* $X = \Sigma = \mathbb{R}^2$ *and*

$$\langle x, t \rangle \xrightarrow{x', t'} \langle x', t' \rangle \;\Leftrightarrow\; f(x) \approx \frac{x' - x}{t' - t} \;\wedge\; t' \approx t \;\wedge\; t' > t.$$

Then $\rightarrow$ *is a continuous transition relation with the property that for each solution* $\underline{x}$ *of the differential equation* $\dot{\underline{x}}(t) = f(\underline{x}(t))$ *there exists a pair* $(x_i, t_i) \in {}^*(\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}})$ *of internal*[1] *sequences such that for all* $i \in {}^*\mathbb{N}$ *we have* $\langle x_i, t_i \rangle \xrightarrow{x_{i+1}, t_{i+1}} \langle x_{i+1}, t_{i+1} \rangle$ *and* ${}^*x({}^*t) \approx x_i$ *whenever* $t_i \leq {}^*t < t_{i+1}$. *Conversely, each such pair of non-standard sequences represents a solution.*

From this conjecture, it becomes clear, that in order to preserve continuous behavior, it is necessary to compare not only the finite sequences but also the infinite ones. This can be obtained by the additional requirement that a bisimulation relation must be internal.

3 Zeno-phenomena

Zeno-phenomena are behaviors of a system, consisting of an infinite number of discrete events that occur in a finite amount of time. Typically, they occur as an artefact of discretisation. A legendary example, once told by the Alean philosopher Zeno himself, is that of Achilles and the tortoise: *"Achilles and a tortoise are involved in a race. And, because an ordinary race would be unfair, the tortoise gets a head start. Now, when Achilles reaches the point where*

[1] *Internal* sequences are sequences in ${}^*(\mathbb{R}^{\mathbb{N}})$ rather than in ${}^*\mathbb{R}^{{}^*\mathbb{N}}$. The advantage of using internal sequences is that we may use induction to obtain conclusions for all elements of ${}^*\mathbb{N}$, including the non-standard (i.e. infinite) elements.

the tortoise started, the tortoise will have moved on a little, and whenever Achilles is there where the tortoise moved to, the tortoise will have moved on again. So, it becomes clear that Achilles never catches up with the tortoise."

Let us assume that Achilles moves twice as fast as the tortoise, then we can model the race between Achilles and the tortoise as the following transition system, where A models the position of Achilles, and T models the position of the tortoise.

$$\langle A,T\rangle \stackrel{\text{racing}}{\rightarrow} \langle A',T'\rangle \;\Leftrightarrow\; A'=T \wedge T'=T+\frac{A'-A}{2} \wedge T>A$$

$$\langle A,T\rangle \stackrel{\text{Achilles catches up}}{\rightarrow} \langle A,T\rangle \;\Leftrightarrow\; A=T$$

If we start at $(0,1)$, where the tortoise has a one-meter head start, we can indeed use induction to show that the "Achilles catches up" transition never occurs. However, looking a little closer at the race, we see that the distance between Achilles and the tortoise decreases by a factor 2 with each transition and that the turtle will never get past the distance of two meters. So, intuitively, Achilles should catch up with the tortoise after he has run 2 meters, but our model does not show this.

If we want our model to show Achilles eventual victory, we have a number of options. Our first option, of course, is to model the race in a completely different manner, in which the discretisation does not take place. If we had not chosen to observe the particular moments at which Achilles has caught up with the turtles previous position, nothing would have gone wrong, probably. But, such a posteriori reasoning does not always work, since the Zeno-phenomena may not always be as obvious as in our example.

Our second option, is to alter the model slightly, by granting Achilles the win whenever the distance between him and the tortoise are approximately equal. So we add the non-standard transitions

$$\langle A,T\rangle \stackrel{\text{Achilles catches up}}{\rightarrow} \langle A,T\rangle \;\Leftrightarrow\; A\approx T.$$

In this non-standard model, Achilles will still need an infinite amount of racing transitions to reach his goal, but the internal sequences over ${}^*\mathbb{N}$ of the non-standard transition system (like the ones we used in the previous section) will all contain an "Achilles catches up" step.

