1 Unbounded linear operators

Let $X,Y$ be normed spaces. The set of bounded linear operators is noted as $\mathcal{L}(X,Y)$.

Let now $D = D(A) \subset X$ be a linear subspace, and $A : D \rightarrow Y$ a linear (not necessarily bounded!) operator.

Notation: $(A,D(A)) : X \rightarrow Y$

**Definition:** $G(A) := \{(x, Ax) \mid x \in D\}$ is called the graph of $A$.

Obviously, $G(A)$ is a linear subspace of $X \times Y$.

The linear operator $A$ is called **closed** if $G(A)$ is closed in $X \times Y$.

The linear operator $A$ is called **closable** if $G(A) = G(\overline{A})$ for some linear operator $(\overline{A}, D(\overline{A})) : X \rightarrow Y$. In this case $\overline{A}$ is called the **closure** of $A$.

If a closure exists, it is obviously unique.

Clearly, if $A$ is closed then $A$ is closable and $A = \overline{A}$.

**Lemma 1.1** Let $(A,D(A)) : X \rightarrow Y$ be a linear operator. The following statements are equivalent:

(i) $A$ is closed.

(ii) For any sequence $(x_n)$ in $D(A)$ that satisfies $x_n \overset{X}{\rightarrow} x$, $Ax_n \overset{Y}{\rightarrow} y$ for some $x \in X$, $y \in Y$, one has $x \in D(A)$, $Ax = y$.

**Proof:** Exercise!

**Lemma 1.2** Let $(A,D(A)) : X \rightarrow Y$ be a linear operator. The following statements are equivalent:

(i) $A$ is closable.

(ii) For any sequence $(x_n)$ in $D(A)$ that satisfies $x_n \overset{X}{\rightarrow} 0$ and $(Ax_n)$ converges, we have $Ax_n \overset{Y}{\rightarrow} 0$.

**Proof:** Exercise!

**Examples:**
Bounded linear operators \((A, D(A)) : X \rightarrow Y\) are closable with \(D(A) = \overline{D(A)}\). In particular, bounded operators are closed if and only if their domain of definition is closed. In particular, operators \(A \in \mathcal{L}(X, Y)\) are closed.

The following example shows that differential operators with smooth coefficients are closable. (This fact is not restricted to the \(L^2\)-context in which it is formulated here.) Let \(\Omega \subset \mathbb{R}^d\) be a domain, \(X = Y = L^2(\Omega)\). Define \((A, D(A))\) by
\[
D(A) = C_0^\infty(\Omega), \quad Au = \sum_{\alpha \leq m} a_\alpha \partial^\alpha u, \quad a_\alpha \in C^\infty(\Omega), \quad u \in D(A).
\]

Then \(A\) is closable. To see this, let \(H^{-m}(\Omega) := (H^m_0(\Omega))'\) and define \(\hat{A} \in \mathcal{L}(L^2(\Omega), H^{-m}(\Omega))\) by
\[
H^{-m}(A \phi, \psi)_{H^m_0} := \sum_{\alpha \leq m} (-1)^{|\alpha|} \int_\Omega \partial^\alpha (a_\alpha \psi) \phi \, dx.
\]
(check that indeed \(\hat{A} \in \mathcal{L}(L^2(\Omega), H^{-m}(\Omega))\) !) Furthermore, \(\hat{A}|_{D(A)} = A\), if functions are identified with (anti)linear forms in the usual way. Note that with this identification we have \(L^2(\Omega) \hookrightarrow H^{-m}(\Omega)\). Assume now that for some sequence \((x_n)\) in \(D(A)\) we have \(x_n \to 0, \ Ax_n \to y\) in \(L^2(\Omega)\). Then \(Ax_n = Ax_n\) for all \(n\) and \(Ax_n \to 0\) in \(H^{-m}(\Omega)\), so \(y = 0\). Therefore, \(A\) is closable by Lemma 1.2.

Let \(X = L^2(\mathbb{R}), Y = \mathbb{K}\), and the operators \((A, D(A)), (B, D(B))\) given by
\[
D(A) = D(B) = C_0(\mathbb{R}), \quad Au = u(0), \quad Bu = \int_\mathbb{R} u \, dx.
\]
(As usual, \(A\) has to be interpreted in the sense of the continuous representative.) Then neither \(A\) nor \(B\) are closable. (Exercise!)

Let \(X = Y = L^2(-1, 1)\) (over \(\mathbb{R}\)) and \((A, D(A))\) given by
\[
D(A) = C_b^1(-1, 1), \quad Au = u'.
\]
Then \(A\) is not closed. Indeed, let \(u, u_n \in X\) be given by
\[
u(t) = |t|, \quad u_n(t) = \frac{t}{\sqrt{t^2 + 1/n}}.
\]
Then \(u_n \in D(A), u_n \to u\) in \(X\) but \(u_n' \to f\) in \(X\), where
\[
f(t) = \begin{cases} 
-1 & (t < 0) \\
1 & (t > 0)
\end{cases}
\]
(check!) So \(f \notin D(A)\) and \(A\) is not closed by Lemma 1.1.
However, $A$ has the closure $(\overline{A}, D(\overline{A}))$ given by

$$D(\overline{A}) = H^1(-1,1), \quad \overline{A}u = u'$$

(in the sense of weak derivatives). To prove this, we have to show that

$$\{ (u, u') \mid u \in C^1_b(-1,1) \} = \{ (u, u') \mid u \in H^1(-1,1) \},$$

where the closure on the left is in the norm of $L^2(-1,1) \times L^2(-1,1)$.

I. “$\subset$”: Let $(u_n)$ be a sequence in $D(A)$ such that $u_n \to u$, $u'_n \to f$ in $L^2(-1,1)$. By the properties of weak derivatives, this implies that $u$ is weakly differentiable and $u' = f$ (recall!). Therefore $u \in H^1(0,1)$.

II. “$\supset$”: Fix $u \in H^1(-1,1)$ and define $u_n := \phi_n * u$ where $\phi_n$ is a standard mollifier. (Extend $u$ by zero and restrict the convolution to $(-1,1)$.) Then $u_n \to u$ and $u'_n = \phi_n * u' \to u'$ in $L^2(-1,1)$ (check!).

• (Closure of elliptic operators)

Let $\Omega \subset \mathbb{R}^d$ be a bounded smooth domain. Let $X = Y = L^2(\Omega)$ and $(A, D(A))$ be given by

$$D(A) = C^\infty_0(\Omega), \quad Au = \Delta u.$$ 

Its closure $(\overline{A}, D(\overline{A}))$ is given by

$$D(\overline{A}) = H^2_0(\Omega), \quad \overline{A}u = \Delta u$$

(again in the sense of weak derivatives.) To prove this, we have to show that

$$\{(u, \Delta u) \mid u \in C^\infty_0(\Omega)\} = \{(u, \Delta u) \mid u \in H^2_0(\Omega)\}.$$ 

I. “$\subset$”: Let $(u_n)$ be a sequence in $D(A)$ such that $(u_n, Au_n) \to (u, f)$ in $L^2(\Omega) \times L^2(\Omega)$. Using the a priori estimate

$$\|v\|_{H^2(\Omega)} \leq C (\|v\|_{L^2(\Omega)} + \|\Delta v\|_{L^2(\Omega)}) \quad \forall v \in C^\infty_0(\Omega) \quad (1.1)$$

we get

$$\|u_n - u_m\|_{H^2(\Omega)} \leq C (\|u_n - u_m\|_{L^2(\Omega)} + \|\Delta u_n - \Delta u_m\|_{L^2(\Omega)}),$$

i.e. $(u_n)$ is a Cauchy sequence in $H^2(\Omega)$. Therefore, $u \in H^2_0(\Omega)$ and $\Delta u = f$ by continuity.

II. “$\supset$”: Fix $u \in H^2_0(\Omega)$. By definition of this space, there is a sequence $(u_n)$ in $C^\infty_0(\Omega)$ such that $u_n \to u$ in $H^2(\Omega)$. By continuity, $\Delta u_n \to \Delta u$ in $L^2(\Omega)$.

Remark: Analogous results hold for more general elliptic operators of order $2m$, given by

$$Au = \sum_{|\alpha| \leq 2m} a_\alpha \partial^\alpha u$$
satisfying the ellipticity condition
\[ \sum_{|\alpha|=2m} a_\alpha \xi^\alpha \geq c|\xi|^{2m}, \quad \xi \in \mathbb{R}^d \]
for some fixed \( c > 0 \). The essential point is that these assumptions imply an \( H^{2m} \)-estimate parallel to (1.1).

**Remark:** The choice of \( D(A) \) is crucial. For example, if we define \( (A, D(A)) \) by
\[ D(A) = \{ u \in C_0^\infty(\Omega) \mid u|_{\partial\Omega} = 0 \}, \quad Au = \Delta u, \]
we get
\[ D(\overline{A}) = H^2 \cap H^1_0(\Omega). \]

- (Closure of nonelliptic operators)
If a differential operator is not elliptic, the closure will in general be larger than the Sobolev space corresponding to the order of the operator.
Consider, for example, \( X = Y = \mathcal{L}^2((0,1)^2) \) and the operator \( (A, D(A)) \) given by
\[ D(A) = C^1_b((0,1)^2), \quad Au = \partial_1 u + \partial_2 u. \]
Let \( \chi_A \) denote the characteristic function of the set \( A \). Let \( (\phi_n) \) be a sequence of smooth functions on \((-1,1)\) such that \( \phi_n \to \chi_{(0,1)} \). Define \( u_n \in C^1_b((0,1)^2) \) by
\[ u_n(x,y) = \phi_n(x-y). \]
Then \( Au_n = 0 \) for all \( n \) and \( u_n \to \chi_{\{x>y\}} \) in \( \mathcal{L}^2((0,1)^2) \) (check!). Therefore, the characteristic function \( \chi_{\{x>y\}} \) belongs to \( D(\overline{A}) \), but obviously not to \( H^1((0,1)^2) \).

**Definition:** Let \( (A, D(A)) : X \longrightarrow Y \) be a linear operator. The norm \( \| \cdot \|_{D(A)} \) on \( D(A) \) given by
\[ \|x\|_{D(A)} := \|x\|_X + \|Ax\|_Y \]
is called the **graph norm** of \( D(A) \).

**Theorem 1.3** Let \( X \) and \( Y \) be Banach spaces. Then \( (D(A), \| \cdot \|_{D(A)}) \) is a Banach space if and only if \( A \) is closed.

**Proof:** The map \( x \mapsto (x, Ax) \) induces an isomorphism from the normed space \( (D(A), \| \cdot \|_{D(A)}) \) to the normed space \( (G(A), \| \cdot \|_{X \times Y}) \). As \( X \times Y \) is a Banach space, the latter (and therefore also the former) is a Banach space if and only if it is closed, i.e. if \( A \) is closed. \( \blacksquare \)
2 Some basic theorems of linear Functional Analysis

In this section, $B_X(x, r)$ will denote the open ball in the normed space $X$ with center $x \in X$ and radius $r > 0$. If no confusion is likely, the space will not be indicated.

**Theorem 2.1 (Baire)** Let $X$ be a Banach space. Assume $X = \bigcup_{n\in\mathbb{N}} M_n$ where all $M_n$ are closed subsets of $X$. Then one of the $M_n$ contains an interior point.

**Proof:** Assume the opposite:

$$\forall x \in X, \varepsilon > 0, n \in \mathbb{N} : \quad B(x, \varepsilon) \setminus M_n \neq \emptyset. \quad (2.1)$$

Fix $x_0 \in X$, $\varepsilon_0 > 0$ arbitrary. The set $B(x_0, \varepsilon_0) \setminus M_0$ is nonempty by (2.1) and open, so there exist $x_1 \in X$, $\varepsilon_1 \in (0, \varepsilon_0/2)$ such that $B(x_1, 2\varepsilon_1) \subset B(\varepsilon_0) \setminus M_0$. Proceeding by induction and using (2.1) repeatedly, we construct sequences $(x_n)$ in $X$ and $(\varepsilon_n)$ in $\mathbb{R}_+$ such that

$$\varepsilon_{n+1} \in (0, \varepsilon_n/2), \quad B(x_{n+1}, 2\varepsilon_{n+1}) \subset B(x_n, \varepsilon_n) \setminus M_n.$$

Obviously, $\varepsilon_n \to 0$. Moreover, $(x_n)$ is Cauchy. Indeed, for given $\eta > 0$ let $n_0$ large enough to ensure $2\varepsilon_{n_0} < \eta$ and let $n, m \geq n_0$. Then by construction $x_n, x_m \in B(x_{n_0}, \varepsilon_{n_0})$ and thus $\|x_n - x_m\| < 2\varepsilon_{n_0} < \eta$. So $x_n \to x^*$ in $X$. For any $N \in \mathbb{N}$ and any $n \geq N + 1$, we have $x_n \in B(x_{N+1}, \varepsilon_{N+1})$ and therefore

$$x^* \in \overline{B(x_{N+1}, \varepsilon_{N+1})} \subset B(x_{N+1}, 2\varepsilon_{N+1}) \subset B(x_N, \varepsilon_N) \setminus M_N.$$ 

In particular, $x^* \notin M_N$ for all $N \in \mathbb{N}$. This is in contradiction to $X = \bigcup_{n\in\mathbb{N}} M_n$. \hfill ■

**Remark:** It is clear from the proof that Theorem 2.1 holds more generally if $X$ is just a complete metric space. However, completeness is necessary, as simple counterexamples show. (See exercises.)

**Definition:** A mapping $T : X \to Y$ is called **open** if for any open set $U \subset X$, the image $T(U)$ is open in $Y$.

**Lemma 2.2** Let $T : X \to Y$ be linear. Then $T$ is open if and only if $0$ in an interior point of $T(B_X(0, 1))$, i.e.

$$\exists \varepsilon > 0 : \quad B_Y(0, \varepsilon) \subset T(B_X(0, 1)).$$

**Proof:** "⇒": trivial.

"⇐" Fix $U \subset X$ open and $y \in T(U)$. Fix $x \in U$ such that $y = Tx$. As $U$ is open, there is $\delta > 0$ such that $B(x, \delta) \subset U$. Then, by linearity of $T$,

$$B_Y(y, \varepsilon \delta) = y + \delta B_Y(0, \varepsilon) \subset T(x + \delta B_X(0, 1)) = T(B_X(x, \delta)) \subset T(U).$$

\hfill ■
Theorem 2.3 (Open Mapping theorem) Let $X$ and $Y$ be Banach spaces, $T \in \mathcal{L}(X,Y)$. Then $T$ is surjective if and only if $T$ is open.

Proof: 
"$\Leftarrow$": straightforward. (Check!)

"$\Rightarrow$": If $T$ is surjective then

$$ Y = \bigcup_{n \in \mathbb{N}_+} T(B_X(0,n)). $$

By Baire’s theorem 2.1, there exist $n_0 \in \mathbb{N}_+$, $\varepsilon_0 > 0$ and $y_0 \in Y$ such that

$$ B_Y(y_0, \varepsilon_0) \subset T(B_X(0,n_0)). \quad (2.2) $$

Fix $x_0 \in X$ such that $Tx_0 = y_0$.

1. We are going to show that

$$ B_Y(0, \varepsilon) \subset T(B_X(0,1)) \quad (2.3) $$

with $\varepsilon = \frac{\varepsilon_0}{n_0 + \|x_0\|}$. Indeed, by shifting we get from (2.2)

$$ B_Y(0, \varepsilon_0) = B_Y(y_0, \varepsilon_0) - y_0 \subset T(B_X(0,n_0)) - Tx_0 = T(B(-x_0,n_0)) $$

$$ \subset T(B(0, n_0 + \|x_0\|)) $$

and upon scaling with the factor $(n_0 + \|x_0\|)^{-1}$ this yields (2.3).

2. For $\varepsilon$ as defined above, we are going to show

$$ B_Y(0, \varepsilon) \subset T(B(0,3)), \quad (2.4) $$

which implies that $T$ is open by Lemma 2.2.

We fix $y \in B_Y(0, \varepsilon)$ and use (2.3) to successively construct approximate solutions to the equation $Tx = y$.

By (2.3),

$$ \exists x_0 \in B_X(0,1) : \|Tx_0 - y\| < \frac{\varepsilon}{2}, \text{ i.e. } 2(Tx_0 - y) \in B_Y(0, \varepsilon), $$

so

$$ \exists x_1 \in B_X(0,1) : \|Tx_1 - 2(Tx_0 - y)\| < \frac{\varepsilon}{2}, \text{ i.e. } 4(Tx_0 + \frac{1}{2}Tx_1 - y) \in B_Y(0, \varepsilon). $$

Continuing in this way, we obtain a sequence $(x_j)$ in $B_X(0,1)$ such that

$$ \left\| \frac{y - T \left( \sum_{j=0}^{k} 2^{-j} x_j \right) }{y} \right\| \leq 2^{-j-1} \varepsilon. $$

As $X$ is a Banach space, we can define $x = \sum_{j=0}^{\infty} 2^{-j} x_j$ and by continuity of $T$ we get $Tx = y$. Moreover,

$$ \|x\| \leq \sum_{j=0}^{\infty} 2^{-j} = 2 < 3. $$

This shows (2.4).
Corollary 2.4 (Inverse Mapping Theorem) Let $X$ and $Y$ be Banach spaces and $T \in \mathcal{L}(X,Y)$ be bijective. Then $T^{-1} \in \mathcal{L}(Y,X)$.

Proof: As $T$ is surjective, $T$ is open by Theorem 2.3. This is equivalent to the continuity of $T^{-1}$. ■

Corollary 2.5 (Equivalence of norms on Banach spaces) Let $X$ be a vector space equipped with two norms $\| \cdot \|_1$ and $\| \cdot \|_2$, such that $X$ is complete under both norms, and there is a $C > 0$ such that

$$\|x\|_1 \leq C \|x\|_2, \quad x \in X.$$ 

Then there is also a constant $C'$ such that

$$\|x\|_2 \leq C' \|x\|_1, \quad x \in X,$$

i.e. both norms are equivalent.

Proof: Apply Corollary 2.4 to the identity $i : (X, \| \cdot \|_2) \longrightarrow (X, \| \cdot \|_1)$. ■

Corollary 2.6 (Closed-Graph theorem)

Let $X$ and $Y$ be Banach spaces and $A : X \longrightarrow Y$ a linear operator with $D(A) = X$. Then $A$ is bounded if and only if $A$ is closed.

