SOS Rule Formats for Determinism and Idempotency

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Structural Operational Semantics (SOS)

Semantics A mapping from a syntax to a semantic domain

Operational semantic domain: transition relation (execution steps of a piece of syntax)

Structural Semantics of a composite object is defined in terms of its parts

1. Has the shape of logical inference rules
2. Proposed by Hennessy and Plotkin in 1979 (later developed in Plotkin 81 and Kahn 87)



SOS: An Informal Example

Operational Semantics

$$\frac{}{\langle x := v, st\rangle \longrightarrow \langle nil, st[x \mapsto v]\rangle} \qquad \frac{}{\langle nil, st\rangle \surd}$$

$$\frac{\langle s_0, st\rangle \longrightarrow \langle s_0', st'\rangle}{\langle s_0; s_1, st\rangle \longrightarrow \langle s_0'; s_1, st'\rangle} \qquad \frac{\langle s_0, st\rangle \surd \quad \langle s_1, st\rangle \longrightarrow \langle s_1', st'\rangle}{\langle s_0; s_1, st\rangle \longrightarrow \langle s_1', st'\rangle}$$

$$\frac{\langle s_0, st\rangle \surd \quad \langle s_1, st\rangle \surd}{\langle s_0; s_1, st\rangle \surd}$$

SOS?

Operational Semantics (cont'd)

$$\frac{b(st) = T \quad \langle s_0, st\rangle \rightarrow \langle s_0', st'\rangle}{\langle \texttt{if}\ b\ \texttt{then}\ s_0\ \texttt{else}\ s_1\ \texttt{fi}, st\rangle \rightarrow \langle s_0', st'\rangle}$$

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Signature

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Signature $\Sigma = (O, ar : O \rightarrow \mathbb{N})$:

- O: set of function symbols O (for operators);
- ar: defining a natural number (arity) for each function symbol.

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Examples

- 0, 1, a: constants (i.e., $ar(0) = ar(1) = 0$);
- $\sigma_{\mathrm{rel}}(\_), \sigma^*_{\mathrm{rel}}$: unary function symbols;
- $f(\_, \_)$, $\_ + \_$: binary function symbols.

Terms

Terms

1. Assume a countable set of variables V;
2. The set of terms $\mathcal{T}(\Sigma, V)$ s.t. $\Sigma = (O, ar)$ is the minimal set satisfying:
   2.1 $V \subseteq \mathcal{T}(\Sigma, V)$;
   2.2 $t_1, \ldots, t_n \in \mathcal{T}(\Sigma, V)$, $f \in O$ and $ar(f) = n \Rightarrow$ $f(t_1, \ldots, t_n) \in \mathcal{T}(\Sigma, V)$.

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Examples

- 0, a, $0 + \sigma_{\mathrm{rel}}(0 + a)$: closed terms
- $x + \sigma_{\mathrm{rel}}(y)$: terms.

Transition Formulae

Transition Formulae

Given a set of labels A and a set of terms $\mathcal{T}$:

1. $\phi = t \xrightarrow{a} t'$ is a positive transition formula and
2. $\phi' = t \overset{a}{\nrightarrow}$ is a negative transition formula

for all $t, t' \in \mathcal{T}$ and $a \in A$.

t is the source of both formulae and t' is the target of the former.

Transition and Predicate Formulae

Examples

1. $x + y \xrightarrow{a} 0$: positive transition formula and
2. $a + \sigma_{\mathrm{rel}}(0) \overset{a}{\nrightarrow}$: negative transition formula

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Closed Formulae

A formula is closed when it mentions no variable.

Transition System Specification (TSS)

TSS

1. A deduction rule is a pair (H, ϕ) where H is a set of formulae (called premises) and ϕ is a positive formula (called conclusion);
2. A TSS is a set of deduction rules.

We denote deduction rule (H, ϕ) by $\frac{H}{\phi}$.

f-defining rule: f is the main operator of the source of conclusion.

