Exercises Evolution Equations (2WA13) 2014-15,
Week 6

1. Let $BC(\mathbb{R})$ and $BUC(\mathbb{R})$ be the spaces of bounded continuous and bounded uniformly continuous functions on $\mathbb{R}$ (under the supremum norm). On both spaces we consider the translation semigroup $\{T(t)\}$ given by

$$(T(t)u)(\xi) = u(\xi - t).$$

a) Show that $\{T(t)\}$ is not strongly continuous on $BC(\mathbb{R})$.

b) Show that $\{T(t)\}$ is strongly continuous but not uniformly continuous on $BUC(\mathbb{R})$.

2. Let $X$ be a Banach space, $A, B \in \mathcal{L}(X)$. Show that if $AB = BA$ then

$$e^{t(A+B)} = e^{tA}e^{tB},$$

3. Let $X$ be a Banach space and let $V \in \mathcal{L}(X)$ be invertible. Let $A \in \mathcal{L}(X)$. Express $e^{t(V^{-1}AV)}$ in terms of $e^{tA}$.

4. a) Let $n \in \mathbb{N}_+$, $\lambda, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and

$$N = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \ddots \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 \\ & \ddots \\ & & \lambda_n \end{bmatrix}, \quad A = \begin{bmatrix} \lambda & 1 \\ & \ddots \\ & & \ddots \\ & & & 1 \\ & & & & \lambda \end{bmatrix}$$

Find $e^{tN}$, $e^{tD}$, $e^{tA}$.

b) Using 3. and a), explain how to calculate $e^{tA}$ for any (complex) $(n,n)$-matrix $A$, using its Jordan decomposition.

5. On the Banach space $\ell^2$, consider the semigroup of operators $\{T(t)\}$ given by

$$(T(t)x)_n = e^{t\lambda_n}x_n, \quad n \in \mathbb{N},$$

where $\lambda_n$ are given complex numbers.

a) Show that if $\{\lambda_n | n \in \mathbb{N}\}$ is bounded then $\{T(t)\}$ is uniformly continuous. Find the generator.

b) Show that if $\{\text{Re} \lambda_n | n \in \mathbb{N}\}$ is bounded above then $\{T(t)\}$ is strongly continuous. Find the generator.

(Finding the generator includes finding its domain of definition!)