Proof: If $A$ is bounded, then $A$ is continuous, and therefore the graph $G(A)$ is closed. On the other hand, if $G(A)$ is closed, then the space $X = D(A)$ can be given the graph norm corresponding to $A$ which will be denoted by $\| \cdot \|_A$ here. The space $(X, \| \cdot \|_A)$ is a Banach space by Theorem 1.3. Moreover, $\|x\|_X \leq \|x\|_A$ for all $x \in X$. Then the norms $\| \cdot \|_X$ and $\| \cdot \|_A$ are equivalent by Corollary 2.5. In particular, there is a constant $C > 0$ such that $\|Ax\|_Y \leq C \|x\|_X$ for all $x \in X$. ■

Definition: Let $\mathcal{T} \subset \mathcal{L}(X,Y)$ be a family of bounded linear operators. $\mathcal{T}$ is called pointwise bounded if for all $x \in X$ the set $\{Tx \ | T \in \mathcal{T}\}$ is bounded in $Y$.

Theorem 2.7 (Uniform boundedness principle, Banach-Steinhaus theorem)

Let $X$ be a Banach space. A family $\mathcal{T} \subset \mathcal{L}(X,Y)$ is pointwise bounded if and only if it is bounded in the norm of $\mathcal{L}(X,Y)$, i.e. if

$$\exists C > 0 : \forall x \in X, T \in \mathcal{T} : \|Tx\|_Y \leq C \|x\|_X.$$ 

Proof: “$\Leftarrow$”: trivial.

“$\Rightarrow$”: For $k \in \mathbb{N}_+$ we define the closed sets

$$A_k := \bigcap_{T \in \mathcal{T}} \{x \in X | \|Tx\| \leq k\}.$$ 

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If $T$ is pointwise bounded, then $X = \bigcup_{k \in \mathbb{N}_+} A_k$, and by Baire’s theorem 2.1 one of the sets $A_k$ contains an interior point, i.e.

$$\exists x_0 \in X, k_0 \in \mathbb{N}_+, \varepsilon > 0 : \forall z \in X : \|z - x_0\| \leq \varepsilon \Rightarrow (\forall T \in T : \|Tz\| \leq k_0).$$

Fix $x \in X \setminus \{0\}$ and define $z = z(x) = \frac{\varepsilon x}{\|x\|} + x_0$. Then $x = \frac{\|x\|}{\varepsilon} (z - x_0)$, $\|z - x_0\| = \varepsilon$, and therefore for any $T \in T$

$$\|Tx\| = \frac{\|x\|}{\varepsilon} \|Tz - Tx_0\| \leq \frac{\|x\|}{\varepsilon} (\|Tz\| + \|Tx_0\|) \leq \frac{2k_0}{\varepsilon} \|x\|. \quad \Box$$

By contraposition, we get:

**Corollary 2.8 (Resonance theorem)**

Let $X$ be a Banach space and let $T \subset \mathcal{L}(X,Y)$ be unbounded. Then there exist an $x \in X$ and a sequence $(T_n)$ in $T$ such that $\|T_n x\| \to \infty$.

## 3 Basic spectral theory

**Definition:** A linear operator $(T, D(T)) : X \to Y$ is called invertible if $T$ is bijective (between $D(T)$ and $Y$) and its inverse is in $\mathcal{L}(Y,X)$. (Note that the range of the inverse is $D(T)$, but in general not $X$.)

In the remainder of this section, let $X$ be a complex Banach space and $(A, D(A)) : X \to X$ an in general unbounded linear operator.

**Definitions:** The set

$$\rho(A) := \{ \lambda \in \mathbb{C} | \lambda I - A \text{ is invertible} \}$$

is called the **resolvent set** of $A$. The function $R(\cdot, A) : \rho(A) \to \mathcal{L}(X)$ given by

$$R(\lambda, A) := (\lambda I - A)^{-1}, \quad \lambda \in \rho(A),$$

is called the **resolvent** of $A$.

The set

$$\sigma(A) := \mathbb{C} \setminus \rho(A)$$

is called the **spectrum** of $A$.

The set

$$\sigma_p(A) := \{ \lambda \in \mathbb{C} | N(\lambda I - A) \neq \{0\} \}$$

is called the **point spectrum** of $A$. Its elements are the eigenvalues of $A$. Obviously, $\sigma_p(A) \subset \sigma(A)$.

**Lemma 3.1** If $\rho(A)$ is nonempty then $A$ is closed.

**Proof:** Exercise!

This shows that spectrum and resolvent are interesting only for closed operators.
Lemma 3.2 (Properties of the resolvent)

(i) \( \rho(A) \) is an open subset of \( \mathbb{C} \).

(ii) The resolvent \( R(\cdot,A) \) is an analytic \( \mathcal{L}(X) \)-valued function on \( \rho(A) \).

(iii) The resolvent has no analytic extension to any subset of \( \mathbb{C} \) larger than \( \rho(A) \).

**Proof:** Fix \( \lambda_0 \in \rho(A) \), let \( \lambda \in \mathbb{C} \) such that

\[
|\lambda - \lambda_0| < \|R(\lambda_0, A)\|^{-1}_{\mathcal{L}(X)}.
\]

then, by using

\[
\lambda I - A = \lambda_0 I - A + (\lambda - \lambda_0)I
\]

and applying a Neumann series for the calculation of the inverse, we get \( \lambda \in \rho(A) \) and

\[
R(\lambda, A) = \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_0)^n R(\lambda_0, A)^{n+1},
\]

where the operator power series on the right converges absolutely in \( L(X) \).

(Exercise!) This shows (i) and (ii).

More precisely, as (3.1) implies \( \lambda \in \rho(A) \) we have for all \( \lambda \in \sigma(A) \)

\[
|\lambda - \lambda_0| \geq \|R(\lambda_0, A)\|^{-1}_{\mathcal{L}(X)}
\]

and hence

\[
\|R(\lambda_0, A)\| \geq \frac{1}{|\lambda - \lambda_0|}.
\]

Taking the supremum over these \( \lambda \) yields

\[
\|R(\lambda_0, A)\| \geq \frac{1}{\text{dist}(\lambda_0, \sigma(A))} \tag{3.2}
\]

which implies (iii).

\[\Box\]

4 Operator semigroups and generators

Let \( X \) be a Banach space.

**Definition:** A family \( \{T(t) \mid t \geq 0\} \subset \mathcal{L}(X) \) of bounded linear operators on \( X \) such that

\[
T(0) = I, \\
T(s + t) = T(s)T(t) \quad s, t \geq 0
\]

is called a semigroup of (linear) operators on \( X \). This semigroup is called
- **uniformly continuous** if the map \( t \mapsto T(t) \) is continuous from \([0, \infty)\) to \( \mathcal{L}(X) \),

- **strongly continuous** or \( C_0\)-semigroup if for all \( u \in X \) the map \( t \mapsto T(t)u \) is continuous from \([0, \infty)\) to \( X \).

We will also use the terms “strongly continuous” and “uniformly continuous” (and obvious modifications of it) for the operator valued mappings themselves. Clearly, any uniformly continuous semigroup is strongly continuous.

**Examples:**

- Let \( X = L^\infty(\mathbb{R}) \) and define for \( t \geq 0 \)
  \[
  (T(t)u)(x) = u(x + t), \quad u \in L^\infty(\mathbb{R}).
  \]
  Then \( \{T(t)\} \) is a semigroup of linear operators on \( X \) which is not strongly continuous. (Exercise!)

- Let \( X = BUC(\mathbb{R}) \) and define for \( t \geq 0 \)
  \[
  (T(t)u)(x) = u(x + t), \quad u \in BUC(\mathbb{R}). \tag{4.1}
  \]
  Then \( \{T(t)\} \) is a semigroup of linear operators on \( X \) which is strongly but not uniformly continuous. (Exercise!)

- For any complex Banach space \( X \) and any \( \lambda \in \mathbb{C} \), the semigroup \( \{T(t)\} \) of operators on \( X \) given by
  \[
  T(t) = e^{\lambda t}I \tag{4.2}
  \]
  is uniformly continuous.

**Definition:** Let \( \{T(t)\} \) be a strongly continuous semigroup of operators on \( X \). The linear (possibly unbounded) operator \( (A, D(A)) : X \rightarrow X \) given by

\[
D(A) := \left\{ u \in X \mid \lim_{t \downarrow 0} \frac{T(t)u - u}{t} \text{ exists in } X \right\},
\]

\[
Au := \lim_{t \downarrow 0} \frac{T(t)u - u}{t}, \quad u \in D(A),
\]

is called the **generator** of the semigroup \( \{T(t)\} \).

**Examples:**

- Let \( X = BUC(\mathbb{R}) \) and let \( \{T(t)\} \) be given by (4.1). The generator \( (A, D(A)) \) is an unbounded operator that satisfies
  \[
  BUC^1(\mathbb{R}) \subset D(A) \neq X, \quad Au = u', \quad u \in BUC^1(\mathbb{R}).
  \]
  (Check! Later we will see that \( D(A) = BUC^1(\mathbb{R}) \).)
Let $X$ be an arbitrary complex Banach space and let $\{T(t)\}$ be given by (4.2). The generator $(A, D(A))$ is the bounded operator given by

$$D(A) = X, \quad A = \lambda I.$$  

(Check!)

The above definitions give rise to a number of questions:

- Which (bounded or unbounded) linear operators on a Banach space generate a (strongly continuous) semigroup of operators? Is this semigroup unique?

- What properties of the semigroup are implied by properties of its generator, and vice versa?

Answering these questions is a core subject of the theory of operator semigroups.

5 Uniformly continuous semigroups

A particularly simple (in some sense, nearly “trivial”) case in point is given by the property of uniform continuity. This property is in “one-to-one correspondence” with boundedness of the generator.

**Theorem 5.1 (Uniformly continuous semigroups)**

Let $X$ be a Banach space.

(i) Any bounded linear operator $A \in \mathcal{L}(X)$ generates precisely one uniformly continuous semigroup of operators on $X$.

(ii) The generator of any uniformly continuous semigroup of operators on $X$ is bounded and satisfies $D(A) = X$.

**Proof:** 1. Fix $A \in \mathcal{L}(X)$ and define for $t \geq 0$

$$T(t) := \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k. \quad (5.1)$$

Then clearly $T(0) = I$, and the series converges absolutely in $\mathcal{L}(X)$, uniformly as $t$ varies in compact subsets of $\mathbb{R}$. This implies $T(t+s) = T(t)T(s)$, $t \mapsto T(t)$ is continuous from $[0, \infty)$ to $\mathcal{L}(X)$, and

$$\lim_{t \downarrow 0} \frac{T(t) - I}{t} = A \quad \text{in} \ \mathcal{L}(X)$$

Therefore, $\{T(t)\}$ is a uniformly continuous semigroup on $X$ generated by $A$.

2. Let $\{T(t)\}$ be a uniformly continuous semigroup on $X$. Then, for sufficiently small $\rho > 0$, the mean value $\frac{1}{\rho} \int_0^{\rho} T(s) \, ds$ is close to $I$ and therefore
invertible, hence \( \int_0^\rho T(s) \, ds \) is invertible as well. So, for any such \( \rho \), the operator

\[
A := (T(\rho) - I) \left( \int_0^\rho T(s) \, ds \right)^{-1}
\]

is a well-defined element of \( \mathcal{L}(X) \). We are going to show that

\[
A = \lim_{t \downarrow 0} \frac{T(t) - I}{t},
\]

so \( A \) generates \( \{T(t)\} \) (and, in particular, is independent of \( \rho \)).

Indeed, for \( t \in (0, \rho) \) we have

\[
\frac{1}{t} (T(t) - I) \int_0^\rho T(s) \, ds = \frac{1}{t} \left( \int_0^\rho T(s + t) \, ds - \int_0^\rho T(s) \, ds \right)
= \frac{1}{t} \left( \int_0^\rho T(s) \, ds - \int_0^\rho T(s) \, ds \right) = \frac{1}{t} \left( \int_0^\rho T(s) \, ds - \int_0^t T(s) \, ds \right),
\]
so

\[
\frac{1}{t} (T(t) - I) = \frac{1}{t} \left( \int_0^\rho T(s) \, ds - \int_0^t T(s) \, ds \right) \left( \int_0^\rho T(s) \, ds \right)^{-1} \rightarrow A.
\]

This proves (5.2).

3. Let \( \{T(t)\}, \{S(t)\} \) be uniformly continuous semigroups on \( X \), both generated by \( A \in \mathcal{L}(X) \). Then, by 2.,

\[
\lim_{t \downarrow 0} \frac{T(t) - I}{t} = \lim_{t \downarrow 0} \frac{S(t) - I}{t} = A.
\]

Fix \( \Theta > 0, \varepsilon > 0 \) arbitrary. By continuity and compactness, there is a \( C > 0 \) such that

\[
\|T(t)\|, \|S(t)\| \leq C, \quad t \in [0, \Theta],
\]
and by (5.3) there is a \( \delta > 0 \) such that for all \( h \in (0, \delta) \)

\[
\frac{\|T(h) - S(h)\|}{h} < \frac{\varepsilon}{C^2 \Theta}.
\]

Fix \( t \in [0, \Theta] \). Choose \( n \in \mathbb{N} \) large enough such that \( t/n < \delta \). Then

\[
\|T(t) - S(t)\| = \|T(t/n)^n - S(t/n)^n\|
\leq \sum_{k=0}^{n-1} \|T(t/n)^{n-k-1} (T(t/n) - S(t/n)) S(t/n) \|^{k+1} \leq \frac{1}{(n-k-1)t/n} \leq C^2 \|T(t/n) - S(t/n)\| \leq nC^2 \|T(t/n) - S(t/n)\| \leq C^2 \frac{\varepsilon t/n}{\Theta} \leq \varepsilon.
\]

Therefore \( T(t) = S(t) \) for \( t \in [0, \Theta] \) and \( \{T(t)\} = \{S(t)\} \) as \( \Theta \) was arbitrary. 

Remarks:
For $A \in \mathcal{L}(X)$, the semigroup given by (5.1) is called the (operator) exponential function of $A$ and is also denoted by $e^{tA}$.

This exponential function extends analytically (with values in $\mathcal{L}(X)$) to the whole complex plane. In particular, the map $t \mapsto T(t)$ is analytic from $\mathbb{R}$ to $\mathcal{L}(X)$ and has the property $T(s + t) = T(t)T(s)$ for all $s, t \in \mathbb{R}$, so the family $\{T(t) \mid t \in \mathbb{R}\}$ is called an (analytic) group of operators on $X$. It satisfies

$$AT(t) = T(t)A, \quad \frac{d^n}{dt^n}T(t) = A^nT(t), \quad t \in \mathbb{R}, \ n \in \mathbb{N},$$

and the bound

$$\|T(t)\| \leq \sum_{k=0}^{\infty} \frac{1}{k!} |t|^k \|A\|^k = e^{\|A\| |t|}.$$

Theorem (5.1) can be straightforwardly generalized to the case where $\mathcal{L}(X)$ is replaced by an arbitrary Banach algebra.

The uniqueness of the semigroup generated by $A$ will later be reproved in the more general context of strongly continuous semigroups. The proof given above is included for self-containment and because the proof for strongly continuous semigroups does not generalize to the Banach-algebra case.

6 Basic properties of strongly continuous semigroups

Let $X$ again be a Banach space. We return to the general situation and recall the definition of strong continuity from Section 4. Let $\{T(t)\}$ be a semigroup of operators on $X$. We will investigate the following properties:

(a) $t \mapsto T(t)$ is strongly continuous,

(b') $t \mapsto T(t)$ is strongly right continuous,

(b) $t \mapsto T(t)$ is strongly right continuous at $t = 0$,

(c1') for all $\Theta > 0$ there is an $M_\Theta > 0$ such that $\|T(t)\| \leq M_\Theta$ for all $t \in [0, \Theta]$,

(c1) there are $\delta > 0, \ M > 0$ such that $\|T(t)\| \leq M$ for all $t \in [0, \delta]$,

(c2) there is a dense subset $D \subset X$ such that $T(t)x \xrightarrow{t\downarrow 0} x$ for all $x \in D$.

Lemma 6.1 (Criteria for strong continuity)

For any operator semigroup $\{T(t)\}$ we have

\[(a) \iff (b) \iff (b') \iff ((c1) \land (c2)).\]
Proof: (a)⇒(b')⇒(b)⇒(c2): trivial.
(c1')⇒(c1): trivial.
(c1)⇒(c1'): see exercises!
(b)⇒(b'): Fix $t_0 > 0$, $x \in X$, and let $h > 0$. Then
\[
T(t_0 + h)x = T(h)T(t_0)x \xrightarrow{h \downarrow 0} T(t_0)x.
\]
(b)⇒(c1): Assume (c1) does not hold: Then there is a sequence $(t_n)$ of positive numbers such that $t_n \downarrow 0$ and $\|T(t_n)\| \to \infty$. By Corollary 2.8, this implies that there is an $x \in X$ such that the set $\{T(t_n)x \mid n \in \mathbb{N}\}$ is unbounded. This is in contradiction to (b).

((c1)∧(c2))⇒(b): Fix $x \in X$, $\varepsilon > 0$. Choose $\tilde{x} \in D$ such that $\|\tilde{x} - x\| < \varepsilon/(2(M+1))$. Choose $\eta \in (0, \delta)$ such that $\|T(t)\tilde{x} - \tilde{x}\| < \varepsilon/2$ for all $t \in [0, \eta]$. For $t \in [0, \eta]$ we have then
\[
\|T(t)x - x\| \leq \underbrace{\|T(t)(x - \tilde{x})\|}_{< \frac{M}{2(M+1)}} + \underbrace{\|T(t)\tilde{x} - \tilde{x}\|}_{< \frac{\varepsilon}{2}} + \|\tilde{x} - x\| < \varepsilon.
\]
This implies (b).

((b')∧(c1'))⇒(a): We have to show that $t \mapsto T(t)$ is strongly left continuous on $(0, \infty)$. Fix $t_0 > 0$, $x \in X$, and let $h \in (0, t_0)$. Then
\[
\|T(t_0 - h)x - T(t_0)x\| \leq \underbrace{\|T(t_0 - h)\|}_{< M_{t_0}} \underbrace{\|x - T(h)x\|}_{\to 0} \to 0 \\
\leq M_{t_0} \to 0
\]
Gathering the above implications, we finally get
\[
((c1)∧(c2)) \Rightarrow ((b)∧(c1')) \Rightarrow ((b')∧(c1')) \Rightarrow (a).
\]
This completes the proof.