TSS: Examples

$$\frac{}{\sigma_{\mathrm{rel}}(x) \xrightarrow{1} x} \qquad \frac{x \xrightarrow{1} x' \quad y \xrightarrow{1} y'}{x + y \xrightarrow{1} x' + y'} \qquad \frac{x \xrightarrow{1} x' \quad y \stackrel{1}{\not\rightarrow}}{x + y \xrightarrow{1} x'} \qquad \frac{x \stackrel{1}{\not\rightarrow} \quad y \xrightarrow{1} y'}{x + y \xrightarrow{1} y'}$$

TSS: Another Examples

$$\frac{a \xrightarrow{l} a}{a \xrightarrow{l} b} \qquad \frac{a \overset{l}{\nrightarrow}}{a \xrightarrow{l} a}$$

The Semantics of Semantics!

Provable Rules

$\mathcal{T}$ proves $\frac{N}{\phi}$, denoted by $\mathcal{T} \vdash \frac{N}{\phi}$, when there is a tree with formulae as nodes and of which

- the root is labelled by ϕ;
- if a node is labelled by ψ and its direct offsprings form the set K then:
  - ψ is a negative formula and $\psi \in N$, or
  - ψ is a positive formula and $\frac{K}{\psi}$ is an instance of a deduction rule in $\mathcal{T}$.

The Semantics of Semantics!

Contradiction and Contingency

Formula $t \xrightarrow{l} t'$ is said to contradict $t \stackrel{l}{\nrightarrow}$, and vice versa.

Φ contradicts Ψ, denoted by $\Phi \nvDash \Psi$, when there is a $\phi \in \Phi$ that contradicts a $\psi \in \Psi$.

Φ is contingent w.r.t. Ψ, denoted by $\Phi \vDash \Psi$, when Φ does not contradict Ψ.

The Semantics of Semantics!

The Least Three-Valued Stable Model

A pair (C, U) of sets of positive closed transition formulae is called a 3V-SM for $\mathcal{T}$ when

- $C \subseteq U$,
- for all $\phi \in C$, $\mathcal{T} \vdash \frac{N}{\phi}$ for a set N such that $U \vDash N$, and
- for all $\phi \in U$, $\mathcal{T} \vdash \frac{N}{\phi}$ for a set N such that $C \vDash N$.

The least 3V-SM is the least w.r.t. $(C, U) \leq (C', U') \doteq C' \subseteq C \wedge U \subseteq U'$.

A TSS is complete when $C = U$.

3V-SM: Examples

$$\frac{}{\sigma_{\mathrm{rel}}(x) \xrightarrow{1} x} \qquad \frac{x \xrightarrow{1} x' \quad y \xrightarrow{1} y'}{x + y \xrightarrow{1} x' + y'} \qquad \frac{x \xrightarrow{1} x' \quad y \stackrel{1}{\nrightarrow}}{x + y \xrightarrow{1} x'} \qquad \frac{x \stackrel{1}{\nrightarrow} \quad y \xrightarrow{1} y'}{x + y \xrightarrow{1} y'}$$

3V-SM: Another Examples

$$\frac{a \xrightarrow{l} a}{a \xrightarrow{l} b} \qquad \frac{a \stackrel{l}{\nrightarrow}}{a \xrightarrow{l} a}$$

Rule Formats

Idea

Put sufficient syntactic conditions on rules to guarantee a semantic property

Our Order of Business

Rule formats to guarantee:

1. Determinism of certain transitions,
2. Idempotency of certain binary operators.

Determinism

Idea

No program can evolve in two different ways.

Formal Definition

Set of formulae T is called deterministic for label l, when if $p \xrightarrow{l} p' \in T$ and $p \xrightarrow{l} p'' \in T$, then $p' \equiv p''$.

Take a Break: Determinism at work!

Source Dependency

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v is source dependent via R, where $\mathcal{R}$ is the collection of labels of premises needed to show its source dependency.

