Structural Congruences may harm Congruence!

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Overview

Guess the transition relation

$$\frac{}{a \xrightarrow{l_0} a} \qquad \frac{a \xrightarrow{l_0} a}{b \xrightarrow{l_1} b}$$

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$$\frac{}{a \xrightarrow{l_0} a} \qquad \frac{a \xrightarrow{l_0} a}{b \xrightarrow{l_1} b}$$

$$\{a \xrightarrow{l_0} a, b \xrightarrow{l_1} b\}$$

Guess the transition relation

$$\frac{}{a \xrightarrow{l_0} a} \qquad \frac{x \xrightarrow{l_0} y}{f(x) \xrightarrow{l_1} z}$$

Guess the transition relation

$$\frac{}{a \xrightarrow{l_0} a} \qquad \frac{x \xrightarrow{l_0} y}{f(x) \xrightarrow{l_1} z}$$

$$\{a \xrightarrow{l_0} a, f(a) \xrightarrow{l_1} a, f(a) \xrightarrow{l_1} b(?)\}$$

Guess the transition relation

$$\frac{b \xrightarrow{l_0} b}{a \xrightarrow{l_0} a} \qquad \frac{a \xrightarrow{l_0} a}{b \xrightarrow{l_0} b}$$

Guess the transition relation

$$\frac{b \xrightarrow{l_0} b}{a \xrightarrow{l_0} a} \qquad \frac{a \xrightarrow{l_0} a}{b \xrightarrow{l_0} b}$$

$$\{\}$$

Guess the transition relation

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$$\{a \xrightarrow{l_0} a\}$$

Provable Transitions

A TSS (transition system specification)

proves a transition $p \xrightarrow{l} q$, if and only if

there is an upwardly branching tree with finite depth

of which the root is $p \xrightarrow{l} q$,

and the nodes are provable from their ancestors

using a deduction rule in TSS.

Guess the transition relation

$$\frac{b \overset{l_0}{\not\rightarrow}}{a \xrightarrow{l_0} a}$$

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Guess the transition relation

$$\frac{b \overset{l_0}{\not\to}}{a \overset{l_0}{\longrightarrow} a} \qquad \frac{a \overset{l_0}{\not\to}}{b \overset{l_0}{\longrightarrow} b}$$

$$\{a \overset{l_0}{\longrightarrow} a\}? \quad or \quad \{b \overset{l_0}{\longrightarrow} b\}?$$

Guess the transition relation

$$\frac{a \stackrel{l_0}{\not\rightarrow}}{a \stackrel{l_0}{\longrightarrow} a}$$

Guess the transition relation

$$\frac{a \overset{l_0}{\nrightarrow}}{a \xrightarrow{l_0} a}$$

?

Stable Model [Bol&Groote96][vanGlabbeek04]

A set of formulae S is a stable model of TSS if and only if

for all $p \xrightarrow{l} q \in S$,

there is an upwardly branching tree with finite depth

of which the root is $p \xrightarrow{l} q$, and

1. if the node is labelled with a positive formula, then it is provable from its ancestors using a deduction rule in TSS;
2. if the node is labelled with a negative formula $p' \stackrel{l'}{\nrightarrow}$, then there is no q' such that $p' \xrightarrow{l'} q' \in S$.

Overview

✓ SOS: A Brief Tour

→ Structural Congruences

3. SCs and SOS: Congruence

4. SCs and SOS: Well-Definedness

5. Conclusion

Structural Congruences: A Motivating Example

$$\frac{x_0 \xrightarrow{l} y}{x_0 \mid\mid x_1 \xrightarrow{l} y \mid\mid x_1} \qquad \frac{x_1 \xrightarrow{l} y}{x_0 \mid\mid x_1 \xrightarrow{l} x_0 \mid\mid y}$$

SCs: A Motivating Example

$$\frac{x_0 \xrightarrow{l} y}{x_0 \parallel x_1 \xrightarrow{l} y \parallel x_1} \qquad x_1 \parallel x_0 \equiv x_0 \parallel x_1$$

SCs: The Meaning of $\equiv$

The set of equations $t \equiv t'$ induces a relation $\equiv_{sc}$ on closed terms.

