# MRI course
# Variational and Topological Methods for PDEs

## Exercises 2

**1.** Consider the problem in $\mathbb{R}$,

$$ax = \sin x,$$

with $a > 0$. Note that $x = 0$ is a solution of this problem; but for the sake of the argument we now assume that we don't already know this.

**a)** Apply each of the fixed-point theorems of Schauder I, Schauder II, and Schäfer to the question of existence.

**b)** For which $a$ can you apply the Banach fixed-point theorem? Can you show that this range is sharp?

**2.** In this exercise we study the differential equation

$$-u'' + u = f \qquad \text{on } (0,1)$$

with boundary data

$$u(0) = u(1) = 0,$$

and $f \in L^2(0,1)$.

**a)** By integrating the equation twice and applying partial integration, one can convert the differential equation into an integral equation of the form

$$u(x) = \int_0^1 k(x,y)(u(y) - f(y))\,dy.$$

**b)** Show that the operator

$$T : L^2(0,1) \to L^2(0,1)$$

given by

$$T(u)(x) := \int_0^1 k(x,y)(u(y) - f(y))\,dy$$

is continuous and compact.

**c)** Show that there exists a solution of the differential equation.