Theorem 6.2 (Basic properties)
Let $\{T(t)\}$ be a strongly continuous semigroup of operators on $X$ with generator $(A, D(A))$. Then:

(i) There are $M, \omega > 0$ such that
\[
\|T(t)\| \leq Me^{\omega t}, \quad t \geq 0.
\]

(ii) For all $x \in X$, $t > 0$ we have
\[
\int_0^t T(s)x \, ds \in D(A), \quad A \int_0^t T(s)x \, ds = T(t)x - x.
\]

(iii) For all $x \in D(A)$, $t \geq 0$ we have
\[
T(t)x \in D(A), \quad AT(t)x = T(t)Ax. \quad (6.1)
\]
Moreover, the mapping \( t \mapsto T(t)x \) is continuously differentiable from \([0, \infty)\) to \(X\), and
\[
\frac{d}{dt} T(t)x = AT(t)x.
\] (6.2)

Caution: In general, (6.1) and (6.2) are not true for all \( x \in X \!\!\!\).

**Proof of Theorem 6.2:**

(i): See exercise!

(ii): Fix \( x \in X \), \( t > 0 \), and let \( h > 0 \). Then
\[
\frac{T(h) - I}{h} \int_0^t T(s)x\,ds = \frac{1}{h} \int_0^t (T(s + h)x - T(s)x)\,ds
\]
\[
= \frac{1}{h} \int_t^{t+h} T(s)x\,ds - \frac{1}{h} \int_0^h T(s)x\,ds \overset{h \downarrow 0}{\rightarrow} T(t)x - x
\]
as the integrand is continuous with values in \(X\).

(iii): Fix \( x \in D(A) \), \( t \geq 0 \), and let \( h > 0 \). Then
\[
\frac{T(h) - I}{h} T(t)x = \frac{T(t + h)x - T(t)x}{h}
\]
by continuity of \( T(t) \). This implies (6.1) and the differentiability of \( t \mapsto T(t)x \) from the right. To show left differentiability and (6.2), fix \( t > 0 \) and let \( h \in (0, t) \). Then
\[
\frac{T(t)x - T(t-h)x}{h} - T(t)Ax
\]
\[
= T(t-h) \left( \frac{T(h)x - x}{h} - Ax \right) + T(t-h)Ax - T(t)Ax \overset{h \downarrow 0}{\rightarrow} 0,
\]
where we used that the operators \( T(t-h) \) are bounded uniformly in \( h \) by (i) and that the map \( t \mapsto T(t)Ax \) is (right) continuous at \( t \). Finally, by (6.1) and (6.2) we have
\[
\frac{d}{dt} T(t)x = T(t)Ax,
\]
so the derivative is continuous in \( t \). \( \blacksquare \)

**Theorem 6.3 (Properties of generators)**

Let \((A, D(A))\) be the generator of a strongly continuous semigroup \( \{T(t)\} \) on \(X\). Then \(D(A)\) is dense in \(X\), and \(A\) is a closed operator on \(X\).

**Proof:** For any \( x \in X \) we have
\[
\frac{1}{t} \int_0^t T(s)x\,ds \overset{t \downarrow 0}{\rightarrow} x
\]
and \( \int_0^t T(s)x\,ds \in D(A) \) for all \( t > 0 \) by Theorem 6.2 (ii). This proves that \(D(A)\) is dense in \(X\).
To prove closedness, fix a sequence \( (x_n) \) in \( D(A) \) such that \( x_n \to x \), \( Ax_n \to y \) in \( X \). Let \( t > 0 \). By Theorem 6.2 (iii), the mapping

\[ s \mapsto \phi(s) := T(s)x_n \]

is continuously differentiable on \([0, t]\), and \( \phi'(s) = T(s)Ax_n \). Integration from 0 to \( t \) yields

\[ T(t)x_n - x_n = \int_0^t T(s)Ax_n \, ds. \tag{6.3} \]

As the operators \( T(s) \), \( s \in [0, t] \) are uniformly bounded in \( s \), we have \( T(s)Ax_n \to T(s)y \) uniformly in \( s \). Therefore, taking \( n \to \infty \) in (6.3),

\[ T(t)x - x = \int_0^t T(s)y \, ds \]

and hence

\[ \frac{T(t)x - x}{t} \to \frac{1}{t} \int_0^t T(s)y \, ds \xrightarrow{t \downarrow 0} y. \]

Consequently, \( x \in D(A) \) and \( Ax = y \). This proves that \( A \) is closed. \( \blacksquare \)

**Theorem 6.4** (Uniqueness)

Let \( \{T(t)\} \), \( \{S(t)\} \) be two strongly continuous semigroups of operators on \( X \) having the same generator. Then \( T(t) = S(t) \) for all \( t \geq 0 \).

**Proof:** See exercises!

**Definition:** For a given linear operator \( (A, D(A)) : X \to X \) we define inductively its powers \( (A^n, D(A^n)) : X \to X \) by

\[ D(A^n) := \{ x \in D(A^{n-1}) | Ax \in D(A^{n-1}) \}, \quad A^n x = AA^{n-1} x, \quad x \in D(A^n) \]

\( (n \geq 2) \). Moreover, let

\[ D(A^\infty) := \bigcap_{n=1}^{\infty} D(A^n). \tag{6.4} \]

(Note that there is no definition of any operator named \( A^\infty \)!)

**Lemma 6.5** Let \( (A, D(A)) \) be the generator of a strongly continuous semigroup of operators \( \{T(t)\} \). Then even \( D(A^\infty) \) is dense in \( X \).

**Proof:** For \( x \in X \) and \( \phi \in C_0^\infty((0, \infty)) \), let

\[ x(\phi) := \int_0^{\infty} \phi(s)T(s)x \, ds \]

and

\[ Y := \{ x(\phi) | x \in X, \phi \in C_0^\infty((0, \infty)) \}. \]

Obviously, \( Y \) is a linear subspace of \( X \).
1. We show $Y \subset D(A^\infty)$. Fix $x \in X$, $\phi \in C_0^\infty((0, \infty))$, and let $h > 0$ be small. Then
\[
\frac{T(h) - I}{h} x(\phi) = \frac{1}{h} \int_0^\infty \phi(s) (T(s + h) - T(s)) x \, ds
\]
\[
= - \int_0^\infty \frac{\phi(s) - \phi(s-h)}{h} T(s) x \, ds \xrightarrow{h \downarrow 0} - \int_0^\infty \phi'(s) T(s) x \, ds,
\]
where we used that the support of $\phi$ is away from zero and $\frac{\phi(s) - \phi(s-h)}{h} \to \phi'(s)$ uniformly for $s \in (0, \infty)$ by compactness of the support of $\phi$.

Hence $x(\phi) \in D(A)$ with $Ax(\phi) = -x(\phi')$. Induction shows then $x\phi \in D(A^n)$ and $A^n x(\phi) = (-1)^n x(\phi^{(n)})$.

2. We show that $Y$ is dense in $X$. Suppose the opposite. Then $Y \neq X$, and the Hahn-Banach theorem\footnote{A proof of this theorem is not part of these lectures. The reader is encouraged to consult any book on basic linear functional analysis.} implies the existence of a nonzero bounded linear functional $x^* \in X^* \setminus \{0\}$ such that
\[
\langle x^*, x(\phi) \rangle = \int_0^\infty \phi(s) \langle x^*, T(s) x \rangle \, ds
\]
for all $\phi \in C_0^\infty((0, \infty))$ and $x \in X$. This implies, by continuity of $s \mapsto \langle x^*, T(s) x \rangle$ that
\[
\langle x^*, T(s) x \rangle = 0 \quad \text{for all } s \geq 0
\]
and in particular $\langle x^*, x \rangle = 0$ for all $x \in X$, in contradiction to $x^* \neq 0$. \hfill \qed

7 The Hille-Yosida theorem for contractions

Let $X$ be a complex Banach space. Here and in the sequel we will write $\mathbb{R}_+ = (0, \infty)$.

**Definition:** A semigroup $\{T(t)\}$ of operators on $X$ is called a contraction semigroup\footnote{Note the different usage of the term “contraction” in contrast to the Banach contraction principle!} if $\|T(t)\| \leq 1$ for all $t \geq 0$.

**Theorem 7.1 (Hille-Yosida)**

A linear operator $(A, D(A))$ on $X$ is the generator of a strongly continuous contraction semigroup $\{T(t)\}$ if and only if

(i) $A$ is closed and densely defined,

(ii) $\mathbb{R}_+ \subset \rho(A)$, and
\[
\|R(\lambda, A)\| \leq \frac{1}{\lambda}, \quad \lambda > 0.
\]
Proof: “\(\Rightarrow\)”: Statement (i) follows by Theorem 6.3. To show (ii), we fix \(\lambda > 0\) and define \(R(\lambda) \in \mathcal{L}(X)\) by

\[
R(\lambda)x := \int_0^\infty e^{-\lambda t}T(t)x dt.
\]

The assumption that \(\{T(t)\}\) is a strongly continuous contraction semigroup implies that \(R(\lambda)\) is well defined and that

\[
\|R(\lambda)\| \leq \int_0^\infty e^{-\lambda t} dt = \frac{1}{\lambda}.
\]

We will show that \(R(\lambda)\) is in fact the resolvent operator \(R(\lambda, A)\). For this, we have to show that

\[
\forall x \in X: R(\lambda)x \in D(A), \quad AR(\lambda)x = \lambda R(\lambda)x - x, \quad (7.1)
\]

and

\[
\forall x \in D(A): R(\lambda)Ax = AR(\lambda)x. \quad (7.2)
\]

(Check!)

To show (7.1), fix \(x \in X\) and let \(h > 0\). Then

\[
\frac{T(h) - I}{h} R(\lambda)x = \frac{1}{h} \int_0^\infty e^{-\lambda t}(T(t + h) - T(t))x dt = \frac{e^{\lambda h} - 1}{h} \int_h^\infty e^{-\lambda t} T(t)x dt - \frac{1}{h} \int_0^\infty e^{-\lambda t} T(t)x dt = \frac{e^{\lambda h} - 1}{h} \int_h^\infty e^{-\lambda t} T(t)x dt \xrightarrow{\lambda \downarrow 0} \lambda R(\lambda)x - x.
\]

To show (7.2), fix \(x \in D(A)\). Then

\[
R(\lambda)Ax = \int_0^\infty e^{-\lambda t} T(t)Ax dt = \int_0^\infty e^{-\lambda t} AT(t)x dt = A \int_0^\infty e^{-\lambda t} T(t)x dt = AR(\lambda)x.
\]

Observe that the closedness of \(A\) is used in the third equality.

“\(\Longleftrightarrow\)”: To shorten notation, we write \(R(\lambda)\) instead of \(R(\lambda, A)\).

1. We show

\[
\forall x \in X: \lim_{\lambda \to \infty} \lambda R(\lambda)x = x. \quad (7.3)
\]

Assume first \(x \in D(A)\). Then

\[
\|\lambda R(\lambda)x - x\| = \|AR(\lambda)x\| = \|R(\lambda)Ax\| \leq \frac{1}{\lambda} \|Ax\| \xrightarrow{\lambda \to \infty} 0.
\]
In the general case, fix \( x \in X \) and \( \varepsilon > 0 \). As \( D(A) \) is dense, there is a \( \tilde{x} \in D(A) \) such that \( \| x - \tilde{x} \| < \varepsilon / 3 \). For large \( \lambda \) we have \( \| \lambda R(\lambda)\tilde{x} - \tilde{x} \| < \varepsilon / 3 \) and thus

\[
\| \lambda R(\lambda)x - x \| \leq \| \lambda R(\lambda)(x - \tilde{x}) \| + \| x - \tilde{x} \| + \| \lambda R(\lambda)\tilde{x} - \tilde{x} \| < \varepsilon.
\]

2. For \( \lambda > 0 \) we define the so-called Yosida approximation

\[
A_\lambda := \lambda AR(\lambda) = \lambda(\lambda R(\lambda) - I).
\]

Note that \( A_\lambda \in \mathcal{L}(X) \) for all \( \lambda > 0 \), and by (7.3), for all \( x \in D(A) \),

\[
A_\lambda x = \lambda AR(\lambda)x = \lambda R(\lambda)Ax \xrightarrow{\lambda \to \infty} Ax.
\] (7.4)

The semigroups \( \{ e^{tA_\lambda} \} \) generate by \( A_\lambda \) are uniformly continuous, and they are contraction semigroups because

\[
\| e^{tA_\lambda} \| = \left\| e^{(\lambda^2 R(\lambda) - \lambda I)} \right\| \leq e^{-t\lambda} \left\| e^{\lambda^2 R(\lambda)} \right\| \leq e^{-t\lambda} e^{t\lambda^2 \| R(\lambda) \|} \leq 1.
\] (7.5)

Furthermore, we can estimate the dependence of \( e^{tA_\lambda}x \) on \( \lambda \) by

\[
\| e^{tA_\lambda}x - e^{tA_\mu}x \| = \left\| \int_0^t \frac{d}{ds} \left[ e^{sA_\lambda} e^{t(1-s)A_\mu} x \right] ds \right\|
\leq t \int_0^1 \left\| e^{sA_\lambda} e^{t(1-s)A_\mu} (A_\lambda - A_\mu) x \right\| ds \leq t \| (A_\lambda - A_\mu) x \|
\]

for \( t \geq 0, x \in X, \lambda, \mu > 0 \). In particular, for \( x \in D(A) \), by (7.4),

\[
\| e^{tA_\lambda}x - e^{tA_\mu}x \| \leq t \| (A_\lambda - A_\mu) x \| \xrightarrow{\lambda,\mu \to \infty} 0.
\]

3. This implies that the limit

\[
T(t)x := \lim_{\lambda \to \infty} e^{tA_\lambda}x
\] (7.6)

exists for all \( t \geq 0 \) and \( x \in D(A) \). (Check!) The operator \( T(t) \) defined by this is clearly linear and bounded with \( \| T(t) \| \leq 1 \) because of (7.5), and therefore it extends by density with the same estimate to all \( x \in X \). We straightforwardly have the semigroup properties \( T(0) = I \) and \( T(s + t) = T(t)T(s) \). Finally, as the convergence in (7.6) is uniform with respect to \( t \) for \( t \) varying in bounded sets, we conclude that \( t \mapsto T(t)x \) is continuous. (Check!)

4. Let \((B, D(B))\) the generator of the strongly continuous semigroup of contractions \( \{ T(t) \} \). We have to show \( B = A \). By assumption and from the first part of the proof we get \( \mathbb{R}_+ \subset \rho(A) \cap \rho(B) \). Therefore it is sufficient to show \( A \subset B \) (see exercise!). To see this, fix \( x \in D(A) \). Then

\[
T(t)x - x = \lim_{\lambda \to \infty} e^{tA_\lambda}x - x = \lim_{\lambda \to \infty} \int_0^t e^{sA_\lambda}A_\lambda x ds = \int_0^t T(s)Ax ds,
\]

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where the last equality follows from the fact that \(e^{sA} A_\lambda x \xrightarrow{\lambda \to \infty} T(s)A x\) uniformly in \(s \in [0, t]\). (Check!) Consequently,

\[
\frac{T(t)x - x}{t} = \frac{1}{t} \int_0^t T(s)Ax \, ds \xrightarrow{t \downarrow 0} Ax,
\]

so \(x \in D(B)\) and \(Bx = Ax\).

\[\text{Remarks:}\]

- The first part of the proof can be easily modified to show that if \(A\) generates a contraction semigroup \(\{T(t)\}\), then the right complex half plane \(C_+ := \{\lambda \in \mathbb{C} \mid \text{Re} \lambda > 0\}\) belongs to \(\rho(A)\), and we have

\[
R(\lambda, A)x = \int_0^\infty e^{-\lambda t}T(t)x \, dt, \quad x \in X, \lambda \in C_+, \quad (7.7)
\]

with an estimate

\[
\|R(\lambda, A)\| \leq \frac{1}{\text{Re} \lambda}.
\]

Note, furthermore, that \(\lambda \mapsto R(\lambda, A)x\) is in fact the (vector valued) Laplace transform of \(t \mapsto T(t)x\).

- The Hille-Yosida theorem and the above remark can be straightforwardly generalized to give a characterization of generators of so-called quasicontraction semigroups, i.e. semigroups that satisfy \(\|T(t)\| \leq e^{\omega t}\) for some \(\omega \in \mathbb{R}\). (See exercises.)

## 8 Dissipative operators and the Lumer-Phillips theorem

Let \(X\) be a complex Banach space.

**Definition:** For \(x \in X\), we call

\[
F(x) := \{x^* \in X^* \mid \langle x^*, x \rangle = \|x^*\|^2 = \|x\|^2\}
\]

the **duality set** of \(x\).

**Remarks:**

- The Hahn-Banach theorem implies \(F(x) \neq \emptyset\) for all \(x \in X\).

- If \(X\) is a Hilbert space, then \(F(x) = \{i(x)\}\), where \(i\) is the Riesz antiisomorphism. (Check!)

- In general, \(F(x)\) is not necessarily a singleton. (Example?)
Definition: A linear operator \((A, D(A))\) on \(X\) is called **dissipative** if
\[
\forall x \in D(A) : \exists x^* \in F(x) \quad \text{Re}\langle x^*, Ax \rangle \leq 0.
\]

In the case of a Hilbert space with inner product \((\cdot, \cdot)\) this just means
\[
\text{Re}(Ax, x) \leq 0, \quad x \in D(A).
\]

This shows the close relationship of dissipativity to energy estimates of the corresponding linear evolution equation \(\dot{x} = Ax\).

**Theorem 8.1** A linear operator \((A, D(A))\) on \(X\) is dissipative if and only if
\[
\|\lambda x - Ax\| \geq \lambda \|x\|, \quad \text{for all } \lambda > 0, \ x \in D(A).
\]

**Proof:** We only show \(\Rightarrow\) here. Let \(A\) be dissipative. Fix \(\lambda > 0, \ x \in D(A)\), and \(x^* \in F(x)\) such that \(\text{Re}\langle x^*, Ax \rangle \leq 0\). Then
\[
\|\lambda x - Ax\| \|x\| \geq |\langle x^*, \lambda x - Ax \rangle| \geq \text{Re}\langle x^*, \lambda x \rangle - \text{Re}\langle x^*, Ax \rangle \geq \lambda \|x\|^2.
\]
This implies the result. \(\blacksquare\)

**Corollary 8.2** Let \((A, D(A))\) be dissipative and assume there is a \(\lambda_0 > 0\) such that \(\lambda_0 I - A\) is surjective. Then \(\mathbb{R}^+ \subset \rho(A)\), and
\[
\|R(\lambda, A)\| \leq \frac{1}{\lambda}, \quad \lambda > 0.
\]

**Proof:** Let \(S := \mathbb{R}^+ \cap \rho(A)\). Then \(S \neq \emptyset\) by assumption and \(S\) is open. By Theorem 8.1, we have \(R(\lambda, A) \geq 1/\lambda\) for all \(\lambda \in S\). Therefore (cf. (3.2)) \(\text{dist}(\lambda, \sigma(A)) \geq \lambda\) for these \(\lambda\), which implies \(S = \mathbb{R}^+\). \(\blacksquare\)

**Theorem 8.3 (Lumer-Phillips)**

Let \((A, D(A))\) be a densely defined linear operator on \(X\).

