Asymptotic behaviour of solutions of a nonlinear transport equation

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1. Introduction.

In this paper we consider the asymptotic behaviour as $t \to \infty$ and as $\varepsilon \downarrow 0$ of the solution of the initial-boundary value problem

$$
\text{I}\begin{cases}
\beta(u)_t + \dfrac{\lambda - 1}{r} u_r - u_{rr} = 0 & \varepsilon < r < \infty,\ t > 0 \qquad \text{(Ia)}\\
u_r = \dfrac{\lambda}{\varepsilon}(u - u_e) & r = \varepsilon,\ t > 0 \qquad \text{(Ib)}\\
u(r,0) = u_0(r) & r > \varepsilon \qquad \text{(Ic)}
\end{cases}
$$

where t and r denote a time and a radial space coordinate, and the subscripts t and r denote partial differentiation with respect to these variables. In this problem, λ and u_e are constants such that $\lambda > 0$ and $u_e \in \{0,1\}$. Further $\beta : [0,\infty) \to [0,\infty)$ and $u_0 : [\varepsilon,\infty) \to [0,\infty)$ are functions to be specified later.

Problem I arises in a two-dimensional model describing the transport of reactive solutes, with scaled concentration u, through a porous medium in which the groundwater flow is induced by well injection. The small parameter ε is related to the well radius and the parameter λ is the Peclet number. The nonlinear function β in equation (Ia) reflects the effect of equilibrium adsorption reactions of the dissolved chemicals on the surface of the soil particles. The boundary condition (Ib) describes the injection of water with solute concentration u_e into the flow domain. The model is described in detail by Van Duijn and Knabner [5]; we refer to [4] for a general discussion of the underlying physical and chemical assumptions.

With the scaling properties of Problem I in mind we only consider the cases

$$\text{Contamination process:} \quad u_e = 1,\ u_0(\infty) = 0, \tag{1.1}$$

and

$$\text{Remediation process:} \quad u_e = 0,\ u_0(\infty) = 1, \tag{1.2}$$

where we suppose that $u_0(\infty) := \lim_{r\to\infty} u_0(r)$ exists. About the function β we shall make the following assumptions (cf. [4]):

(H$_\beta$1) $\beta \in C^\infty(0,\infty) \cap C([0,\infty))$;
(H$_\beta$2) $\beta(0) = 0,\ \beta'(s) > 0$ and $\beta''(s) \le 0$ for $s > 0$;

The canonical example of functions β that we have in mind is $\beta(u) = u + \psi(u)$, where ψ is the so-called adsorption isotherm [4]. Two typical isotherms are the Langmuir isotherm

$$\psi(u) = \frac{k_1 u}{1 + k_2 u}, \qquad k_1, k_2 > 0,$$

and the Freundlich isotherm

$$\psi(u) = k u^p, \qquad k > 0,\ 0 < p < 1.$$

In the absence of convection, i.e. $\lambda = 0$, equation (Ia) reduces to the radially symmetric version of the nonlinear diffusion equation

$$\beta(u)_t = \Delta u. \tag{1.3}$$

If $\beta'(\cdot)$ is not bounded from above, this equation is degenerate parabolic: the diffusion coefficient vanishes when u vanishes. In particular, when $1/\beta(\cdot)$ is integrable at zero, the equation is similar to the well-known porous media equation

$$u_t = \phi(u)_{xx}, \tag{1.4}$$

having the property that compactly supported initial data gives rise to solutions with compact support for all time. This is the case with a Freundlich isotherm for all $0 < p < 1$.

A Cauchy-Dirichlet problem for equation (Ia) with $\lambda > 1$ was studied by Goncerzewicz [9], generalising results by Gilding [8] and Díaz and Kersner [3] who considered general nonlinear one-dimensional convection-diffusion equations without r-dependence. Following these authors we introduce weak solutions in the following sense. Let T be some fixed positive number, which eventually will tend to infinity, and consider the half strip $S_T^\varepsilon = \{(r,t) : \varepsilon < r < \infty,\ 0 < t < T\}$.

