MasterMath course

Variational and Topological Methods for PDEs

Exercises 8

1. For which p can you prove that the functional $W(u) = \int_U \frac{1}{p+1}|u|^{p+1}$ is continuously differentiable from H_0^1 to $\mathbb{R}$?

2. (answered in the lecture, by accident) For E a Banach space, let $F \in C^1(E, \mathbb{R})$ satisfy the Palais-Smale condition. Why is every generalized critical value of F also a (true) critical value?

3. For any $F \in C^1(E, \mathbb{R})$, prove that the set of generalized critical values is closed.

4. Let $K_\beta = \{u \in E \,|\, dF(u) = 0, F(u) = \beta\}$ be the critical points at level β. Show that K_β is compact if F satisfies the Palais-Smale condition.

5. (last week) Let E_1, E_2 be Banach spaces and $F : E_1 \to E_2$ be continuous. Suppose that for all $u \in E_1$ there exist a function $\tilde{T}_u F \in \mathcal{L}(E_1, E_2)$, depending continuously on u, such that, for some dense subspace $X \subset E_1$

$$\lim_{t \to 0} \frac{F(u + tv) - F(u)}{t} = \tilde{T}_u F(v) \qquad \text{for all } u, v \in X.$$

[This means that $\tilde{T}_u F$ is the Gateaux derivative when tested in points and directions in a dense subspace (but $\tilde{T}_u F$ is defined for all u and is continuous on E_1).] Prove that F is Fréchet differentiable (by adapting the result/proof presented in the lecture).