(i) If \(A\) is dissipative and there is a \(\lambda_0 > 0\) such that \(\lambda_0 I - A\) is surjective then \(A\) generates a strongly continuous semigroup of contractions.

(ii) If \(A\) generates a strongly continuous semigroup of contractions \(\{T(t)\}\) then
\[
\forall x \in D(A) : \forall x^* \in F(x) : \ \text{Re}\langle x^*, Ax \rangle \leq 0. \quad (8.1)
\]

(Recall that \(F(x) \neq \emptyset\), hence (8.1) implies that \(A\) is dissipative.)

**Proof:** (i) follows by Corollary 8.2 and the Hille-Yosida theorem.

(ii): Fix \(x \in D(A)\) and \(x^* \in F(x)\). Let \(t > 0\). Then
\[
\text{Re}\langle x^*, T(t)x - x \rangle = \text{Re}\langle x^*, T(t)x \rangle - \|x\|^2 \leq |\langle x^*, T(t)x \rangle| - \|x\|^2 \leq 0.
\]
The result follows by dividing by \(t\) and taking the limit \(t \downarrow 0\). \(\blacksquare\)
9 The Hille-Yosida theorem for bounded semigroups

**Definition:** A semigroup of operators \( \{T(t)\} \) on the Banach space \( X \) is called **bounded** if there is an \( M > 0 \) such that \( \|T(t)\| \leq M \) for all \( t \geq 0 \).

Observe that if \( A \) generates a strongly continuous semigroup of operators then there is an \( \omega > 0 \) such that \( A - \omega I \) generates a bounded strongly continuous semigroup. (Check!) Hence, up to a spectral shift, a characterization of generators of bounded strongly continuous semigroups covers the general case. The main result of this section gives such a characterization, although its direct practical applicability is limited.

The proof rests on the idea of introducing equivalent norms with respect to which given bounded operators have a norm not exceeding 1.

**Lemma 9.1** Let \( (A, D(A)) \) be a linear operator on \( X \) satisfying \( R_+ \in \rho(A) \) and 
\[ \exists M > 0: \quad \forall n \in \mathbb{N}_+, \lambda > 0: \quad \|\lambda^n R(\lambda, A)^n\| \leq M. \]

Then there is an equivalent norm \( \|\cdot\| \) on \( X \) such that 
\[ \|\lambda R(\lambda, A)x\| \leq \|x\|, \quad \|x\| \leq \|\cdot\| \leq M \|x\| \]
for all \( x \in X, \lambda > 0 \).

**Proof:** For \( \mu > 0 \) we define the norm \( \|\cdot\|_{\mu} \) on \( x \) by
\[ \|x\|_{\mu} := \sup_{n \geq 0} \|\mu^n R(\mu, A)^n x\|_{\mu}. \]

Then, for all \( x \in X \), by assumption,
\[ \|x\| \leq \|x\|_{\mu} \leq M \|x\| \quad (9.1) \]
and by definition of \( \|\cdot\|_{\mu} \)
\[ \|\mu R(\mu, A)x\|_{\mu} \leq \|x\|_{\mu}. \quad (9.2) \]

Fix \( \lambda \in (0, \mu] \) and \( x \in X \). Define \( y := R(\lambda, A)x = R(\mu, A)(x + (\mu - \lambda)y) \). Then, by (9.2),
\[ \|y\|_{\mu} \leq \frac{1}{\mu} \|x + (\mu - \lambda)y\|_{\mu} \leq \frac{1}{\mu} \|x\|_{\mu} + \left(1 - \frac{\lambda}{\mu}\right) \|y\|_{\mu}. \]

Subtracting the second term on the right and multiplication by \( \mu \) yields \( \lambda \|y\|_{\mu} \leq \|x\|_{\mu} \), or
\[ \|\lambda R(\lambda, A)\|_{\mu} \leq \|x\|_{\mu}, \quad 0 < \lambda \leq \mu. \quad (9.3) \]

Define now
\[ \|x\| := \lim_{\mu \to \infty} \|x\|_{\mu}. \]

The lemma follows by taking the upper limit \( \mu \to \infty \) in (9.1) and (9.3).
Theorem 9.2 (Hille-Yosida)
A linear operator \((A, D(A))\) on \(X\) is the generator of a strongly continuous semigroup \(\{T(t)\}\) that satisfies \(\|T(t)\| \leq M\) for all \(t \geq 0\) if and only if

(i) \(A\) is closed and densely defined,

(ii) \(\mathbb{R}_+ \subset \rho(A)\), and

\[
\|R(\lambda, A)^n\| \leq \frac{M}{\lambda^n}, \quad n \in \mathbb{N}, \lambda > 0.
\]

Proof: “\(\Rightarrow\)”: Define on \(X\) the norm \(\|\cdot\|\) by

\[
\|x\| := \sup_{s \geq 0} \|T(s)x\|.
\]

This norm is obviously equivalent to \(\|\cdot\|\), so \(\{T(t)\}\) is a strongly continuous semigroup of operators on \((X, \|\cdot\|)\), and it is even a semigroup of contractions on this space, as

\[
\|T(t)x\| = \sup_{s \geq 0} \|T(t+s)x\| = \sup_{s \geq t} \|T(s)x\| \leq \|x\|.
\]

The generator of the semigroup and its topological properties (closedness, denseness of \(D(A)\), spectrum) do not depend on the norm. It remains to show the resolvent estimate. We will denote the operator norm of \(L(X)\) corresponding to \(\|\cdot\|\) by that symbol as well. Fix \(\lambda > 0, n \in \mathbb{N}\), and \(x \in X\). Theorem 7.1 yields that \(\|\lambda^n R(\lambda, A)^n\| \leq 1\), hence \(\|\lambda^n R(\lambda, A)^n x\| \leq \|x\|\).

\[
\|\lambda^n R(\lambda, A)^n x\| \leq \|\lambda^n R(\lambda, A)^n x\| \leq \|x\| \leq M\|x\|.
\]

“\(\Leftarrow\)”: Let \(\|\cdot\|\) be the norm from Lemma 9.1. By Theorem 7.1, \(A\) generates a strongly continuous semigroup of contractions \(\{T(t)\}\) on \((X, \|\cdot\|)\). By equivalence of norms, \(\{T(t)\}\) is also strongly continuous (and also generated by \(A\)) on \((X, \|\cdot\|)\). Moreover, for \(t \geq 0, x \in X\),

\[
\|T(t)x\| \leq \|T(t)x\| \leq \|x\| \leq M\|x\|.
\]

Remark: It is not hard to see that the first part of the proof can be generalized to yield \(\lambda \in \rho(A)\) for all complex \(\lambda\) satisfying \(\text{Re}\lambda > 0\) and for these \(\lambda\),

\[
\|R(\lambda, A)\| \leq \frac{M}{\text{Re}\lambda}.
\]

Moreover, inspection of the arguments that prove the representation formula (7.7) show that it remains valid if for (generators of) bounded semigroups.

In view of Theorem 6.2(i) and the remarks after Theorem 7.1 we can give now the following characterization of generators of strongly continuous semigroups:
Corollary 9.3 A linear operator \((A, D(A))\) on a Banach space \(X\) generates a strongly continuous semigroup of operators if and only if \(A\) is closed and densely defined and there are constants \(M > 0\) and \(\omega \in \mathbb{R}\) such that \((\omega, \infty) \in \rho(A)\) and

\[
\forall n \in \mathbb{N}, \lambda > 0: \quad \|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n}.
\]

In this case, we even have \(\lambda \in \rho(A)\) and

\[
\forall n \in \mathbb{N}: \quad \|R(\lambda, A)^n\| \leq \frac{M}{(\text{Re}\lambda - \omega)^n}
\]

for all complex \(\lambda\) that satisfy \(\text{Re}\lambda > \omega\).

The semigroup \(\{T(t)\}\) generated by \(A\) satisfies

\[
\|T(t)\| \leq Me^{\omega t}, \quad t \geq 0.
\]

10 Inverse Laplace transform

As we have seen above, the resolvent of the generator can (in a certain sense) be obtained as Laplace transform of the semigroup. (See the first part of the proof of Theorem 7.1. The resolvent representation given there generalizes straightforwardly to bounded semigroups.) Therefore, it is natural to construct the semigroup by “inverting the Laplace transform”. This corresponds to a classical technique of solving initial value problems for linear evolution equations.

We start with an informal discussion of the idea behind the inversion of the Laplace transform.

Let \(\{T(t)\}\) be a strongly continuous semigroup of operators on the Banach space \(X\) which we assume (without loss of generality) to be bounded. For \(f \in L^1(\mathbb{R}_+)\) we define \(T[f] \in L(X)\) by

\[
(T[f])x := \int_0^\infty f(s)T(s)x \, dx, \quad x \in X.
\]

Is is easy to check that

\[
T \in L(L^1(\mathbb{R}_+), L(X)).
\]

Observe that for \(x \in X\)

\[
R(\lambda, A)x = T[\exp_{-\lambda}]x, \quad \lambda \in \mathbb{C}_+,
\]

\[
S(t)x := \int_0^t T(s)x \, ds = T[1_{[0,t]}]x, \quad t \geq 0.
\]

where \(1_{[0,t]}\), \(\exp_{-\lambda} \in L^1(\mathbb{R}_+)\) are given by

\[
1_{[0,t]}(s) := \begin{cases} 1 & (s < t) \\ 0 & (s > t) \end{cases}, \quad \exp_{-\lambda}(s) = e^{-\lambda s}.
\]
Fix $\varepsilon > 0$. We are going to approximate $1_{[0,t]}$ by functions $H_{n,t}$ given by the complex line integral

$$H_{n,t}(s) := \frac{1}{2\pi i} \int_{\varepsilon - i n}^{\varepsilon + i n} \frac{e^{\lambda t}}{\lambda} \exp(-\lambda(s)) d\lambda = \frac{1}{2\pi i} \int_{\varepsilon - i n}^{\varepsilon + i n} \frac{e^{\lambda(t-s)}}{\lambda} d\lambda$$

with $n \to \infty$. (The integration is along the straight line from $\varepsilon - in$ to $\varepsilon + in$).

For $t$ and $s \neq t$ fixed, the improper complex integral arising for $n \to \infty$ can be calculated using the residue theorem, giving 1 for $t > s$ and 0 for $t < s$, respectively:

The next lemma provides the precise result we will need:

**Lemma 10.1** For fixed $\varepsilon > 0$ we have

$$\lim_{n \to \infty} H_{n,t} = 1_{[0,t]} \text{ in } L^1(\mathbb{R}_+),$$

uniformly with respect to $t \in [0, \Theta]$.

**Proof:** We split

$$\|H_{n,t} - 1_{[0,t]}\|_{L^1(\mathbb{R}_+)} = \int_0^t \left| \frac{1}{2\pi i} \int_{\varepsilon - i n}^{\varepsilon + i n} \frac{e^{\lambda(t-s)}}{\lambda} d\lambda - 1 \right| ds + \int_t^\infty \left| \frac{1}{2\pi i} \int_{\varepsilon - i n}^{\varepsilon + i n} \frac{e^{\lambda(t-s)}}{\lambda} d\lambda - 1 \right| ds$$

where we substituted $u = t - s > 0$ in the first integral and $u = s - t > 0$ in the second.

1. As $J_1(n,t)$ is increasing in $t$, it is sufficient to show $J_1(n,t) \to 0$ for all $t \in [0,\Theta]$. In view of the structure $J_1(n,t) = \int_0^t \phi_n(u) du$ this follows by Lebesgue’s dominated convergence theorem if we show $\phi_n(u) \to 0$ for all $u \in (0,t]$ and $\phi_n(u) \leq C$ with $C$ independent of $u$ and $n$.
Let \( \Gamma' := \{ \varepsilon + ne^{i\theta} \mid \theta \in (\pi/2, 3\pi/2) \} \). The (oriented) union \( \Gamma' \) and the line from \( \varepsilon - in \) to \( \varepsilon + in \) is a closed, simple, positively oriented curve that encloses the origin if \( n > \varepsilon \). Therefore, the residue theorem yields
\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{e^{\lambda u}}{\lambda} \, d\lambda = 1
\]
and therefore
\[
\phi_n(u) = \left| \frac{1}{2\pi i} \int_{\varepsilon - in}^{\varepsilon + in} \frac{e^{\lambda u}}{\lambda} \, d\lambda - 1 \right| = \left| \frac{1}{2\pi i} \int_{\varepsilon - in}^{\varepsilon + in} \exp(u(\varepsilon + ne^{i\theta})) \, d\lambda \right|.
\]
Using the estimates \( |\varepsilon + ne^{i\theta}| \geq n - \varepsilon \), \( |\exp(u(\varepsilon + ne^{i\theta}))| = e^{n(\varepsilon + \cos \theta)} \) we get
\[
\phi_n(u) \leq \frac{ne^{\varepsilon u}}{2\pi(n - \varepsilon)} \int_{\varepsilon - in}^{\varepsilon + in} \frac{e^{\lambda u}}{\lambda} \, d\lambda = \frac{ne^{\varepsilon u}}{\pi(n - \varepsilon)} \int_{0}^{\pi/2} e^{-un \cos \theta} \, d\theta.
\]
By the concavity of the cosine function, we have \( \cos \theta \geq 1 - \frac{1}{2} \theta^2 \) for \( \theta \in (0, \pi/2) \). Using this and elementary calculations, we get
\[
0 \leq \phi_n(u) \leq \frac{ne^{\varepsilon u}}{\pi(n - \varepsilon)} \int_{0}^{\pi/2} e^{-uin(1 - \frac{1}{2} \theta^2)} \, d\theta = \frac{e^{\varepsilon u}}{2} \frac{n - \varepsilon}{n - \varepsilon} \frac{1 - e^{-nu}}{nu} \xrightarrow{n \to \infty} 0
\]
for \( n > 2\varepsilon \). This shows that \( J_{1}(n, t) \xrightarrow{n \to \infty} 0 \) uniformly in \( t \) as announced.

2. Similarly, we have \( J_2(n) = \int_{0}^{\infty} \psi_n(u) \, du \), \( \psi_n(u) \geq 0 \) and will show that \( \psi_n(u) \to 0 \) as \( n \to \infty \) for all \( u \in (0, \infty) \) and \( \psi_n(u) \leq \psi(u) \) with \( \psi \in L^1(\mathbb{R}_+) \). This implies \( J_2(n) \to 0 \) by dominated convergence.

Let \( \Gamma'' := \{ \varepsilon + ne^{i\theta} \mid \theta \in (0, \pi/2) \} \). Then, by the Cauchy integral theorem,
\[
\int_{\varepsilon - in}^{\varepsilon + in} \frac{e^{-u\lambda}}{\lambda} \, d\lambda = -\int_{\varepsilon - in}^{\varepsilon + in} \frac{e^{-u\lambda}}{\lambda} \, d\lambda
\]
and analogous estimates as in 1. (with |\( \lambda | \geq n \) yield
\[
\psi_n(u) = \left| \frac{1}{2\pi i} \int_{\varepsilon - in}^{\varepsilon + in} \frac{e^{-u\lambda}}{\lambda} \, d\lambda \right| \leq \frac{e^{-uz}}{u} \frac{1 - e^{-nu}}{nu} \xrightarrow{n \to \infty} 0.
\]
Moreover, as the function \( x \mapsto \frac{1 - e^{-x}}{x} \) is decreasing on \( \mathbb{R}_+ \),
\[
\psi_n(u) \leq \frac{e^{-uz}}{u} \frac{1 - e^{-u}}{u} =: \tilde{\psi}(u)
\]
with \( \tilde{\psi} \in L^1(\mathbb{R}_+) \). This proves \( J_2(n) \to 0 \). □

To simplify notation for we set for \( \varepsilon > 0 \)
\[
\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} f(\lambda) \, d\lambda := \lim_{n \to \infty} \int_{\varepsilon - in}^{\varepsilon + in} f(\lambda) \, d\lambda.
\]
For further reference we recall that we have shown in particular
\[
\frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{e^{\lambda u}}{\lambda} d\lambda = \left\{ \begin{array}{ll} 1 & \text{if } u > 0, \\
0 & \text{if } u < 0. \end{array} \right. \tag{10.2}
\]

**Theorem 10.2 (Complex inversion formula)**
For \( \varepsilon > 0 \), \( x \in X \), \( t > 0 \) we have
\[
\int_0^t T(s) x ds = \frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{e^{\lambda t}}{\lambda} R(\lambda, A) x d\lambda.
\]
The convergence of the improper integral is uniform with respect to \( t \in [0, \Theta] \).

**Proof:** For fixed \( n \in \mathbb{N}_+ \), \( t \in [0, \Theta] \), and \( x \in X \) we have
\[
T[H_{n,t}] x = \int_0^\infty H_{n,t}(s) T(s) x ds = \frac{1}{2\pi i} \int_0^{\infty} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{e^{\lambda(t-s)}}{\lambda} T(s) x d\lambda ds.
\]
As the integrand \((\lambda, s) \mapsto \frac{e^{\lambda(t-s)}}{\lambda} T(s) x\) is in \( L^1((\varepsilon + i(-n,n)) \times (0, \infty), X)\) we get by (the vector valued version of) Fubini’s theorem
\[
T[H_{n,t}] x = \frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{e^{\lambda t}}{\lambda} R(\lambda, A) x d\lambda.
\]
Hence, by (10.1) and Lemma 10.1,
\[
\int_0^t T(s) x ds = T[1_{[0,t]}] x = T[ \lim_{n \to \infty} H_{n,t}] x = \lim_{n \to \infty} T[H_{n,t}] x
\]
\[
= \frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{e^{\lambda t}}{\lambda} R(\lambda, A) x d\lambda.
\]
\]

Note, however, that the integral does not converge absolutely. For a similar reason, the validity of the following classical inversion formula for Laplace transforms is restricted to \( x \in D(A) \) and cannot be extended by a density argument to the complete space.