Source dependency: Examples

$$\frac{}{\sigma_{\mathrm{rel}}(x) \xrightarrow{1} x} \qquad \frac{x \xrightarrow{1} x' \quad y \xrightarrow{1} y'}{x + y \xrightarrow{1} x' + y'} \qquad \frac{x \xrightarrow{1} x' \quad y \stackrel{1}{\nrightarrow}}{x + y \xrightarrow{1} z'} \qquad \frac{x \stackrel{1}{\nrightarrow} \quad z \xrightarrow{1} y'}{x + y \xrightarrow{1} y'}$$

Normalized TSSs w.r.t. L

For $l \in L$, $f \in \Sigma$ and each two f-defining rules:

$$(\mathrm{r})\frac{\Phi_r}{f(\overrightarrow{s}) \xrightarrow{l} s'} \quad (\mathrm{r}')\frac{\Phi_{r'}}{f(\overrightarrow{t}) \xrightarrow{l} t'}$$

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1. $f(\overrightarrow{s}) \equiv f(\overrightarrow{t})$,
2. each $v \in vars(s')$ is source dependent in (r) via some subset of L,
3. for each variable $v \in vars(r) \cap vars(r')$ there is a set of formulae in $\Phi_r \cap \Phi_{r'}$ proving its source dependency (both in (r) and $(\mathrm{r'})$) via some subset of L.

Normalized TSSs: Example

$$\frac{}{\sigma_{\mathrm{rel}}(x) \xrightarrow{1} x} \qquad \frac{x \xrightarrow{1} x' \quad y \xrightarrow{1} y'}{x + y \xrightarrow{1} x' + y'} \qquad \frac{x \xrightarrow{1} x' \quad y \stackrel{1}{\nrightarrow}}{x + y \xrightarrow{1} x'} \qquad \frac{x \stackrel{1}{\nrightarrow} \quad y \xrightarrow{1} y'}{x + y \xrightarrow{1} y'}$$

Normalized TSSs: Another Example

$$\frac{x \xrightarrow{a} x'}{f(x,y) \xrightarrow{a} x'} \qquad \frac{y \overset{a}{\nrightarrow} \quad x \xrightarrow{b} x'}{f(y,x) \xrightarrow{a} x'}$$

Determinism Format

A complete and normalized TSS is in the determinism format w.r.t. L, when for each two rules $\frac{\Phi_0}{f(\overrightarrow{s}) \xrightarrow{l} s'}$ and $\frac{\Phi_1}{f(\overrightarrow{s}) \xrightarrow{l} s''}$, it holds that $s' \equiv s''$ or Φ_0 contradicts Φ_1.

Determinism Format: Example

$$\frac{}{\sigma_{\mathrm{rel}}(x) \xrightarrow{1} x} \qquad \frac{x \xrightarrow{1} x' \quad y \xrightarrow{1} y'}{x + y \xrightarrow{1} x' + y'} \qquad \frac{x \xrightarrow{1} x' \quad y \overset{1}{\nrightarrow}}{x + y \xrightarrow{1} x'} \qquad \frac{x \overset{1}{\nrightarrow} \quad y \xrightarrow{1} y'}{x + y \xrightarrow{1} y'}$$

Determinism Format: Another Example

$$\frac{x \overset{1}{\nrightarrow}}{\sigma^*_{\mathrm{rel}}(x) \overset{1}{\rightarrow} \sigma^*_{\mathrm{rel}}(x)} \qquad \frac{x \overset{1}{\rightarrow} x'}{\sigma^*_{\mathrm{rel}}(x) \overset{1}{\rightarrow} x' + \sigma^*_{\mathrm{rel}}(x)}$$

Determinism Format: Completeness Requirement

$$\frac{a \xrightarrow{l} a}{a \xrightarrow{l} b} \qquad \frac{a \stackrel{l}{\nrightarrow}}{a \xrightarrow{l} a}$$

Idempotency

Idea

Composing two identical program does not influence their behavior.

