# MasterMath course

# Variational and Topological Methods for PDEs

# Exercises 7

1. Does direct minimisation (in $H_0^1$ say) work for (watch the signs)

$$I(u) = \int_U \frac{1}{2}|\nabla u|^2 - \frac{1}{4}u^4 \quad ?$$

   Why (not)?

2. Does direct minimisation (in $H_0^1$ say) work for (watch the signs)

$$I(u) = \int_U -\frac{1}{2}|\nabla u|^2 + \frac{1}{4}u^4 \quad ?$$

   Why (not)?

3. Prove uniqueness of the minimiser (if it exists) of

$$\int_U \frac{1}{q}|\nabla u|^q + F(u),$$

   where $F$ is convex and $q > 1$.

4. Let $H$ be a Hilbert space and $J : H \to \mathbb{R}$ is defined by $J(u) = \frac{1}{2}\|u\|_H^2$. Show that $J$ is $C^1$ and calculate the derivative.

5. If $J : E \to \mathbb{R}$ is strictly convex, then $J$ has at most one critical point. Why?

6. Exercise 8.6.10 from Evans. Additionally, show that you can in this way in fact find a solution with $u \geq 0$. [Use that if $u \in W^{1,2}(U)$, then $|u| \in W^{1,2}(U)$ and $\|u\| = \||u|\|$.]

7. Exercise 8.6.11 from Evans.