**Corollary 10.3** For \( x \in D(A) \), \( \varepsilon > 0 \), \( t > 0 \) we have
\[
T(t) x = \frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{\lambda t} R(\lambda, A) x d\lambda. \tag{10.3}
\]

**Proof:** We have
\[
T(t) x - x = \int_0^t T(s) A x ds = \frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{e^{\lambda t}}{\lambda} R(\lambda, A) A x d\lambda
\]
\[
= \frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{\lambda t} R(\lambda, A) x d\lambda - \frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{e^{\lambda t}}{\lambda} x d\lambda
\]
and the result follows from (10.2).
11 An exponential formula

Another way of approximating the semigroup with generator $A$ powers of the resolvent and is closely related to solving the initial value problem

\[ \dot{u} = Au, \quad u(0) = x, \]

approximately by the Euler-backward method. For fixed $t > 0$, $n \in \mathbb{N}$ large, $h := t/n$ one approximates $u(t) := T(t)x$ by $u_n$, where

\[ \frac{u_{k+1} - u_k}{h} = Au_{k+1}, \quad k = 0, \ldots, n - 1, \quad u_0 = x. \]

Then, as one easily checks,

\[ u_n = \left( \frac{n}{t} R \left( \frac{n}{t}, A \right) \right)^n x. \]

The following theorem can be considered as a convergence result for this approximation.

**Theorem 11.1** Let $t \mapsto T(t)$ be a strongly continuous semigroup of operators on $X$ generated by $A$. Then

\[ T(t)x = \lim_{n \to \infty} \left( \frac{n}{t} R \left( \frac{n}{t}, A \right) \right)^n x \quad (11.1) \]

uniformly for $t \in (0, \Theta]$.

Eqn. (11.1) is called an exponential formula or the Post-Widder inversion formula. Note that it makes sense because large real numbers belong to $\rho(A)$.

**Proof:** Fix $M, \omega \geq 0$ large enough to ensure $\|T(t)\| \leq Me^{\omega t}$, $t \geq 0$. Then for $\lambda > 0$ we have

\[ R(\lambda, A)x = \int_0^{\infty} e^{-\lambda s} T(s)x \, ds. \]

Taking the $n$-th order derivative with respect to $\lambda$ yields

\[ (-1)^n n! R(\lambda, A)^{n+1} x = (-1)^n \int_0^{\infty} s^n e^{-\lambda s} T(s)x \, ds. \]

Substituting $s = vt$, setting $\lambda := \frac{n}{t}$, and multiplication by $n^{n+1}$ gives

\[ \left( \frac{n}{t} R \left( \frac{n}{t}, A \right) \right)^{n+1} x = \frac{n^{n+1}}{n!} \int_0^{\infty} (ve^{-v})^n T(vt)x \, dv. \]

Using

\[ \frac{n^{n+1}}{n!} \int_0^{\infty} (ve^{-v})^n \, dv = 1 \quad (11.2) \]

we find

\[ \Delta := \left( \frac{n}{t} R \left( \frac{n}{t}, A \right) \right)^{n+1} x - T(t)x = \frac{n^{n+1}}{n!} \int_0^{\infty} (ve^{-v})^n (T(vt)x - T(t)x) \, dv, \]

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and, in view of (7.3), it is sufficient to show $\Delta \to 0$ as $n \to \infty$.

Fix $\varepsilon > 0$. By continuity, there are $a < 1$, $b > 1$ such that

$$
\|T(vt)x - T(t)x\| < \varepsilon \quad \text{for all } v \in (a,b) \text{ and all } t \in [0,\Theta].
$$

(11.3)

We split

$$
\Delta = \frac{n^{n+1}}{n!} \left( \int_0^a \ldots + \int_a^b \ldots + \int_b^\infty \ldots \right)
$$

and estimate the terms on the right separately.

1. For $v \in (0,a)$ we have $0 < ve^{-v} < ae^{-a} := \alpha < e^{-1}$. Therefore

$$
\frac{n^{n+1}}{n!} \left\| \int_0^a (ve^{-v})^n (T(vt)x - T(t)x) \, dv \right\| \leq \frac{n^{n+1}}{n!} \alpha^n a M (e^{a \omega \Theta} + e^{\omega \Theta}) \|x\|
$$

$$
\leq C \alpha^{n} n^{n-\infty} 0.
$$

2. In view of (11.2) and (11.3) we directly have

$$
\frac{n^{n+1}}{n!} \left\| \int_a^b (ve^{-v})^n (T(vt)x - T(t)x) \, dv \right\| \leq \frac{n^{n+1}}{n!} \varepsilon \int_a^b (ve^{-v})^n \, dv < \varepsilon.
$$

3. We have $\beta := be^{-b} < e^{-1}$. So

$$
\frac{n^{n+1}}{n!} \left\| \int_a^\infty (ve^{-v})^n (T(vt)x - T(t)x) \, dv \right\|
$$

$$
\leq \frac{n^{n+1}}{n!} 2M \int_a^\infty v^n e^{-vn} e^{\Theta v} \, dv \|x\|
$$

$$
eq C \alpha^n \int_a^\infty (v' + b)^n e^{-v'\sigma} e^{\Theta \sigma} \, dv'
$$

$$
= C \alpha^n \int_a^\infty \left( \frac{\sigma}{n} + b \right)^n e^{-\sigma} e^{\frac{\Theta \sigma}{n}} \, d\sigma
$$

$$
= C \alpha^n \beta \int_a^\infty \left( 1 + \frac{\sigma/t}{n} \right)^n e^{-\Theta \sigma} \, d\sigma \xrightarrow{n \to \infty} 0,
$$

as the integral exists and is bounded independently of $n$ for $n$ large.

12 Analytic Semigroups of Operators

For $\delta \in (0,\pi)$ we define the sector

$$
\Sigma_\delta := \{ z \in \mathbb{C} \setminus \{0\} \mid \arg z < \delta \}
$$

in the complex plane. Note that $\Sigma_\delta$ is open, $(0,\infty) \subset \Sigma_\delta$, and that for $\lambda > 0$, $z,w \in \Sigma_\delta$ we have $\lambda z, z + w \in \Sigma_\delta$. 29
Let $X$ be a Banach space.

**Definition:** A mapping $T : \Sigma_\delta \rightarrow \mathcal{L}(X)$ is called an analytic semigroup of operators (in $\Sigma_\delta$) if

- the mapping $z \mapsto T(z)$ is analytic,
- $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Sigma_\delta$,
- $\lim_{z \to 0} T(z)x = x$ for all $x \in X$.

For some purposes it is convenient to define additionally $T(0) = I$. (Observe, however, that analyticity of the map $T$ does in general not hold on $\Sigma_\delta \cup \{0\}$.) It is clear that with this definition, the restriction of $T$ to the real half axis $[0, \infty)$ is a strongly continuous semigroup of operators $t \mapsto T(t)$. Moreover, for any $x \in X$ the map $t \mapsto T(t)x$ has derivatives of all orders on $(0, \infty)$. This implies (recall (6.4) and check!)

$$T(t)x \in D(A^\omega), \quad \frac{d^n}{dt^n} T(t)x = A^n T(t)x$$  \hspace{1cm} (12.1)

for all $x \in X$, $t > 0$, $n \in \mathbb{N}$.

In applications, analytic semigroups arise from parabolic evolution problems. The property (12.1) then translates into the “smoothing property” which is characteristic for these equations.

As we will show, semigroups of this type are generated by so-called sectorial operators (and only by these). To simplify notation, we define for $\omega \in \mathbb{R}$, $\delta \in (0, \pi/2)$ the sector $S_{\omega, \delta} := \omega + \Sigma_{\pi/2+\delta}$.

**Definition:** A densely defined linear operator $(A, D(A))$ on $X$ is called sectorial (or $(\omega, \delta)$-sectorial) if there are $\omega \in \mathbb{R}$, $M, \delta > 0$ such that $\rho(A) \supset S_{\omega, \delta}$ and

$$||R(\lambda, A)|| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \in S_{\omega, \delta}. \hspace{1cm} (12.2)$$

---

3Recall that analyticity of a map on an open subset of $\mathbb{C}$ can be defined either as (complex) differentiability or by the demand that any point of the domain of definition has a neighborhood in which the function is represented by a convergent power series. Both definitions are equivalent and can be carried over to maps that take values in Banach spaces.
As \( A - \omega I \) is \((0, \delta)\)-sectorial if \( A \) is \((\omega, \delta)\)-sectorial we can restrict ourselves to the case \( \omega = 0 \) without loss of generality.

We introduce the following notation for paths of integration: For \( r > 0, \theta \in (0, \pi/2) \), let

\[
\begin{align*}
\gamma_{r, \theta} & := \gamma_{r, \theta}^{(1)} \cup \gamma_{r, \theta}^{(2)} \cup \gamma_{r, \theta}^{(3)}, \\
\gamma_{r, \theta}^{(1)} & := \{ \rho e^{i(\pi/2 - \theta)} | \rho \in (-\infty, -r) \}, \\
\gamma_{r, \theta}^{(2)} & := \{ re^{i\phi} | \phi \in (-\theta + \pi/2, \theta + \pi/2) \}, \\
\gamma_{r, \theta}^{(3)} & := \{ \rho e^{i(\theta + \pi/2)} | \rho \in (r, \infty) \}.
\end{align*}
\]

The orientation is with increasing \( \rho \) and \( \phi \), respectively.

Let \( A \) be a \((0, \delta)\)-sectorial operator, \( \delta' \in (0, \delta) \), \( \varepsilon \in (0, \delta - \delta') \) and \( r > 0 \). We construct the analytic semigroup \( z \mapsto T(z) \) on \( \Sigma_{\delta'} \) generated by \( A \) by means of a so-called Dunford integral, given by

\[
T(z) := \frac{1}{2\pi i} \int_{\gamma_{r, \delta - \varepsilon}} e^{\lambda z} R(\lambda, A) d\lambda \tag{12.3}
\]

Observe the formal similarity to the complex Laplace inversion formula (10.3). Note, however, that due to the stronger assumptions on \( A \) and and the different integration path, here we will obtain absolute convergence of the integral, even in \( \mathcal{L}(X) \). Moreover, the function \( \lambda \mapsto e^{\lambda z} \) can be replaced by an arbitrary analytic function \( \lambda \mapsto f(\lambda) \) from a certain class, to obtain a functional calculus, i.e. a linear map \( f \mapsto f(A) \) that translates products of functions in this class into compositions of operators in \( \mathcal{L}(X) \). Here we content ourselves with \( \omega T(z) = e^{A z} \).

**Theorem 12.1** Under the assumptions given above, the following holds:

(i) The integral (12.3) exists in \( \mathcal{L}(X) \) for all \( z \in \Sigma_{\delta'} \). It is independent of \( r \in (0, \infty) \) and \( \varepsilon \in (0, \delta - \delta') \). Moreover, \( \|T(z)\| \leq C_{\delta'} \) for all \( z \in \Sigma_{\delta'} \), and the mapping \( z \mapsto T(z) \) is analytic on \( \Sigma_{\delta'} \).
(ii) We have \( T(z_1 + z_2) = T(z_1)T(z_2) \), \( z_1, z_2 \in \Sigma_{\delta'} \).

(iii) For all \( x \in X \),
\[
\lim_{\Sigma_{\delta'} \ni z \to 0} T(z)x = x. \tag{12.4}
\]

(iv) The strongly continuous semigroup \( t \mapsto T(t) \), \( t \in [0, \infty) \), has generator \( A \).

Remark: As the theorem holds for all \( \delta' \in (0, \delta) \), \( T \) can be extended uniquely and analytically to \( \Sigma_{\delta} = \bigcup_{\delta' < \delta} \Sigma_{\delta'} \), and the semigroup property is preserved. However, \( T \) may be unbounded on the intersection of any neighborhood of 0 with \( \Sigma_{\delta} \), and in (12.4), \( \delta' \) cannot be replaced by \( \delta \). Therefore, \( z \mapsto T(z) \) is not necessarily an analytic semigroup on \( \Sigma_{\delta} \) in the sense of our definition.

Proof of Theorem 12.1: (i) We will first show the convergence of the integral and the bound for \( r = 1/|z| \) and \( \varepsilon = (0, \delta - \delta') \).

For \( \lambda \in \gamma_{r,\delta-\varepsilon}^{(3)} \) we have
\[
\arg \lambda + \arg z > \pi/2 + \delta - \varepsilon - \delta' = \pi/2 + \varepsilon, \quad \Re(\lambda z) = |\lambda z| \cos(\arg \lambda + \arg z) \leq |\lambda z| \cos(\pi/2 + \varepsilon) = -|\lambda z| \sin \varepsilon, \quad |e^{\lambda z}| \leq e^{-|\lambda z| \sin \varepsilon},
\]
and with (12.2)
\[
\|e^{\lambda z} R(\lambda, A)\| \leq \frac{M}{|\lambda|} e^{-|\lambda z| \sin \varepsilon}. \tag{12.5}
\]

Now, parameterizing \( \lambda = e^{i(\pi/2+\delta-\varepsilon)} \rho \),
\[
\int_{\gamma_{r,\delta-\varepsilon}^{(3)}} e^{\lambda z} R(\lambda, A) \, d\lambda = e^{i(\pi/2+\delta-\varepsilon)} \int_{r}^{\infty} e^{\lambda z} R(\lambda, A) \, d\rho
\]
and the integral converges because from (12.5) we get
\[
\|e^{\lambda z} R(\lambda, A)\| \leq \frac{M}{\rho} e^{-\rho |z| \sin \varepsilon}.
\]

Additionally, we use \( r |z| = 1 \) to get the estimate
\[
\left\| \int_{\gamma_{r,\delta-\varepsilon}^{(3)}} e^{\lambda z} R(\lambda, A) \, d\lambda \right\| \leq M \int_{r}^{\infty} e^{-\rho |z| \sin \varepsilon} \frac{d\rho}{\rho}
\]
\[
\quad \quad \quad \int_{r |z|}^{\infty} e^{-\rho' \sin \varepsilon} \frac{d\rho'}{\rho'} = C_{\delta'}.
\]

The integral over \( \gamma_{r,\delta-\varepsilon}^{(1)} \) can be discussed analogously.

For \( \lambda \in \gamma_{r,\delta-\varepsilon}^{(2)} \) we have \( |\lambda z| = 1 \), \( |e^{\lambda z}| \leq e \), \( \|R(\lambda, A)\| \leq M/r \), and therefore
\[
\left\| \int_{\gamma_{r,\delta-\varepsilon}^{(2)}} e^{\lambda z} R(\lambda, A) \, d\lambda \right\| \leq 2\pi Me.
This proves that the integral (12.3) exists in \( L(X) \), and
\[
\left\| \int_{\gamma_{r,\delta-\varepsilon}} e^{\lambda z} R(\lambda, A) \, d\lambda \right\| \leq C_{\delta'}
\]
for our choices of \( \varepsilon \) and \( r \). However, as the map \( \lambda \mapsto e^{\lambda z} R(\lambda, A) \) is analytic for fixed \( z \), we can show by “deforming the path of integration” (and using the exponential decay) that (12.3) is independent of \( r \in (0, \infty) \) and \( \varepsilon \in (0, \delta - \delta') \).

The integrand \( \lambda \mapsto e^{\lambda z} R(\lambda, A) \) depends analytically on the parameter \( z \), with derivative \( \lambda \mapsto \lambda e^{\lambda z} R(\lambda, A) \). Similar to what has been shown for (12.3) above, we have absolute convergence of the integral
\[
\int_{\gamma_{r,\delta-\varepsilon}} \lambda e^{\lambda z} R(\lambda, A) \, d\lambda.
\]
(12.6)

This shows that \( z \mapsto T(z) \) given by (12.3) is complex differentiable and therefore analytic in \( \Sigma_{\delta'} \).

(ii) In the sequel, we fix \( r \) and \( \varepsilon \) and write \( \gamma := \gamma_{r,\varepsilon} \). Let \( c > 0 \) be such that \( \gamma' := \gamma + c \) lies completely to the right of \( \gamma \).

Fix \( z_1, z_2 \in \Sigma_{\delta'} \). By shifting the path of integration, applying the resolvent identity, and Fubini’s theorem,
\[
T(z_1)T(z_2) = \frac{1}{(2\pi i)^2} \int_{\gamma} \int_{\gamma'} e^{\mu z_1} e^{\lambda z_2} \frac{R(\mu, A)R(\lambda, A)}{R(\mu, A) - R(\lambda, A)} \frac{1}{\lambda - \mu} \, d\lambda \, d\mu
\]
\[
= \frac{1}{(2\pi i)^2} \left[ \int_{\gamma} e^{\mu z_1} R(\mu, A) \, d\mu \int_{\gamma'} e^{\lambda z_2} \frac{1}{\lambda - \mu} \, d\lambda \right]
\]
\[
- \int_{\gamma} e^{\lambda z_2} R(\lambda, A) \, d\lambda \int_{\gamma'} e^{\mu z_1} \frac{1}{\lambda - \mu} \, d\mu
\]
\[
= \frac{1}{(2\pi i)^2} \int_{\gamma} e^{\mu (z_1 + z_2)} R(\mu, A) \, d\mu = T(z_1 + z_2).
\]
The integrals over the single paths have been calculated by “closing the integration contour” and applying the Residue theorem and Cauchy’s theorem; see sketches:

(The integrand is $O(e^{-\alpha R})$ with some $\alpha > 0$ on the closing arc with radius $R$.)

(iii) As $\|T(z)\|$ is bounded uniformly in $z \in \Sigma_{\delta'}$, it is sufficient to show the limit relation for $x \in D(A)$. In view of

$$\frac{1}{2\pi i} \int_{\gamma} \frac{e^{\mu z}}{\mu} d\mu = 1$$

we have

$$T(z)x - x = \frac{1}{2\pi i} \int_{\gamma} e^{\mu z} \left( R(\mu, A) - \frac{I}{\mu} \right) x d\mu = \frac{1}{2\pi i} \int_{\gamma} \frac{e^{\mu z}}{\mu} R(\mu, A)Ax d\mu.$$  

Because of the estimate

$$\left\| \frac{e^{\mu z}}{\mu} R(\mu, A)Ax \right\| \leq \Phi(\mu) := \begin{cases} \frac{M}{|\mu|^\alpha} \|Ax\| & \mu \in \gamma^{(1)} \cup \gamma^{(3)}, \\ \frac{M}{|\mu|^\beta} e^{|\mu|} \|Ax\| & \mu \in \gamma^{(2)} \end{cases}$$

for $z \in \Sigma_{\delta'}$, $|z| \leq 1$ with $\Phi \in L^1(\gamma)$ we can take the limit in (12.7) by dominated convergence and obtain

$$\lim_{\Sigma_{\delta'} \ni z \to 0} T(z)x - x = \frac{1}{2\pi i} \int_{\gamma} \frac{R(\mu, A)}{\mu} Ax d\mu = 0,$$

where the last integral is calculated by closing the integration contour by a circular arc of radius $R$ on the right.
(The integrand is $O(R^{-2})$ on this arc, while its length is $O(R)$, so that the integral on the arc vanishes for $R \to \infty$. Cauchy’s theorem yields 0 for the integral over the closed contour.)

