# MasterMath course

# Variational and Topological Methods for PDEs

# Exercises 9

1. We are going to discuss billiards. For our purposes a billiard is a bounded, strictly convex set in the plane with a smooth boundary. The idea is that a frictionless ball is bouncing around in this set. The orbits are sequences of line segments where each two successive segments share a point on the boundary, and at this points the two segments make the same angle with the tangent to the boundary (angle of incidence equals angle of reflection). Let $s \in S^1$ parametrize the boundary, i.e., the points on the boundary are given by $p(s) = (x(s), y(s)) \in \mathbb{R}^2$. One may take $|\frac{dp}{ds}| = 1$: arc-length parametrization.

   Let $H(s, s')$ be defined as the distance in $\mathbb{R}^2$ between the points $p(s)$ and $p(s')$.

   (a) Show that the line segments $p(s)$ to $p(s')$ and $p(s')$ to $p(s'')$ are part of an orbit if and only if
   $$\frac{\partial}{\partial s'} H(s, s') + \frac{\partial}{\partial s'} H(s', s'') = 0.$$

   (b) An $n$-periodic orbit is a closed orbit on the billiard with $n$ boundary points. Show that the variational formulation of the problem of finding $n$-periodic orbits is finding critical points of
   $$H_n = \sum_{k=1}^{n} H(s_k, s_{k+1}),$$
   where $s_{n+1} \equiv s_1$.

   (c) Let us look at 2-periodic orbits, represented by $(s_1, s_2)$. Show that there exists a 2-periodic orbit corresponding to a maximum of $H_2$.

   (d) Argue that there are in fact two maxima. Then use the mountain pass theorem to obtain a different 2-periodic orbit.

   (e) What can you say, using variational methods, about 3-periodic orbits?

***There are more exercises on the next page!***

2. Prove the following deformation lemma for critical levels. Let $F \in C^1(E, \mathbb{R})$ [or, to avoid technicalities, $F \in C^1(H, \mathbb{R})$ having Lipschitz derivative on bounded sets (with $H$ Hilbert)] satisfy the Palais-Smale condition. Let $K_\beta$ be the critical points at level $\beta$. Prove that for any neighborhood $N$ of $K_\beta$ and arbitrarily small $\varepsilon$ there is a deformation $\Phi \in C([0, 1] \times E, E)$ such that

(a) $\Phi(0, u) = u$ for all $u \in E$

(b) $\Phi(t, u) = u$ for all $t$ if $F(u) \neq [\beta - 2\varepsilon, \beta + 2\varepsilon]$ or if $dF(u) = 0$

(c) $\Phi(1, u) \in F^{\beta-\varepsilon}$ for $u \in F^{\beta+\varepsilon} - N$

(d) $\Phi(1, u) \in F^{\beta-\varepsilon} \cup N$ for $u \in F^{\beta+\varepsilon}$

(e) $F(\Phi(t, u))$ is non-increasing in $t$ for any $u \in E$

(f) $\Phi(t, \cdot)$ is a homeomorphism from $E$ to $E$ for every $t \in [0, 1]$.

3. Prove the following. Let $F \in C^1(E, \mathbb{R})$ satisfy the Palais-Smale condition. Suppose 0 is a local minimizer of $F$ with $F(0) = 0$, and suppose that $F$ admits a second local minimizer $u_1 \neq 0$. Then

(a) either there exists a critical point $u$ of $F$ which is not of minimum type; or

(b) the origin and $u_1$ can be connected by a path in any neighborhood of the set of local minimizers $u$ of $F$ with $F(u) = 0$. Necessarily then $\beta = F(u_1) = F(0) = 0$, where

$$\beta = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} F(\gamma(t)),$$

and $\Gamma$ is the usual space of continuous paths connecting 0 to $u_1$.

4. Prove the following. Let $F \in C^1(E, \mathbb{R})$ satisfy the Palais-Smale condition. Suppose $F$ has two local minima (or maxima), then $F$ possesses a third critical point.