MRI course
Variational and Topological Methods for PDEs

Exercises 4

1. Show that weak (i.e. $\sigma(X, X')$) limits are unique.

2. Do exercise 8.6.1 of Evans.

3. Give an example of a sequence of functions $u_n : (0, 1) \to \mathbb{R}$ such that

- $u_n \rightharpoonup u$ in $L^2(0, 1)$
- $u_n^2 \rightharpoonup v$ in $L^1(0, 1)$

and $v \neq u^2$.

4. Prove the following statement (known as the weak-strong convergence property):

> If H is a Hilbert space, and $x_n \to x$, $y_n \rightharpoonup y$ in H, then $(x_n, y_n) \to (x, y)$.

5. Generalize the previous statement to the duality $\langle \cdot, \cdot \rangle$ between X and X'. Think carefully about the roles of X and X'.

6. Apply the previous result to $L^p(\Omega)$ and $L^{p'}(\Omega)$, where $1/p + 1/p' = 1$. Which values of p can you deal with in this way?

7. Prove the following statement:

> If X and Y are two Banach spaces, and $T : X \to Y$ is a compact bounded linear operator, then T converts weakly converging sequences into strongly converging sequences, i.e, $T(x_n) \to T(x)$ whenever $x_n \rightharpoonup x$ in $\sigma(X, X')$.

You might want to use the result of the next exercise.

8. (Tricky) Use the Banach-Steinhaus and Hahn-Banach theorems to prove the following property:

> If $x_n \rightharpoonup x$ in $\sigma(X, X')$, then the set $\{x_n\}_n$ is bounded in X.