(iv) Let $B$ be the generator of the strongly continuous semigroup $t \mapsto T(t)$, $t \in [0, \infty)$. We will show $R(\lambda, B) = R(\lambda, A)$ for real $\lambda > \max_{\mu \in \gamma} \Re \mu + 1$. Fix $t_0 > 0$ and $x \in X$. By Fubini’s theorem,

\[
\int_0^{t_0} e^{-\lambda t} T(t) x \, dt = \frac{1}{2\pi i} \int_\gamma \int_0^{t_0} e^{\mu - \lambda} R(\mu, A) x \, dt \, d\mu
\]

\[
= \frac{1}{2\pi i} \int_\gamma \left( \frac{e^{(\mu - \lambda)t_0} - 1}{\mu - \lambda} \right) R(\mu, A) x \, d\mu
\]

\[
= - \frac{1}{2\pi i} \int_\gamma \frac{R(\mu, A)}{\mu - \lambda} x \, d\mu + \frac{e^{(\mu - \lambda)t_0}}{2\pi i} \int_\gamma \frac{R(\mu, A)}{\mu - \lambda} x \, d\mu.
\]

The first integral can be calculated by closing the contour as in (iii), taking into account the residuum $R(\lambda, A)$ of $\mu \mapsto \frac{R(\mu, A)}{\mu - \lambda}$ at $\mu = \lambda$ and the clockwise orientation of the contour.

To estimate the second integral we use

\[
\left\| \frac{e^{(\mu - \lambda)t_0}}{\mu - \lambda} R(\mu, A) \right\| \leq \frac{M}{|\mu||\mu - \lambda|} e^{-t_0} \leq C|\mu|^{-2} e^{-t_0}.
\]
So this vanishes as $t_0 \to \infty$, and in this limit we obtain

$$R(\lambda, B)x = \int_0^\infty e^{-\lambda t}T(t)x \, dt = R(\lambda, A)x.$$ 

**Remark:** Sectoriality of an operator may be defined without the demand that $D(A)$ is dense in $X$. Then (i) and (ii) remain true while (12.4) holds only for $x \in D(A)$. Accordingly, $A$ generates a strongly continuous semigroup of operators on $D(A)$. In this sense, the restriction to densely defined operators is without loss of generality (upon replacing $X$ by $D(A)$).

Our next result provides the reverse of the generation result from Theorem 12.1 together with an equivalent characterization of sectoriality. We define $\mathbb{C}_+ := \{ z \in \mathbb{C} \mid \text{Re } z > 0 \}$.

**Theorem 12.2** Let $(A, D(A))$ be a densely defined linear operator on a Banach space $X$. The following statements are equivalent:

(i) $A$ is a $(0, \delta)$-sectorial operator for some $\delta > 0$,

(ii) $A$ generates a strongly continuous semigroup of operators that extends to a bounded analytic semigroup in a sector $\Sigma_{\delta'}$ for some $\delta' > 0$,

(iii) We have $\mathbb{C}_+ \subset \rho(A)$ and there is a $C > 0$ such that

$$\|R(\lambda, A)\| \leq \frac{C}{\text{Re } \lambda}, \quad \lambda \in \mathbb{C}_+$$

and

$$\|R(\lambda, A)\| \leq \frac{C}{|\text{Im } \lambda|}, \quad \lambda \in \mathbb{C}_+ \setminus (0, \infty)$$

**Proof:** (i)$\Rightarrow$(ii): This follows (with $\delta' \in (0, \delta)$) from Theorem 12.1.

(ii)$\Rightarrow$(iii): The inclusion $\mathbb{C}_+ \subset \rho(A)$ and (12.8) follow from the Hille-Yosida theorem 9.2 and (9.4). To show (12.9), assume first $\text{Re } \lambda > 0$ and fix $\theta \in (-\delta, 0)$, so that $e^{i\theta} = a - bi$, $a, b > 0$. As the integrand $z \mapsto \exp(-\lambda z)T(z)x$ is analytic in $\Sigma_{\delta'}$ and decays exponentially for $\text{Re } z$ large in the part of $\Sigma_{\delta'}$ below the positive real axis we can shift the path of integration in (9.4) to the ray $P_\theta := \{ e^{i\theta}r \mid r \in (0, \infty) \}$ and obtain for $x \in X$

$$R(\lambda, A)x = \int_{P_\theta} e^{-\lambda z}T(z)z \, dz = e^{i\theta} \int_0^\infty \exp(-\lambda r e^{i\theta})T(re^{i\theta})x \, dr,$$

and from this the estimate

$$\|R(\lambda, A)\| \leq \frac{C}{\text{Re } (\lambda e^{i\theta})} = \frac{C}{a \text{Re } \lambda + b \text{Im } \lambda} \leq \frac{C}{b \text{Im } \lambda}.$$

(iii)$\Rightarrow$(i): We will show the result for $\delta \in (0, \arctan(1/C))$.

Fix $\lambda \in S_{0, \delta}$. If $\text{Re } \lambda > 0$ then $|\lambda| \leq \sqrt{2} \max(\text{Re } \lambda, |\text{Im } \lambda|)$ and this implies

$$\|R(\lambda, A)\| \leq \frac{\sqrt{2}C}{|\lambda|}.$$
If Re $\lambda \leq 0$ then $\mu := i \text{Im} \lambda \neq 0$ and

$$|\lambda - \mu| = |\text{Re} \lambda| \leq \frac{q}{C}|\mu|$$

for some $q \in (0, 1)$. As

$$\|R(\mu + i \varepsilon, A)\| \leq \frac{C}{|\mu|}, \quad \text{for all } \varepsilon > 0,$$

we also have $\mu \in \rho(A)$ and $\|R(\mu, A)\| \leq \frac{C}{|\mu|}$. As

$$\text{dist}(\mu, \sigma(A)) \geq \frac{1}{\|R(\mu, A)\|} \geq \frac{|\mu|}{C}$$

we have $\lambda \in \rho(A)$, and from the Neumann series representation

$$R(\lambda, A) = \sum_{k=0}^{\infty} (\mu - \lambda)^k R(\mu, A)^{k+1}$$

and (12.10) we get

$$\|R(\lambda, A)\| \leq \sum_{k=0}^{\infty} q^{|\mu|^k} C^{k+1} \leq \frac{C'}{|\mu|} \leq \frac{C''}{|\lambda|}.$$  

The following corollary gives another characterization for sectoriality which is easier to apply in practical applications:

**Corollary 12.3** A densely defined operator $(A, D(A))$ is sectorial if and only if there are constants $M \geq 1$, $\omega' \geq 0$ such that $\rho(A) \supset \{\omega'\} + C_+$ and

$$\|\lambda R(\lambda, A)\| \leq M, \quad \lambda \in \{\omega'\} + C_+.$$  

**Proof:** "⇒": Let $A$ be $(\omega, \delta)$-sectorial. Choose $\omega' > \max(0, \omega)$. Then

$$\frac{|\lambda|}{|\lambda - \omega|} \leq C_1, \quad \lambda \in \{\omega'\} + C_+,$$

and, as these $\lambda$ are also in $S_{\omega', \delta}$,

$$\|R(\lambda, A)\| \leq \frac{C_2}{|\lambda - \mu|} \leq \frac{M}{|\lambda|}.$$  

"⇐": Let $A$ satisfy $\rho(A) \supset \{\omega'\} + C_+$ and (12.11). Then, for $\lambda \in \{\omega'\} + C_+$, $\lambda - \omega' \in \rho(A - \omega'I)$ and

$$\|R(\lambda - \omega', A - \omega'I)\| = \|R(\lambda, A)\| \leq \frac{M}{|\lambda|}.$$  

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Replacing $\lambda - \omega'$ by $\lambda$ we conclude $\rho(A - \omega'I) \supset \mathbb{C}_{+}$ and

$$\|R(\lambda, A - \omega'I)\| \leq \frac{M}{|\lambda + \omega'|} \leq \frac{M}{|\lambda|}, \quad \lambda \in \mathbb{C}_{+}.$$ 

So by Theorem 12.2, $A - \omega'I$ is $(0, \delta)$-sectorial for some $\delta > 0$, and therefore $A$ is $(\omega', \delta)$-sectorial.

Recall that $T(t)x \to x$ as $t \to 0$ for $x \in X$, but $AT(t)x \to Ax$ only if $x \in D(A)$. In general, $\|AT(t)x\| \to \infty$ in this limit, i.e. the time derivative of $T(t)x$ blows up near $t = 0$. The following theorem provides a bound for the order of this blowup, while at the same time describing the decay for large $t$.

**Theorem 12.4 (Asymptotics of analytic semigroups)**

Let $(A, D(A))$ be a $(0, \delta)$-sectorial operator generating the (bounded, analytic) semigroup $t \mapsto T(t)$. There is a $C > 0$ such that

$$\|tAT(t)\| \leq C, \quad t \geq 0.$$ 

**Proof:** Assume $t > 0$ without loss of generality. Recall the definition (12.3) and the absolute convergence of this integral and the integral (12.6). Recall further that $AR(\lambda, A) = \lambda R(\lambda, A) - I$ implying the estimate

$$\|AR(\lambda, A)\| \leq M + 1, \quad \lambda \in S_{0, \delta}.$$ 

This yields

$$AT(t) = \frac{1}{2\pi i} \int_{\gamma_{r, \delta - \epsilon}} e^{\lambda t} AR(\lambda, A) d\lambda$$

and estimates (cf. the proof of Theorem 12.1 (i))

$$\|AT(t)\| \leq C \left( \int_{r}^{\infty} e^{-\rho t \sin \epsilon} d\rho + r \right)$$

for $r > 0$ with $C$ independent of $r$ and $t$, which imply the result upon taking $r \to 0$.

**Corollary 12.5**

(i) Under the assumptions of Theorem 12.4, there are constants $C_n$ such that

$$\|t^n A^n T(t)\| \leq C_n, \quad n \in \mathbb{N}_+, \quad t \geq 0.$$ 

(ii) Let $(A, D(A))$ be a $(\omega, \delta)$-sectorial operator generating the semigroup $t \mapsto T(t)$, let $\epsilon > 0$, $n \in \mathbb{N}_+$. There are constants $C_{n, \epsilon}$ (independent of $t$) such that

$$\|t^n A^n T(t)\| \leq C_{n, \epsilon} e^{(\omega + \epsilon)t}, \quad n \in \mathbb{N}_+, \quad t \geq 0.$$ 

**Proof:** (i) By the semigroup property and commuting,

$$t^n A^n T(t) = (tAT(t/n))^n = n^n \left( \frac{t}{n} AT \left( \frac{t}{n} \right) \right)^n$$

and thus, by Theorem 12.4,

$$\|t^n A^n T(t)\| \leq n^n C_n^n := C_n.$$ 

(ii) Exercise!
13 Perturbation theorems

Let \((A, D(A))\) be a generator of a semigroup of operators and \((B, D(B))\) (its "perturbation") a linear operator on \(X\) with \(D(B) \supset D(A)\). Perturbation theorems provide sufficient conditions under which \((A + B, D(A))\) (the “perturbed operator”) also generates a semigroup of operators, and describe the properties of this semigroup. We give two simple, important results, one for strongly continuous semigroup and another for analytic semigroups.

The following preliminary lemma deals with “perturbations of the resolvent” and is guided by the algebraic identity

\[
\frac{1}{\lambda - (a + b)} = \frac{1}{\lambda - a} \frac{1}{1 - \frac{b}{\lambda - a}}.
\]

**Lemma 13.1** Let \((A, D(A)), (B, D(B))\) be two linear operators on a Banach space \(X\) with \(D(B) \supset D(A)\). Let \(\lambda \in \rho(A)\). If \(BR(\lambda, A) \in \mathcal{L}(X)\) and

\[\|BR(\lambda, A)\| < 1\]

then \(\lambda \in \rho(A + B)\) and

\[\|R(\lambda, A + B)\| \leq \frac{1}{1 - \|BR(\lambda, A)\|} R(\lambda, A).\]

**Proof:** Recall that \(I - BR(\lambda, A)\) is invertible and

\[\|(I - BR(\lambda, A))^{-1}\| \leq \frac{1}{1 - \|BR(\lambda, A)\|}.
\]

Define

\[R := R(\lambda, A)(I - BR(\lambda, A))^{-1}.\]

It remains to show that \(R = R(\lambda, A + B)\). Indeed, we have

\[(\lambda I - A - B)R = (\lambda I - A)R - BR = (I - BR(\lambda, A))^{-1} - BR(\lambda, A)(I - BR(\lambda, A))^{-1} = I\]

and for \(x \in D(A)\), using the Neumann series representation of \((I - BR(\lambda, A))^{-1}\),

\[R(\lambda I - A - B)x = \sum_{k=0}^{\infty} R(\lambda, A)[BR(\lambda, A)]^k(\lambda I - A)x
\]

\[= \sum_{k=0}^{\infty} R(\lambda, A)[BR(\lambda, A)]^k x - \sum_{k=0}^{\infty} [R(\lambda, A)B]^k x = x.\]
Theorem 13.2 (Perturbation of generators of strongly continuous semigroups)

Let \((A, D(A))\) be the generator of the strongly continuous semigroup of operators \(t \mapsto T(t)\) on \(X\), satisfying \(|T(t)| \leq Me^{\omega t}\). Let \(B \in \mathcal{L}(X)\). Then \((A + B, D(A))\) also generates a strongly continuous semigroup of operators \(t \mapsto S(t)\) on \(X\), satisfying

\[
\|S(t)\| \leq Me^{(\omega + M\|B\|)t}.
\]

Proof: Introducing the equivalent norm \(||\cdot||\) on \(X\) by

\[
||x|| := \sup_{s \geq 0} \|e^{-\omega s}T(s)x\|, \quad x \in X,
\]

we find that \(t \mapsto T(t)\) is a semigroup of quasicontractions with respect to this norm, i.e.

\[
\|T(t)\| \leq e^{\omega t}, \quad t \geq 0,
\]

cf. the proof of Theorem 9.2. Consequently, by the Hille-Yosida theorem for quasicontractions, in the corresponding operator norm we have \((\omega, \infty) \subset \rho(A)\) and

\[
||R(\lambda, A)|| \leq \frac{1}{\lambda - \omega}, \quad \lambda > \omega.
\]

Fix \(\lambda > \omega + \|B\|\). Then \(\lambda \in \rho(A)\), and \(\|BR(\lambda, A)\| < 1\) because of \(\|B\| < \lambda - \omega\). So Lemma 13.1 (in the space \((X, ||\cdot||)\)) yields \(\lambda \in \rho(A + B)\), and

\[
||R(\lambda, A + B)|| \leq \frac{1}{1 - \frac{\|B\|}{\lambda - \omega}} \frac{1}{\lambda - \omega} = \frac{1}{\lambda - (\omega + \|B\|)}.
\]

Consequently, by the Hille-Yosida theorem 7.1, \(A + B\) generates a strongly continuous semigroup \(t \mapsto S(t)\), and

\[
\|S(t)\| \leq e^{(\omega + \|B\|)t}.
\]

The result follows by translating this back to the original norms. \(\blacksquare\)

Theorem 13.3 (Perturbation of generators of analytic semigroups)

Let \((A, D(A))\) be a sectorial operator. There is an \(\varepsilon > 0\) with the following property: If a linear operator \((B, D(B))\) satisfies \(D(B) \supset D(A)\) and

\[
\|Bx\| \leq \varepsilon \|Ax\| + C_B \|x\|, \quad x \in D(A),
\]

for some \(C_B > 0\), then \((A + B, D(A))\) is sectorial.

Remarks: Note that \(\varepsilon\) depends only on \(A\) while \(C_B\) may depend on \(B\). In practical applications, the value of \(\varepsilon\) for a concrete sectorial operator \(A\) is usually not explicitly known. To apply Theorem 13.3 to concrete operators \(A\) and \(B\) one usually shows that for all \(\varepsilon > 0\) there is a constant \(C_{\varepsilon}\) such that

\[
\|Bx\| \leq \varepsilon \|Ax\| + C_{\varepsilon} \|x\|, \quad x \in D(A).
\]
This is often done by means of so-called interpolation inequalities. The situation of the theorem occurs, for example, if \( A \) is a suitable differential operator and \( B \) is a differential operator of order lower than \( A \). (See exercises.)

**Proof of Theorem 13.3:** By Corollary 12.3, there are \( M \geq 1, \ r \geq 0 \) such that \( \rho(A) \ni \{r\} + \mathbb{C}_+ \) and

\[
\|R(\lambda, A)\| \leq \frac{M}{|\lambda|}, \quad \lambda \in \{r\} + \mathbb{C}_+.
\]

We will show that \( \varepsilon := 1/(4(M+1)) \) has the property announced in the theorem.

Indeed, let an operator \( B \) satisfy the assumptions of that property. Let \( r' := \max(r, 4C_B M) \). Then, for \( \lambda \in \{r'\} + \mathbb{C}_+, x \in X \), we have \( \lambda \in \rho(A) \) and

\[
\|BR(\lambda, A)x\| \leq \varepsilon \|AR(\lambda, A)x\| + C_B \|R(\lambda, A)x\|
\leq \left( \varepsilon(M+1) + \frac{C_B M}{|\lambda|} \right) \|x\| \leq \frac{1}{2} \|x\|.
\]

Therefore \( BR(\lambda, A) \in \mathcal{L}(X) \) and \( \|BR(\lambda, A)\| \leq \frac{1}{2} \). Now Lemma 13.1 yields \( \lambda \in \rho(A + B) \) and

\[
\|R(\lambda, A + B)\| \leq 2R(\lambda, A) \leq \frac{2M}{|\lambda|}.
\]

Therefore, Corollary 12.3 implies that \( A + B \) is sectorial.

\[\Box\]

**14 Homogeneous Cauchy problems**

Now we turn to the investigation of (abstract) evolution equations, which is the main aim of the theory of operator semigroups. The simplest equation of this type is the so-called **homogeneous Cauchy problem** which has the following informal formulation: Let \((A, D(A))\) be a linear operator on a Banach space \( X \) and \( x \in X \). We are looking for a function \( u \) on the (time) interval \([0, \Theta]\) such that

\[
\begin{align*}
\dot{u} &= Au, \\
u(0) &= x.
\end{align*}
\]

**(HCP)**

To make this precise, we introduce a hierarchy of solution concepts to (HCP).

**Definitions:**

- A function \( u \in C^1([0, \Theta], X) \) is called **strict solution** to (HCP) if \( u(0) = x \) and for all \( t \in [0, \Theta] \) we have \( u(t) \in D(A) \) and \( \dot{u}(t) = Au(t) \).