Formal Definition

A binary operator $f \in \Sigma$ is idempotent w.r.t. $\sim$ when for each p, $f(p, p) \sim p$.

Strong Bisimulation

A relation $\mathcal{R}$ on closed terms is a bisimulation relation when

1. $\mathcal{R}$ is symmetric,
2. and for each p_0, p_1, p_0' and l
   $(p_0 \,\mathcal{R}\, p_1 \wedge p_0 \xrightarrow{l} p_0') \Rightarrow \exists_{p_1'} (p_1 \xrightarrow{l} p_1' \wedge p_0' \,\mathcal{R}\, p_1')$

Two closed terms p_0, p_1 are bisimilar, $p_0 \leftrightarrow p_1$, when there exists a bisimulation relation $\mathcal{R}$ such that $p_0 \,\mathcal{R}\, p_1$.

Idempotency Format: Rules

1_l. Choice rules

$$\frac{\{x_i \xrightarrow{l} t\} \cup \Phi}{f(x_0, x_1) \xrightarrow{l} t} \; i \in \{0, 1\}$$

$2_{l_0,l_1}$. Communication rules

$$\frac{\{x_0 \xrightarrow{l_0} t_0,\ x_1 \xrightarrow{l_1} t_1\} \cup \Phi}{f(x_0, x_1) \xrightarrow{\gamma(l_0,l_1)} f(t_0, t_1)} \; t_0 \equiv t_1 \text{ or } (l_0 = l_1 \text{ and } \xrightarrow{l_0} \text{ is deterministic })$$

where $\gamma(l_1, l_2) \in \{l_1, l_2\}$ and Φ is a set of (positive or negative) formulae.

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where $\gamma(l_1, l_2) \in \{l_1, l_2\}$ and Φ is a set of (positive or negative) formulae.

1^*_l and $2^*_{l_0,l_1}$: the same as unstarred ones except that t, t_0 and t_1 are all variables and $\Phi = \emptyset$.

Idempotency Format: Constraints

A set of rules in the shapes given before is in idempotency format w.r.t. a binary operator f if for each label l there exists at least one rule of the forms 1^*_l or $2^*_{l,l}$.

Idempotency Format: Example

$$\frac{}{\sigma_{\mathrm{rel}}(x) \xrightarrow{1} x} \qquad \frac{x \xrightarrow{1} x' \quad y \xrightarrow{1} y'}{x + y \xrightarrow{1} x' + y'} \qquad \frac{x \xrightarrow{1} x' \quad y \stackrel{1}{\nrightarrow}}{x + y \xrightarrow{1} x'} \qquad \frac{x \stackrel{1}{\nrightarrow} \quad y \xrightarrow{1} y'}{x + y \xrightarrow{1} y'}$$

Idempotency Format: Example (cont'd)

$$\frac{x_0 \xrightarrow{a} x_0'}{x_0 + x_1 \xrightarrow{a} x_0'} \qquad \frac{x_1 \xrightarrow{a} x_1'}{x_0 + x_1 \xrightarrow{a} x_1'}$$

Idempotency Format: Another Example

$$\frac{x_0 \xrightarrow{a} x_0'}{x_0 \oplus x_1 \xrightarrow{a} x_0'} \qquad \frac{x_1 \xrightarrow{a} x_1'}{x_0 \oplus x_1 \xrightarrow{a} x_1'} \qquad \frac{x_0 \xrightarrow{\chi} x_0' \quad x_1 \xrightarrow{\chi} x_1'}{x_0 \oplus x_1 \xrightarrow{\chi} x_0' \oplus x_1'}$$

Conclusions

Done

Two rule formats for:

1. determinism and ;
2. idempotency w.r.t. strong bisimilarity.

To Be Done

1. Adding state
2. Structural congruences

That's All Folks!

Thank you! Any questions?