The third option, is to leave the model intact, but to alter the equivalence. If we take bisimulation equivalence as an example, we could add the following requirement to the witnessing relations on a non-standard topological transition system:

Definition 2 (Limit preserving) *A relation $R\subseteq {}^*X\times{}^*X$ is* limit preserving *if for all $x,y\in{}^*X$ and $x'\in X$*

$$xRy \wedge x\approx {}^*x' \;\Rightarrow\; \exists_{y'\in X}\ {}^*x'R{}^*y' \wedge y\approx{}^*y'.$$

This requirement models that if limit points are related, then the standard points that are close to these limit points are also related. In a sense, this resembles the notion of topological bisimulation of [5].

4 Imprecise measurements

The third topological problem in the study of hybrid systems, is that measurements in physics are never precise. This means that models need to cope with small changes in variables. One consequence of this, is that two models can only be considered equivalent if small changes in one model can be mimicked by small changes in the other model. Another consequence, is that we can hardly ever speak of actual equivalence of models. Often, the best we can do is to obtain (arbitrarily precise) approximations (see for example [15, 4]).

If we want to deal with the fact that an imprecise measurement may occur, then this means that the notion of equivalence must be robust against small changes in the state. We therefore propose to extend the notion of bisimulation on non-standard topological transition systems with the following requirement.

Definition 3 (Robust against imprecision) *A relation $R \subseteq {}^*X \times {}^*X$ is* robust against imprecision *if for all $x, y \in X$ and $x' \in {}^*X$*

$$^*xR^*y \wedge {}^*x \approx x' \Rightarrow \exists_{y' \in {}^*X}\, x'Ry' \wedge {}^*y \approx y'.$$

This is a kind of dual to the notion of limit preservation suggested in the previous section. As a matter of fact, we expect that the combination of robustness against imprecision and limit preservation is closely related to the notion of continuity of a relation as defined in [2].

If we want to deal with the fact that an imprecise measurement forces us to compare transition systems only approximately, we could consider using the following notion of approximate (bi-)simulation, which replaces the familiar simulation requirement.

Definition 4 (Approximate simulation) *A relation $R \subseteq {}^*X \times {}^*X$ is an* approximate simulation *if for all $x, y \in {}^*X$ and $x' \in {}^*X$ and $\sigma \in {}^*\Sigma$*

$$^*xR^*y \wedge \langle x \rangle \xrightarrow{\sigma} \langle x' \rangle \Rightarrow \exists_{y' \in {}^*X, \sigma' \in {}^*\Sigma}\, x'Ry' \wedge \langle y \rangle \xrightarrow{\sigma'} \langle y' \rangle \wedge \sigma \approx \sigma'.$$

Note, that if we take the so-called discrete topology on Σ then $\sigma \approx \sigma'$ implies $\sigma = \sigma'$, and we obtain the usual definition of simulation. As a matter of fact, a similar observation holds for the preservation of limits and robustness against imprecision.

5 Conclusion

In order to use process algebras effectively for the specification and analysis of hybrid systems, the topological structure of both the state space and the observation space (signal space) cannot be neglected. In this note, we have proposed to add topological structure to transition systems, and to analyse these topological transition systems by means of non-standard analysis methods. Of course, many combinations of the equivalence relations suggested in this note can be constructed, and certainly there are also other definitions thinkable. The work described is only intended to sketch a direction of research that is largely unexplored, and, in our opinion, possibly of great value to the development of timed and hybrid process theory. We need to study the new equivalences in the usual way, by showing their relation with existing equivalences, by showing congruence for process algebraic operators, by finding axioms to reason about them, and so on, and so on. Our hopes are that the non-standard approach we sketched in this note, will provide us with a flexible way of modeling, that allows us to vary the level of abstraction between completely discrete and complete continuous behavior, just as it was outlined in, for example [13].

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