- A function \( u \in C([0, \Theta], X) \cap C^1((0, \Theta], X) \) is called **classical solution** to (HCP) if \( u(0) = x \) and for all \( t \in (0, \Theta] \) we have \( u(t) \in D(A) \) and \( \dot{u}(t) = Au(t) \).
A function \( u \in C([0, \Theta], X) \) is called **mild solution** to (HCP) if \( u(0) = x \) and for all \( t \in [0, \Theta] \) we have \( \int_0^t u(s) \, ds \in D(A) \) and

\[
u(t) = x + A \int_0^t u(s) \, ds, \quad t \in [0, \Theta].\]

We clearly have (for a given problem (HCP))

\[ u \text{ strict solution} \Rightarrow u \text{ classical solution} \Rightarrow u \text{ mild solution} \]

whenever \( A \) is closed. Moreover, if \( u \) is a mild solution to (HCP) then

\[
x = u(0) = \lim_{t \to 0} \frac{1}{t} \int_0^t u(s) \, ds \in \overline{D(A)}, \tag{14.1}
\]

but for the sake of generality we will not assume in this section that \( A \) is densely defined.

In the case that \( A \) generates a strongly continuous semigroup of operators we immediately have, from previous results,

\[
u(t) := T(t)x \text{ is a } \begin{cases} \text{strict solution for } x \in D(A), \\ \text{mild solution for } x \in X. \end{cases}
\]

If \( A \) generates an analytic semigroup then

\[
u(t) := T(t)x \text{ is a classical solution for } x \in X.
\]

These solutions are unique, as the following results shows:

**Lemma 14.1 (Uniqueness of solutions to (HCP))**

Assume that \( A \) generates a strongly continuous semigroup of operators, and \( u \) is a mild solution to (HCP) with \( x = 0 \). Then \( u \equiv 0 \).

**Proof:** Exercise!

Note, however, that existence and uniqueness of (even strict) solutions to (HCP) for all \( x \in D(A) \) does not imply that \( A \) generates a strongly continuous semigroup of operators, even if \( A \) is closed. (See the exercises for a counterexample.) Before we can clarify the situation completely we need the following lemma:

**Lemma 14.2 (Cores for \( A \))**

Let \( (A, D(A)) \) be the generator of the strongly continuous semigroup \( t \mapsto T(t) \). Let \( F \) be a dense subspace of \( D(A) \) which satisfies \( T(t)F \subset F \) for all \( t \geq 0 \). Then \( F \) is dense in \( D(A) \) even with respect to the graph norm \( \| \cdot \|_{D(A)} \).

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Any subspace $F$ with these properties is called a **core** for $A$.

**Proof of Lemma 14.2:** Fix $x \in D(A)$ and $\varepsilon > 0$. As the map $s \mapsto T(s)x$ is continuous with respect to $\| \cdot \|_{D(A)}$, we have

$$
\left\| x - \frac{1}{t} \int_0^t T(s)x \, ds \right\|_{D(A)} = z < \frac{\varepsilon}{2}
$$

for $t > 0$ sufficiently small. Fix such a $t$. Denote by $\hat{F}$ the closure of $F$ with respect to $D(A)$. Choose a sequence $(x_n)$ in $F$ such that $x_n \xrightarrow{X} x$. As $s \mapsto T(s)x_n$ is continuous with respect to $\| \cdot \|_{D(A)}$ and $T(s)x_n \in F$ for $s \in [0,t]$ we have (as the integral is a limit of Riemann sums)

$$
z_n := \frac{1}{t} \int_0^t T(s)x_n \, ds \in \hat{F}.
$$

Now

$$
\| z_n - z \|_{D(A)} = \| z_n - z \|_X + \| Az_n - Az \|_X

= \left\| \frac{1}{t} \int_0^t T(s)(x_n - x) \, ds \right\|_X + \left\| \frac{1}{t} A \int_0^t T(s)x_n - x \, ds \right\|_X

\leq \frac{C}{t} \| x - x_n \|_X < \frac{\varepsilon}{2}
$$

for $n$ large, hence for such $n$ we have $\| x - z_n \|_{D(A)} < \varepsilon$. So $\hat{F}$ and consequently $F$ are dense in $D(A)$ with respect to $\| \cdot \|_{D(A)}$. $\blacksquare$

We will say that a densely defined linear operator $A$ satisfies property (EU) if for all $x \in D(A)$, (HCP) has a unique strict solution. We will denote this solution by $u(\cdot, x)$.

**Theorem 14.3** *(Homogeneous Cauchy problems and strongly continuous semigroups)*

Let $(A, D(A))$ be a closed linear operator on $X$. The following conditions are equivalent:

(i) $A$ is generator of a strongly continuous semigroup of operators $t \mapsto T(t)$,

(ii) $A$ satisfies property (EU) and $\rho(A) \neq \emptyset$.

(iii) $A$ satisfies property (EU), is densely defined, and for any sequence $(x_n)$ in $D(A)$ with $x_n \xrightarrow{X} 0$ we have $u(\cdot, x_n) \to 0$ in $C[[0, \Theta], X)$ for all $\Theta > 0$. 

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Observe that condition (iii) can be interpreted as a well-posedness statement for (HCP) as it provides, additionally to existence and uniqueness of the solution, continuity of the map \( x \mapsto u(\cdot, x) \) in a precise sense.

**Proof of Theorem 14.3:** (i)\(\Rightarrow\) (ii): follows directly from previous results on semigroups.

(ii)\(\Rightarrow\) (iii): The proof proceeds by constructing *mild* solutions \( v(\cdot, x) \) for any \( x \in X \), showing that these are unique, and proving the continuous dependence on \( x \) for these. By uniqueness, for \( x \in D(A) \), these solutions are the strict solutions \( u(\cdot, x) \). By (14.1) we also get \( D(A) \) dense in \( X \) once the existence of a weak solution for any \( x \in X \) is established.

I. Construction of the mild solution: Fix \( \lambda \in \rho(A) \). For \( x \in X \), let \( y := R(\lambda, A) \in D(A) \) and \( v := u(\cdot, y) \) the corresponding strict solution. Then \( v := v(\cdot, x) := (\lambda I - A)u(\cdot, y) \) is a mild solution to (HCP). Indeed, \( v(0) = x, v = \lambda u - Au = \lambda u - \dot{u} \in C([0, \Theta], X) \), and, using the closedness of \( A \), we get from \( u(t) = y + A \int_0^t u(s) \, ds \) by applying \( (\lambda I - A) \) (which is closed as well)

\[
v(t) = x + A \int_0^t v(s) \, ds.
\]

II. Uniqueness of the mild solution: Let \( v = v(\cdot, 0) \) be a mild solution to (HCP) with \( x = 0 \). This means that for all \( t \in [0, \Theta] \), the integral \( \int_0^t v(s) \, ds \) is in \( D(A) \) and

\[
v(t) = A \int_0^t v(s) \, ds.
\]

Therefore, \( u \) given by

\[
u(t) = \int_0^t v(s) \, ds
\]

is a strong solution to (HCP) with \( x = 0 \) and hence, by assumption, \( u \equiv 0 \). This implies in turn \( v \equiv 0 \).

III. Continuous dependence on the initial value: Define the linear map \( \Phi : X \rightarrow C([0, \Theta], X) \) by \( \Phi x = v(\cdot, x) \), the mild solution to (HCP) with \( v(0) = x \). To show that \( \Phi \) is closed, fix \( x \in X \), assume \( x_n \xrightarrow{X} x \), and \( \Phi x_n \rightarrow y \) in \( C([0, \Theta], X) \). By integration

\[
\int_0^t v(s, x_n) \, ds \xrightarrow{X} \int_0^t y(s) \, ds, \quad t \in [0, \Theta],
\]

and, as the \( v(\cdot, x_n) \) are mild solutions,

\[
A \int_0^t v(s, x_n) \, ds = v(t, x_n) - x \xrightarrow{X} y(t) - x.
\]
As $A$ is closed, this implies
\[
\int_0^t y(s) \, ds \in D(A) \quad \text{and} \quad A \int_0^t y(s) \, ds = y(t) - x,
\]
i.e. $y$ is the mild solution with $y(0) = x$, so $y = \Phi x$. This shows that $\Phi$ is closed, and continuity follows by the Closed Graph theorem (Corollary 2.6).

(iii)$\Rightarrow$(i): Define $T(t)x := u(t,x)$ for $x \in D(A)$. It follows directly from our assumptions that $T(t) \in \mathcal{L}(D(A),X)$ for all $t \geq 0$, and $T(t)$ extends by density to an operator $T(t) \in \mathcal{L}(X)$. Moreover, there is a constant $M$ such that $\|T(t)\| \leq M$ for $t \in [0,1]$. (Otherwise, there would be sequences $(t_n)$ in $[0,1]$ and $(x_n)$ in $D(A)$ such that $x_n \xrightarrow{X} 0$ and
\[
\|T(t_n)x_n\| = \|u(t_n,x_n)\| \geq 1,
\]
in contradiction to the assumption $u(\cdot,x_n) \to 0$ uniformly in $[0,1]$.)

Uniqueness yields, for $t,s \geq 0$, $x \in D(A)$,
\[
T(t+s)x = u(t+s,x) = u(t,u(s,x)) = T(t)T(s)x,
\]
and this extends by continuity to all $x \in X$. Moreover for $x$ in the dense set $D(A)$ we have continuity of the map $t \mapsto T(t)x$. By Lemma 6.1, this implies that $t \mapsto T(t)$ is a strongly continuous semigroup of operators. Let $(B,D(B))$ be its generator. We clearly have $A \subset B$. As $D(A)$ is dense in $X$, it is also dense in the smaller set $D(B)$, and because $T(t)D(A) \subset D(A)$, $D(A)$ is a core for $B$. Now Lemma 14.2 implies that the set $\{A x \mid x \in D(A)\}$ is dense in $\{B x \mid x \in D(B)\}$, and both are closed in $X \times X$. Thus $A = B$.

15 Inhomogeneous Cauchy problems

Let $X$ be a Banach space and $A$ the generator of the strongly continuous $T \mapsto T(t)$. Let $f : [0,\Theta] \to X$ be a function of which we will demand at least local integrability, further demands will depend on the situation. The (abstract) inhomogeneous Cauchy problem can be informally stated as

\[
\begin{align*}
\dot{u} &= Au + f, \\
u(0) &= x.
\end{align*}
\]

The following definitions are essentially analogous to the homogeneous case (we exclude the right boundary point for technical reasons):

Definitions:

- A function $u \in C^1([0,\Theta),X)$ is called strict solution to (ICP) if $u(0) = x$ and for all $t \in [0,\Theta]$ we have $u(t) \in D(A)$ and $\dot{u}(t) = Au(t) + f(t)$.

- A function $u \in C([0,\Theta),X) \cap C^1((0,\Theta),X)$ is called classical solution to (ICP) if $u(0) = x$ and for all $t \in (0,\Theta]$ we have $u(t) \in D(A)$ and $\dot{u}(t) = Au(t) + f(t)$.
Again, strict solutions are classical solutions.

The definition of a mild solution is based on the following important representation formula:

**Lemma 15.1** (*Variation of constants formula, Duhamel’s principle*)

Assume \( f \in L^1((0, \Theta), X) \). Let \( u \) be a classical solution to (ICP). Then

\[
  u(t) = T(t)x + \int_0^t T(t-s)f(s)\,ds, \quad t \in [0, \Theta).
\]

**Proof:** The representation formula obviously holds for \( t = 0 \). Now fix \( t \in (0, \Theta) \) and define \( g \in C([0, t], X) \cap C^1((0, t), X) \) by

\[
  g(s) := T(t-s)u(s), \quad s \in [0, t].
\]

Then

\[
  \dot{g}(s) = -AT(t-s)u(s) + T(t-s)\dot{u}(s)
  = -AT(t-s)u(s) + T(t-s)(Au(s) + f(s)) = T(t-s)f(s),
\]

and by integration

\[
  u(t) - T(t)x = g(t) - g(0) = \int_0^t T(t-s)f(s)\,ds.
\]

**Corollary 15.2** *Classical solutions to (ICP) with \( f \in L^1((0, \Theta), X) \) are unique.*

Observe, however, that Lemma 15.1 does not provide existence of classical solutions. Rather, it gives rise to a generalized solution concept:

**Definition:** For \( x \in X, \ f \in L^1((0, \Theta), X) \), the function \( u \in C([0, \Theta], X) \) given by

\[
  u(t) = T(t)x + \int_0^t T(t-s)f(s)\,ds, \quad t \in [0, \Theta] \quad \text{(MS)}
\]

is called the **mild solution** to (ICP).

In view of our results on homogeneous Cauchy problems, this definition is consistent with the earlier one in the case \( f \equiv 0 \).

Lemma 15.1 shows that (MS) provides the only “candidate” for a classical solution. A counterexample shows that even for \( x = 0, \ f \in C((0, \Theta], X) \) is not sufficient for (MS) to provide a classical solution (see exercise).

The next result provides criteria (in terms of the inhomogeneous part of (MS)) under which (MS) actually is a classical solution.

---

\[4\text{Readers unfamiliar with this space can replace this assumption by the demand that } f \text{ should be regulated on } (0, \Theta), \text{ and } \|f\|_{L^1((0, \Theta), X)} := \int_0^\Theta \|f(s)\|_X \,ds \text{ should be finite.}\]
Lemma 15.3 (Conditions for classical solutions)
Assume $f \in L^1((0, \Theta), X) \cap C((0, \Theta], X)$ and define $v \in C([0, \Theta], X)$ by

$$v(t) = \int_0^t T(t - s)f(s)\, ds, \quad t \in [0, \Theta].$$

(15.1)

The following statements are equivalent:

(i) $v \in C^1((0, \Theta), X)$,

(ii) $v(t) \in D(A)$ for all $t \in (0, \Theta)$ and $Av(\cdot) \in C((0, \Theta), X)$,

(iii) (MS) is the classical solution to (IVP) for some $x \in D(A)$,

(iv) (MS) is the classical solution to (IVP) for all $x \in D(A)$.

Proof: (iii)$\Rightarrow$(i): Fix $x \in D(A)$ such that (MS) is the classical solution to (IVP). Note that then both $t \mapsto T(t)x$ and $u$ are in $C^1((0, \Theta), X)$. Statement (i) follows from $v(t) = u(t) - T(t)x$.

(iii)$\Rightarrow$(ii): With $x \in D(A)$ as in the previous part of the proof, $t \in (0, \Theta)$, we have $v(t) = u(t) - T(t)x \in D(A)$ and

$$Av(t) = \dot{u}(t) - f(t) - T(t)Ax$$

for $t \in (0, \Theta]$. Hence $Av(\cdot) \in C((0, \Theta], X)$.

(i)$\Rightarrow$(iv): Let $t \in (0, \Theta)$. For small positive $h$ we have (check!)

$$\frac{T(h) - I}{h}v(t) = \frac{v(t + h) - v(t)}{h} - \frac{1}{h} \int_t^{t+h} T(t + h - s)f(s)\, ds. \quad (15.2)$$

Taking $h \to 0$ (first on the right) yields $v(t) \in D(A)$, $Av(t) = \dot{v}(t) - f(t)$. From this and $v(0) = 0$ we directly conclude that for any $x \in D(A)$, the classical solution $u$ is given by $u(t) = T(t)x + v(t)$.

(ii)$\Rightarrow$(iv): Let $t \in (0, \Theta)$. From $v(t) \in D(A)$ we conclude by (15.2) that $v$ is right differentiable in $t$, with

$$\frac{d}{dt}^+ v(t) = Av(t) + f(t), \quad t \in (0, \Theta),$$

and, as the right derivative is continuous, $v$ is differentiable on $(0, \Theta)$ (see exercise). From this and $v(0) = 0$ we again conclude that for any $x \in D(A)$, the classical solution $u$ is given by $u(t) = T(t)x + v(t)$.

(iv)$\Rightarrow$(iii): trivial.

Remark: An analogous theorem holds for strict solutions when the open intervals $(0, \Theta)$ are replaced by half-open intervals $[0, \Theta)$. (Check!)

Theorem 15.4 Let $f \in C^1([0, \Theta], X)$, $x \in D(A)$. Then (ICP) has a unique strict solution given by (MS).
Proof: In the notation of Lemma 15.3,

$$v(t) = \int_0^t T(t-s)f(s)\,ds = \int_0^t T(s)f(t-s)\,ds,$$

so $v \in C^1([0,\Theta],X)$, and by this lemma and the remark, (MS) yields a strict solution for all $x \in D(A)$.

Due to the "smoothing effect" of analytic semigroups, the situation is better in that case. Recall that a function $f : [0,\Theta] \to X$ is called Hölder continuous with exponent $\alpha \in (0,1)$ if there is a constant $C > 0$ such that

$$\|f(t) - f(s)\| \leq C|t-s|^\alpha, \quad t,s \in [0,\Theta].$$

We will write $C^\alpha([0,\Theta],X)$ for the set of these functions.

**Theorem 15.5** Let $\alpha \in (0,1)$, and let $A$ be sectorial. Let $f \in C^\alpha([0,\Theta],X)$, $x \in X$. Then (ICP) has a unique classical solution given by (MS).

**Proof:** As $t \mapsto T(t)x$ provides a classical solution to (HCP), we can restrict ourselves to (ICP) with $x = 0$ and are going to apply Lemma 15.3. For $t \in (0,\Theta)$ we define

$$v(t) := \int_0^t T(t-s)f(s)\,ds = \int_0^t T(t-s)(f(s) - f(t))\,ds + \int_0^t T(t-s)f(t)\,ds =: v_1(t) + \int_0^t T(t-s)f(t)\,ds =: v_2(t)$$

and will show that for $i = 1,2$, $v_i(t) \in D(A)$ and $t \mapsto Av_i(t)$ continuous. Then the result will follow from Lemma 15.3. As

$$v_2(t) = \int_0^t T(s)f(t)\,ds$$

we immediately have $v_2(t) \in D(A)$ and $Av_2(t) = T(t)f(t) - f(t)$ which is continuous in $t$.

