Take home exam Continuum Mechanics (Mastermath) 2016

1. ***The Boussinesq approximation***
   In the lectures we talked about the Boussinesq approximation for the density of a fluid heated from below. Here we delve a little deeper into this problem.

a.) Starting from the Navier-Stokes equations for a compressible fluid (chapter 9.1 in the book of Temam) and the equation for heat convection and diffusion ((9.13) in the book of Temam) in $\Omega = (0, L_1) \times (0, L_2)$ (for horizontal coordinate x and vertical coordinate z) argue that the Boussinesq approximation amounts to

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \rho_0[\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}] &= -\nabla p + \eta \nabla^2 \mathbf{u} + \rho_0(1 - \alpha T)\mathbf{g} \\ \partial_t T + (\mathbf{u} \cdot \nabla) T &= \kappa \nabla^2 T \end{aligned} \tag{1}$$

where $\mathbf{g} = [0, -g]^T$. Explain carefully all approximations and assumptions you make.

b.) If the temperature is prescribed at $z = 0$ and $z = L_2$ as

$$T(x, 0, t) = T_0 \quad \text{and} \quad T(x, L_2, t) = T_1 := T_0 - \beta L_2. \tag{1b}$$

Give the boundary conditions for $\mathbf{u} = [u, w]^T$ for the following two cases
1) fixed wall
2) free fluid wall
Explain your answers.

In this exercise we will analyse the free fluid wall case analytically and determine the onset of instability.

c.) Find the stationary solution of Eqs. (1-1b).

d.) Derive the equations for a perturbation (denoted with $\tilde{}$) of the stationary state found in c) by linearising the Boussinesq equations around the stationary state.

e.) Taking suitable derivatives of the equations derived in d) and combining this with the incompressibility condition show that the following equation for $\tilde{p}$ and $\tilde{T}$ can be derived

$$\nabla^2 \tilde{p} = \alpha g \partial_z \tilde{T}. \tag{2}$$

Express in words what this equation is saying.

f.) Apply ∇^2 to the equation for $\tilde{w}$ derived in d) and use Eq. (2) to derive the following equation for $\tilde{w}$

$$\partial_t \nabla^2 \tilde{w} = \alpha \partial_x^2 \tilde{T} + \nu \nabla^4 \tilde{w}, \tag{3}$$

where $\nu = \mu/\rho$. The linearized equation for the temperature $\tilde{T}$, which was derived in d), reads

$$\partial_t \tilde{T} - \beta \tilde{w} = \kappa \nabla^2 \tilde{T} \tag{4}$$

g.) Assuming separable solutions of the form

$$\tilde{w}(x, y, t) = f(x)\bar{w}(z)e^{st}, \quad \tilde{T}(x, z, t) = f(x)\theta(z)e^{st}$$

determine the equations for $\bar{w}(z)$ and θz and show that in order to find solutions in this separable form the condition

$$\partial_x^2 f + a^2 f = 0,$$

has to be satisfied.

h.) Show that by using the condition on f from g), the equations for θ and $\bar{w}$ read

$$\kappa\theta'' - a^2\kappa\theta - s\theta = -\beta\bar{w} \tag{5}$$

$$\left[\frac{d^2}{dz^2} - a^2\right]\left[\frac{d^2}{dz^2} - a^2 - \frac{s}{\nu}\right] = \alpha g a^2 \frac{\theta}{\nu} \tag{6}$$

i.) Nondimensionalize the equations in h) by the following transformation

$$\hat{z} = z/L_2, \qquad \hat{a} = aL_2, \qquad \hat{s} = \frac{sL_2^2}{\kappa}, \qquad \hat{w} = \frac{\kappa}{L_2}\bar{w}$$

and combine the equations to a sixth order differential equation

$$\left[\frac{d^2}{d\hat{z}^2} - \hat{a}^2 - \hat{s}\right]\left[\frac{d^2}{d\hat{z}^2} - \hat{a}^2\right]\left[\frac{d^2}{d\hat{z}^2} - \hat{a}^2 - \frac{\hat{s}}{\mathrm{P}}\right] = -\hat{a}^2 \mathrm{Ra}\ \hat{w},$$

where $\mathrm{P} = \frac{\nu}{\kappa}$ is the Prandtl number and $\mathrm{Ra} = \frac{\alpha L_2^4 g\beta}{\kappa\nu}$ is the Rayleigh number.

j.) Show that for free boundaries in $\hat{z} = 0$ and $\hat{z} = 1$, the following is true

$$\hat{w}^{(2n)}(0) = \hat{w}^{(2n)}(1) = 0, \qquad n = 1, 2, \cdots$$

k.) Use the functions $\hat{w} = A\sin(n\pi\hat{z})$ to find a quadratic equation for $\hat{s}$ and obtain a relation between n, $\hat{a}$ and the Rayleigh number Ra for instability to occur.

l.) As the Rayleigh number increases the basic state with $n = 1$ turns unstable at Ra_{cr}. Determine the value of Ra_{cr}.

m.) Show that a roll state, that is circle motion in the (x, z)-plane, is one of the possible solutions for the unstable state.

n.) Can you give an explanation in physical terms of the instability that arises ? Do you expect other stabilities to be present ? What do you think would change if instead of stress free boundary conditions, fixed walls would be imposed?

2. ***A spherical body in its own gravitation field.***
A solid isotropic spherical body is deformed by its own gravitational field. We assume that the body is alone in the universe. You may apply linear elasticity theory and assume a constant density ρ throughout the solid body. The gravitational field $\mathbf{g}$ can be shown to increase linearly with the radial distance r, hence $\mathbf{g} = -g\mathbf{r}/R$, where g a gravitational constant.

a.) Derive the following equation for the deformation of the body with mass density ρ

$$\nabla[\nabla \cdot \mathbf{u}] - \frac{1-2\nu}{2(1-\nu)}\nabla \wedge [\nabla \wedge \mathbf{u}] = -\rho\mathbf{g}\frac{(1+\nu)(1-2\nu)}{E(1-\nu)}.$$

b.) What are appropriate boundary conditions? Impose these and solve the equation of part (a) using spherical symmetry.

c.) Interpret you results. Discuss dependence on parameters, the pressure in the center of the sphere. Is the material in the sphere compressed or stretched ?