Exercises Evolution Equations (2WA13) 2014-5, Week 1

Let $X, Y, Z$ be Banach spaces.

1. Prove Lemma 1.2 in the notes.

2. Prove that
   \begin{enumerate}
   \item Bounded linear operators $(A, D(A)) : X \rightarrow Y$ are closable with $D(\overline{A}) = \overline{D(A)}$.
   \item In particular, bounded operators are closed if and only if their domain of definition is closed.
   \end{enumerate}

3. Let $(A, D(A)) : X \rightarrow Y$ be a closed injective linear operator. Show that $A^{-1}$ is closed.

4. Let $(A, D(A)) : X \rightarrow Y$ be a linear operator. Assume that for any sequence $(x_n)$ in $D(A)$ that satisfies $x_n \rightarrow 0$ in $X$ and $Ax_n \rightarrow y$ in $Y$, we have $y = 0$. Show that $A$ is closable.

5. Let $(A, D(A)) : X \rightarrow Y$ be a closed linear operator and $B \in \mathcal{L}(X, Y)$. Show that $(A + B, D(A))$ is closed.

6. Let $(A, D(A)) : L^2(\mathbb{R}) \rightarrow \mathbb{R}$ be given by
   \[
   D(A) = C_0^\infty(\mathbb{R}), \quad Af = f(0)
   \]
   (in the sense of the continuous representative). Show that $A$ is not closable.