To get the same results for $v_1$, we have to discuss “interchanging $A$ and an improper integral.” For $\varepsilon > 0$, define

$$v_{1,\varepsilon}(t) = \int_0^{t-\varepsilon} T(t-s)(f(s) - f(t))\,ds = \int_\varepsilon^t T(s)(f(t-s) - f(t))\,ds.$$

As $t \mapsto T(t)$ is an analytical semigroup, the map $s \mapsto AT(s)(f(t-s) - f(t))$ is continuous on $[t,\varepsilon]$. Consequently, this function is integrable there, and, by closedness of $A$,

$$Av_{1,\varepsilon}(t) = \int_\varepsilon^t AT(s)(f(t-s) - f(t))\,ds.$$

By Theorem 12.4 and the Hölder continuity of $f$,

$$\|AT(s)(f(t-s) - f(t))\| \leq Cs^{\alpha-1} \quad (15.3)$$

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with $C$ independent of $s, t \in (0, \Theta]$. Therefore, by dominated convergence,

$$Av_{1, \varepsilon}(t) \to \int_0^t AT(s)(f(t - s) - f(t)) \, ds$$

as $\varepsilon \downarrow 0$. Using $v_{1, \varepsilon}(t) \to v_1(t)$ as $\varepsilon \downarrow 0$ and the closedness of $A$ we conclude $v_1(t) \in D(A)$ and

$$Av_1(t) = \int_0^t AT(s)(f(t - s) - f(t)) \, ds.$$

Finally, (15.3) implies that for any $\eta > 0$ there is a $\delta > 0$ such that

$$\left\| \int_0^\delta AT(s)(f(t - s) - f(t)) \, ds \right\| < \eta, \quad t \in (0, \Theta].$$

Now from

$$Av_1(t) = \int_0^\delta AT(s)(f(t - s) - f(t)) \, ds + \int_\delta^t AT(s)(f(t - s) - f(t)) \, ds$$

and the continuity of the second term in $t$, it follows that $t \mapsto Av_1(t)$ is continuous (check!).

Our next result will be important when we discuss semilinear parabolic problems and concerns Hölder continuity of the mild solution to (ICP) if $f$ is only continuous.

**Lemma 15.6** Let $f \in C([0, T], X), \alpha \in (0, 1)$. Then $v$ given by (15.1) is Hölder continuous with exponent $\alpha$ on $[0, \Theta]$.

**Proof:** Pick $s, t \in [0, \Theta]$, assume without loss of generality $s < t$. Then

$$v(t) - v(s) = \int_0^s (T(t - \sigma) - T(s - \sigma)) f(\sigma) \, d\sigma + \int_s^t T(t - \sigma)f(\sigma) \, d\sigma$$

$$= \int_0^s \int_{s - \sigma}^{t - \sigma} AT(\tau)f(\sigma) \, d\tau d\sigma + \int_s^t T(t - \sigma)f(\sigma) \, d\sigma.$$

We estimate the integrals separately. Using standard bounds for $T(t)$ we immediately get

$$\left\| \int_s^t T(t - \sigma)f(\sigma) \, d\sigma \right\| \leq C\|f\|_{\infty}(t - s) \leq C'\|f\|_{\infty}(t - s)^\alpha.$$

(Check!)
For the first term we find by Theorem 12.4
\[
\left\| \int_0^s \int_{s-\sigma}^{t-\sigma} A(T) f(\sigma) \, d\tau d\sigma \right\|
\]
\[
\leq C \| f \|_{\infty} \int_0^s \int_{s-\sigma}^{t-\sigma} \tau^{-1} \, d\tau d\sigma
\leq C \| f \|_{\infty} \int_0^s (s-\sigma)^{-\alpha} \int_{s-\sigma}^{t-\sigma} \tau^{\alpha-1} \, d\tau d\sigma
\leq C \| f \|_{\infty} \int_0^s (s-\sigma)^{-\alpha} \int_{s-\sigma}^{t-s} (\tau + s - \sigma)^{\alpha-1} \, d\tau d\sigma
\leq C \| f \|_{\infty} \int_0^s (s-\sigma)^{-\alpha} \int_{s-\sigma}^{t-s} \tau^{\alpha-1} \, d\tau d\sigma
\leq C \| f \|_{\infty} \int_0^s (s-\sigma)^{-\alpha} \int_0^{\tau} (\tau + s - \sigma)^{\alpha-1} \, d\tau d\sigma
\leq C \| f \|_{\infty} \alpha^{-1} (1 - \alpha)^{-1} \Theta^{\alpha-1} (t-s)^{\alpha}.
\]
This completes the proof.

16 A simple class of semilinear Cauchy problems

Let \((A, D(A))\) generate a strongly continuous semigroup of operators \(t \mapsto T(t)\) on the Banach space \(X\). Fix \(\Theta > 0\), and let
\[ F : [0, \Theta] \times X \to X \]
be a continuous function for which we additionally demand that it is Lipschitz continuous on bounded sets in the second argument, i.e.:
\[
(A1) \quad \text{For all } R > 0 \text{ there is an } L_R > 0 \text{ such that}
\]
\[ \| F(t, x) - F(t, y) \| \leq L_R \| x - y \|, \quad x, y \in B_X(0, R), \ t \in [0, \Theta]. \]

Under these assumptions we consider for \(x \in X\) the (abstract) \textbf{semilinear Cauchy problem}
\[
\begin{align*}
\dot{u}(t) &= Au(t) + F(u(t), t), \\
u(0) &= x.
\end{align*}
\]
(SCP)

The concepts of \textbf{strong solution} and \textbf{classical solution} are defined as in the linear case. A function \(u \in C([0, \Theta], X)\) is called a \textbf{mild solution} of (SCP) if
\[ u(t) = T(t)x + \int_0^t T(t-s)F(s, u(s)) \, ds. \]
(Note that this is this is a nonlinear integral equation defining \(u\) implicitly, so existence and uniqueness are nontrivial issues here!) Again, it is straightforward to see that strong solutions are classical solutions, and classical solutions are mild solutions.
Theorem 16.1 (Short-time well-posedness of (SCP) for mild solutions) Let $F$ satisfy (A1). For any $\bar{x} \in X$, there are $r, \delta, K > 0$ such that

- for all $x \in B(\bar{x}, r)$, (SCP) has a unique mild solution $u = u(\cdot, x)$ on the interval $[0, \delta]$,
- we have
  \[
  \|u(t, x_1) - u(t, x_2)\| \leq K\|x_1 - x_2\| \quad x_1, x_2 \in B(\bar{x}, r), \ t \in [0, \delta].
  \]

Observe that (16.1) provides, in a certain sense, continuous dependence of the mild solution on the initial data.

**Proof:**

1. Let $M := \sup_{t \in [0, \Theta]} \|T(t)\|$, $R := R/(8M)$,
   \[
   \delta := \min\left\{\Theta, \frac{1}{2MLR}, \frac{R}{4M \sup_{s \in [0, \Theta]} \|F(s, 0)\| + 1}\right\}.
   \]

Define
\[
Y := Y_R := \{v \in \mathbb{C}([0, \delta], X) \mid \sup_{t \in [0, \delta]} \|v(t)\| \leq R}\}
\]
As $Y$ is a closed ball in the Banach space $C([0, \delta], X)$ (with respect to the supremum norm which we will denote by $\| \cdot \|_\infty$), $Y$ is a complete metric space.

Define further the map $\Gamma : Y \rightarrow \mathbb{C}([0, \delta], X)$ by
\[
(\Gamma v)(t) = T(t)x + \int_0^t T(t-s)F(s, v(s)) \, ds, \quad t \in [0, \delta].
\]
Clearly, a function $u \in Y$ is a mild solution to (SCP) on $[0, \delta]$ if and only if $u = \Gamma u$. We will show that $\Gamma|Y \subset Y$ and that $\Gamma$ is $(\frac{1}{2})$-contracting, hence the existence of a mild solution will follow from the Banach fixed point theorem.

Indeed, because for $v_1, v_2 \in Y$ we have
\[
\Gamma v_1(t) - \Gamma v_2(t) = \int_0^t (T(t-s)(F(s, v_1(s)) - F(s, v_2(s)))) \, ds \leq MLR\delta\|v_1 - v_2\|_\infty,
\]
our assumptions on $\delta$ imply
\[
\|\Gamma v_1 - \Gamma v_2\|_\infty \leq \frac{1}{2}\|v_1 - v_2\|_\infty.
\]
Furthermore, for $v \in Y$ we have, due to the contraction property,
\[
\|\Gamma v\|_\infty \leq \|\Gamma x - \Gamma 0\|_\infty + \|\Gamma 0\|_\infty \\
\leq \frac{1}{2}\|v\|_\infty + M(\|\bar{x}\| + r) + \delta M \sup_{s \in [0, \delta]} \|F(s, 0)\| \\
\leq \frac{R}{2} + \frac{R}{4} + \frac{R}{4} = R
\]
and therefore $\Gamma v \in Y$. 

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In particular, due to the uniqueness of the fixed point in \( Y \), we have the following intermediate uniqueness result: For any sufficiently large \( R \), there is a \( \delta = \delta(R, x) > 0 \) such that among all functions in \( C([0, \delta], X) \) that satisfy \( \|u\|_\infty \leq R \), there is precisely one mild solution to (SCP).

2. Fix \( x \in X \) and let \( u_{1,2} \in C([0, \delta], X) \) be two mild solutions to (SCP). Let
\[ t_0 := \sup(\{0\} \cup \{t \in (0, \delta] \mid u_1|_{[0,t]} = u_2|_{[0,t]}\}). \]
We will show by contradiction that \( t_0 = \delta \). Assume \( t_0 < \delta \). Define \( \tilde{u}_{1,2} \in C([0, \delta - t_0], X) \) by \( \tilde{u}_i(t) = u_i(t + t_0), t \in [0, \delta - t_0]. \) Then \( \tilde{u}_{1,2} \) are mild solutions to the semilinear Cauchy problem
\[
\begin{aligned}
\dot{\tilde{u}}(t) &= A\tilde{u}(t) + F(t, \tilde{u}(t)) \\
\tilde{u}(0) &= u(t_0)
\end{aligned}
\]
on \([0, \delta - t_0]\) (exercise!). Choose \( R \geq \max\{\|u_1\|_\infty, \|u_2\|_\infty, 8M\|x\|\}. \) By the uniqueness result of Step 1, we find that \( \tilde{u}_1 \) and \( \tilde{u}_2 \) coincide on some interval \([0, \delta']\) of positive length, contradicting the definition of \( t_0 \).

3. Fix \( x_{1,2} \in B(\bar{x}, r) \) and write \( u_i := u(\cdot, x_i) \) for the corresponding mild solutions of (SCP). Then
\[
u_1(t) - u_2(t) = T(t)(x_1 - x_2) + \int_0^t T(t - s)(F(s, u_1(s)) - F(s, u_2(s))) \, ds.
\]
and, because of \( \|u_i\|_\infty \leq R \),
\[
\|u_1 - u_2\|_\infty \leq M\|x_1 - x_2\| + \delta MLR \|u_1 - u_2\|_\infty \leq \frac{1}{2}
\]
and so
\[
\|u_1 - u_2\|_\infty \leq 2M\|x_1 - x_2\|.
\]

Using arguments similar to the ones given in Step 2 of the proof, one can also show existence and uniqueness of a \textbf{maximal solution} to (SCP), i.e. that there is a mild solution \( \bar{u} \) on an interval \( I \subset [0, \Theta] \) such that for any mild solution \( u \) on some interval \( I \subset [0, \Theta] \) we have \( I \subset \bar{I} \) and \( u = \bar{u}|_I \). (Exercise!)

To discuss regularity of the mild solutions, we additionally assume
\begin{enumerate}[label=(A\arabic*)]
\item \( A \) is sectorial, i.e. the semigroup \( t \mapsto T(t) \) is analytic,
\item \( F \) is Hölder continuous with respect to \( t \), more precisely, there is an \( \alpha \in (0, 1) \) such that for all \( R > 0 \) there is a constant \( C_R \) such that
\[
\|F(t, x) - F(s, x)\| \leq C_R|t - s|^{\alpha}, \quad x \in B(0, R).
\]
\end{enumerate}

\textbf{Theorem 16.2} \textit{(Classical solutions to (SCP) in the parabolic case)}

Let (A1)–(A3) be satisfied, \( x \in X \). Let \( u \) be the mild solution to (SCP) on some interval \([0, \Theta]\)
Proof: Observe first that the map \( s \mapsto f(s) := F(s, u(s)) \) is continuous, so \( u \) is the mild solution the linear problem (ICP) with that function \( f \), and Lemma (15.3) applies. We will show that \( u \in C^1(\varepsilon, \Theta) \) for all \( \varepsilon > 0 \). By the analyticity of \( t \mapsto T(t) x \) and Lemma 15.6 we find that \( u \in C^\alpha([\varepsilon, \Theta], X) \). Define now \( \tilde{u}, \tilde{f} \) by \( \tilde{u}(x, t) = u(x, t + \varepsilon), \tilde{f}(t) = F(t + \varepsilon, \tilde{u}(t)), t \in [0, \Theta - \varepsilon] \), and observe that \( \tilde{u} \) is the mild solution to

\[
\dot{\tilde{u}} = A \tilde{u} + \tilde{f}, \quad \tilde{u}(0) = u(\varepsilon). \tag{16.2}
\]

(Check, see exercise.) As \( \tilde{f} \in C^\alpha([0, \Theta - \varepsilon], X) \) because of (A1), (A3) and the Hölder continuity of \( u \) (exercise!) we find by Theorem 15.5 applied to (16.2) that \( \tilde{u} \in C^1([0, \Theta - \varepsilon], X) \) which implies our claim. 

17 Global existence and stability: a toy example

Instead of giving general results in an abstract context, we want to demonstrate the application of the theory of (analytic) semigroups to the stability analysis of an equilibrium of a simple semilinear parabolic initial-boundary value problem in one space variable.

We are seeking functions \( u = u(x, t), x \in [0, 1], t \geq 0 \) such that

\[
\begin{align*}
u_t(x, t) &= u_{xx}(x, t) + u(x, t)^2 & x \in [0, 1], t \geq 0, \\
u(0, t) &= u(1, t) = 0 & t \geq 0 \\
u(x, 0) &= u_0(x) & x \in [0, 1], \end{align*}
\]

where \( u_0 \) is a given continuous function that satisfies the compatibility conditions \( u_0(0) = u_0(1) = 0 \).

Obviously \( u \equiv 0 \) is a trivial stationary solution. We are going to show its stability, i.e. we will show that solutions with initial value near zero exist for all positive times and decay towards zero for large times.

**Proposition 17.1** For any \( \delta \in (0, 2\pi) \) there is an \( \varepsilon > 0 \) such that (17.1) has a unique classical solution on \((0, \infty)\). It satisfies the estimate

\[
\sup_x |u(x, t)| = \|u(\cdot, t)\| \leq 2\varepsilon e^{-\delta t}. \tag{17.2}
\]

**Proof:** We choose the following setting: Let

\[
X = \{ v \in C([0, 1]) | v(0) = v(1) = 0 \},
\]

with the supremum norm \( \| \cdot \| \), let \( (A, D(A)) \) be given by

\[
D(A) = \{ v \in C^2([0, 1]) | v(0) = v(1) = 0 \}, \quad Av = v_{xx}.
\]

Then \((A, D(A))\) generates a strongly continuous analytic semigroup of operators \( t \mapsto T(t) \) on \( X \). Its spectrum consists of negative real eigenvalues, the largest being \( -\pi^2 \). Moreover, \( \|T(t)\| \leq e^{-\pi^2 t} \). (Exercises!) Further, let \( F : ([0, \infty) \times X) \rightarrow X \) be given by

\[
F(t, v)(x) = v(x)^2.
\]

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It is not hard to check that $F$ satisfies (A1) and (A3). With these choices for $X$, $A$, and $F$, and $x := u_0 \in X$, Problem (17.1) belongs to the class of problems given by (SCP) (on an infinite time interval).

(Danger: The restriction to one space dimension makes it possible to choose particularly simple function spaces. To treat the spatially multidimensional analogon to (17.1), different spaces have to be chosen.)

In the sequel, we identify continuous functions $u : [0,1] \times [0,\infty) \to \mathbb{R}$ that satisfy (17.1) with $X$-valued functions $t \mapsto u(\cdot,t) := u(t) \in X$ without changing the notation.

For $\delta \in (0,\pi^2)$, $\varepsilon > 0$ we define

$$M_{\delta,\varepsilon} := \{ v \in C([0,\infty),X) \mid t \mapsto e^{\delta t} v(t) \text{ bounded} , \|v\|_\delta \leq 2\varepsilon \}$$

where

$$\|v\|_\delta := \sup_{t \geq 0} \|e^{\delta t} v(t)\|.$$ 

Note that $M_{\delta,\varepsilon}$, with the metric induced by $\| \cdot \|_\delta$ is a complete metric space of continuous functions that satisfy the decay estimate (17.2). In view of Theorems 16.1, 16.2, the proposition is established once we find a mild solution to (17.1) on $[0,\infty)$ in $M_{\delta,\varepsilon}$, i.e. a fixed point of the operator

$$\Lambda : M_{\delta,\varepsilon} \to C([0,\infty),X)$$

given by

$$(\Lambda v)(t) := T(t)u_0 + \int_0^t T(t-s)v(s)^2 \, ds.$$ 

To apply Banach’s fixed point theorem, we have to show that for $\varepsilon$ sufficiently small, $\Lambda$ is a $\frac{1}{2}$-contraction and $\Lambda[M_{\delta,\varepsilon}] \subset M_{\delta,\varepsilon}$. Indeed, for $v, w \in M_{\delta,\varepsilon}$, $t \geq 0$, $\varepsilon < \delta/8$, we have

$$\|e^{\delta t}(\Lambda v(t) - \Lambda w(t))\| \leq e^{\delta t} \int_0^t \|T(t-s)\| \|v(s)^2 - w(s)^2\| \, ds$$

$$\leq e^{\delta t} \int_0^t e^{-\pi^2(t-s)} \|v(s) + w(s)\| \|v(s) - w(s)\| \, ds$$

$$\leq 4\varepsilon \int_0^t e^{-\frac{\pi^2}{2}(t-s)} \|v(s) - w(s)\| \, ds$$

$$\leq 1 \|v - w\|_\delta e^{-\delta s}$$

$$\leq 1 \|v - w\|_\delta e^{-\delta s}$$

$$\leq \frac{1}{2} \|v - w\|_\delta,$$

and so, taking the supremum over $t$,

$$\|\Lambda v - \Lambda w\|_\delta \leq \frac{1}{2} \|v - w\|_\delta.$$

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Moreover, for $v \in M_{\delta, \varepsilon}$,
\[
\|\Lambda v\|_\delta \leq \|\Lambda v - \Lambda 0\|_\delta + \|\Lambda 0\|_\delta \leq \frac{1}{2} \|v\|_\delta + \sup_t e^{\delta t}\|T(t)u_0\|
\leq \varepsilon + \sup_t e^{(-\omega-\delta)t}\|u_0\| = 2\varepsilon.
\]
References