$p \equiv_{sc} q$ iff one of the following holds:

0. $p = q$ (refl.)
1. $p \equiv_{sc} q$ is a closed instantiation of an equation $t \equiv t'$; (eq.)
2. $q \equiv_{sc} p$; (symm.)
3. $p \equiv_{sc} r$ and $r \equiv_{sc} q$; (trans.)
4. $p = f(p_0, \ldots, p_{ar(f)-1})$, $q = f(q_0, \ldots, q_{ar(f)-1})$ and $p_i \equiv_{sc} q_i$ $(0 \leq i \leq n)$. (cong.)

SCs: A Motivating Example

$$\frac{x_0 \xrightarrow{l} y}{x_0 \mid\mid x_1 \xrightarrow{l} y \mid\mid x_1}$$

$$x_1 \mid\mid x_0 \equiv x_0 \mid\mid x_1$$

$$\frac{x \equiv x' \quad x' \xrightarrow{l} y' \quad y' \equiv y}{x \xrightarrow{l} y}$$

SCs: Stable Model (extended)

A set of formulae S is a stable model of TSS if and only if

for all $p \xrightarrow{l} q \in S$,

there is an upwardly branching tree with finite depth

of which the root is $p \xrightarrow{l} q$, and

1. if the node is labelled with a positive formula, then it is provable from their ancestors using a deduction rule in TSS;
2. if the node is labelled with a negative formula $p' \stackrel{l'}{\nrightarrow}$, then there is no q' such that $p' \xrightarrow{l'} q' \in S$.
3. if the node is labelled with a structural congruence $p \equiv q$, then it holds that $p \equiv_{sc} q$.

SCs: Guess the Transition Relation

$$\frac{}{f(a) \xrightarrow{l} a} \qquad a \equiv b$$

$$\frac{x \equiv x' \quad x' \xrightarrow{l} y' \quad y' \equiv y}{x \xrightarrow{l} y}$$

SCs: Guess the Transition Relation

$$\frac{}{f(a) \xrightarrow{l} a} \qquad a \equiv b$$

$$\frac{x \equiv x' \quad x' \xrightarrow{l} y' \quad y' \equiv y}{x \xrightarrow{l} y}$$

$$\{f(a) \xrightarrow{l} a, f(a) \xrightarrow{l} b, f(b) \xrightarrow{l} a, f(b) \xrightarrow{l} b\}$$

Overview

Bisimilarity

Relation R on closed terms is a bisimulation relation iff for all $(p, q) \in R$:

1. $\forall_{l,p'} p \xrightarrow{l} p' \Rightarrow \exists_{q'} q \xrightarrow{l} q' \ \wedge \ (p', q') \in R$;
2. $\forall_{l,q'} q \xrightarrow{l} q' \Rightarrow \exists_{p'} p \xrightarrow{l} p' \ \wedge \ (p', q') \in R$.

p and q are bisimilar, denoted by $p \,\underline{\leftrightarrow}\, q$, iff there exists a bisimulation relation, relating p and q.

Congruence

Relation R on terms is a congruence iff for all arbitrary operators f:

1. $\forall_{0 \leq i < ar(f)} \forall_{p_i, q_i} (p_i, q_i) \in R \Rightarrow$
   $(f(p_0, \ldots, p_{ar(f)-1}), f(q_0, \ldots, q_{ar(f)-1})) \in R$.

Congruence Formats

$$\frac{\{t_i \xrightarrow{l_i} y_i | i \in I\}}{f(x_0, \ldots, x_{ar(f)-1}) \xrightarrow{l} t}$$

Congruence Formats

$$\frac{\{t_i \xrightarrow{l_i} y_i | i \in I\}}{f(x_0, \ldots, x_{ar(f)-1}) \xrightarrow{l} t}$$

Theorem [Groote&Vaandrager92]:
If a TSS comprises of deduction rules in tyft format, then bisimilarity with respect to its induced transition relation is a congruence.

SCs: Congruence

$$\frac{}{a \xrightarrow{l_0} a} \qquad \frac{}{b \xrightarrow{l_0} b}$$

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$$a \equiv f(b)$$

SCs: Congruence

$$\frac{}{a \xrightarrow{l_0} a} \qquad \frac{}{b \xrightarrow{l_0} b}$$

$$a \equiv f(b) \qquad \frac{x \equiv x' \quad x' \xrightarrow{l} y' \quad y' \equiv y}{x \xrightarrow{l} y}$$

$$f(b) \xrightarrow{l_0} a$$

$$f(a) \overset{l_0}{\not\rightarrow}$$

Congruence Format for SCs (CFSC)

$$f(\overrightarrow{x}) \equiv g(\overrightarrow{y})$$
$$f(\overrightarrow{x}) \equiv t \qquad \text{if } f \text{ is a fresh function symbol.}$$

Theorem [Mousavi&Reniers04]:
If a TSS comprises of deduction rules in tyft format,
and structural congruences are in CFSC (with respect to the TSS)
then bisimilarity with respect to its induced transition relation
is a congruence.

SCs: A Motivating Example (revisited)

$$\frac{x_0 \xrightarrow{l} y}{x_0 \mid\mid x_1 \xrightarrow{l} y \mid\mid x_1}$$

$$x_1 \mid\mid x_0 \equiv x_0 \mid\mid x_1$$

$$\frac{x \equiv x' \quad x' \xrightarrow{l} y' \quad y' \equiv y}{x \xrightarrow{l} y}$$

SCs: A Motivating Example (revisited)

$$\frac{x_0 \xrightarrow{l} y}{x_0 \mid\mid x_1 \xrightarrow{l} y \mid\mid x_1}$$

$$x_1 \mid\mid x_0 \equiv x_0 \mid\mid x_1 \qquad !x \equiv x \mid\mid !x$$

$$\frac{x \equiv x' \quad x' \xrightarrow{l} y' \quad y' \equiv y}{x \xrightarrow{l} y}$$

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Guess the transition relation (revisited)

$$\frac{b \overset{l_0}{\not\rightarrow}}{a \overset{l_0}{\longrightarrow} a} \qquad \frac{a \overset{l_0}{\not\rightarrow}}{b \overset{l_0}{\longrightarrow} b}$$

$$\{a \overset{l_0}{\longrightarrow} a\}? \quad or \quad \{b \overset{l_0}{\longrightarrow} b\}?$$

Guess the transition relation (revisited)

$$\frac{a \overset{l_0}{\not\to}}{a \overset{l_0}{\longrightarrow} a}$$

?

Stratification [Groote93] (simplified)

A function $\mathcal{S}$ is a stratification for a TSS in ntyft format if for all rules:

$$\frac{\{t_i \xrightarrow{l_i} y_i | i \in I\} \quad \{t_j \stackrel{l_j}{\nrightarrow} | j \in J\}}{f(\overrightarrow{x}) \xrightarrow{l} t}$$

and for all closed substitutions σ:

1. $\forall_{i \in I} \mathcal{S}(\sigma(f(\overrightarrow{x}))) \leq \mathcal{S}(\sigma(t_i))$
2. $\forall_{j \in I} \mathcal{S}(\sigma(f(\overrightarrow{x}))) < \mathcal{S}(\sigma(t_j))$

Stratification

Theorem [Bol&Groote96]:

If a TSS is in ntyft format and it is stratified then

it has a unique stable model

and bisimilarity with respect to this model is a congruence.

Guess the transition relation (revisited)

$$\frac{b \overset{l_0}{\nrightarrow}}{a \xrightarrow{l_0} a}$$

$$\{a \xrightarrow{l_0} a\}$$

Guess the transition relation

$$\frac{b \overset{l_0}{\nrightarrow}}{a \xrightarrow{l_0} a}$$

$$a \equiv b$$

$?$

Stratification (extended and simplified)

A combination of TSS in ntyft format and SCs in CFSC format is stratified if TSS is stratified by function $\mathcal{S}$ and

1. for all equations of the form $f(\overrightarrow{x}) \equiv g(\overrightarrow{y})$
   $\mathcal{S}(\sigma(f(\overrightarrow{x}))) = \mathcal{S}(\sigma(g(\overrightarrow{y})))$
2. for all equations of the form $f(\overrightarrow{x}) \equiv t$ then
   $\mathcal{S}(\sigma(f(\overrightarrow{x}))) \leq \mathcal{S}(\sigma(t))$

Stratification (extended and simplified)

Theorem [Mousavi&Reniers04]:

If a combination of a TSS and SCs is stratified then

it has a unique stable model

and bisimilarity with respect to this model is a congruence.

Overview

Conclusion

Done:

- Structural congruences may harm well-definedness of the semantics and congruence of strong bisimilarity, if not used with care;
- Syntactic criteria for making them safe were propose;
- Conservative (operational and equational) extensions of semantics with structural congruences were studied (not presented).

To be done:

- Studying congruence for other notions of bisimilarity;
- Investigating structural congruences in the categorical model of [Turi&Plotkin97].

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Thank You!