Definition. A non-negative function $u : \overline{S_T^\varepsilon} \to \mathbf{R}$ is called a weak solution of Problem I if
(i) $u \in C(\overline{S_T^\varepsilon})$ and u has a bounded weak derivative u_r in S_T^ε;
(ii) for every test function $\varphi \in H^1(S_T^\varepsilon)$ that vanishes for large r and at $t = T$,

$$\int_{S_T^\varepsilon} \{\beta(u)\varphi_t r + (\lambda u - r u_r)\varphi_r\}\, dr dt + \int_\varepsilon^\infty \beta(u_0(r))\varphi(r,0)\, r dr + \lambda u_e \int_0^T \varphi(\varepsilon, t)\, dt = 0. \tag{1.5}$$

If u satisfies (1.5) with the equality replaced by $\geq$ ($\leq$) and with $\varphi \geq 0$ in S_T^ε then we call u a sub(super)solution.

Observe that the parameter λ specifies the two-dimensional convection: the water flow is outward when $\lambda > 0$, and zero when $\lambda = 0$. However, when equation (Ia) is interpreted as a one-dimensional convection-diffusion equation the corresponding convection term $(\lambda - 1)r^{-1}u_r$ implies a flow which is directed towards the origin when $\lambda < 1$ and away from the origin when $\lambda > 1$. This distinction will turn out to be important when studying the asymptotic behaviour as $\varepsilon \downarrow 0$.

Besides the conditions on β we will assume concerning u_0 that

(A1) $u_0 \in C^{0,1}([\varepsilon,\infty))$, such that $ru_0'(r)$ is uniformly bounded on (ε,∞).

Together with hypotheses H_β1-2, hypothesis A1 ensures the existence of a unique weak solution u which is smooth in the set $\{u>0\}$. This is proved in Section 2.

Our aim is to show that under certain conditions solutions of Problem I converge to self-similar solutions when either $\varepsilon \downarrow 0$ or $t \to \infty$. The combination of these two limit processes is explained by the following transformation:

$$x = \frac{r}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon^2}$$

under which Problem I becomes

$$\mathrm{II}\begin{cases} \beta(u)_\tau + \dfrac{\lambda-1}{x}u_x - u_{xx} = 0 & x>1,\ \tau>0 \quad \text{(IIa)} \\ u_x = \lambda(u-u_e) & x=1,\ \tau>0 \quad \text{(IIb)} \\ u(x,0) = u_0(\varepsilon x) & x>1. \quad \text{(IIc)} \end{cases}$$

Obviously, the behaviour of solutions of Problem II for large τ is strongly linked to that of solutions of Problem I for either $\varepsilon \downarrow 0$ or $t \to \infty$.

A scaling argument leads us to investigate self-similar solutions of equation (Ia) of the form

$$u(r,t) = f(\frac{r}{\sqrt{t}}) = f(\eta), \tag{1.6}$$

satisfying the equation

$$\tfrac{1}{2}\eta^2\{\beta(f)\}' + (\eta f' - \lambda f)' = 0, \tag{1.7}$$

where primes denote differentiation with respect to η. Since these self-similar solutions are expected to arise in the limit $\varepsilon \downarrow 0$, we solve equation (1.6) in the domain $0<\eta<\infty$ with the combinations (1.1) and (1.2) as boundary conditions:

$$\text{Contamination process:} \quad f(0)=1,\ f(\infty)=0, \tag{1.8}$$

$$\text{Remediation process:} \quad f(0)=0,\ f(\infty)=1. \tag{1.9}$$

Note that $\eta = r/\sqrt{t} = x/\sqrt{\tau}$, and therefore the self-similar solution satisfies both equation (Ia) and equation (IIa).

The boundary value problems (1.7)-(1.8) and (1.7)-(1.9) are studied in Section 3. Finally, in Section 4 we prove the main results of this paper. They are concerned with the asymptotic behaviour of weak solutions of Problem I. We shall need some additional hypotheses on u_0 and β in order to prove these results. They are:

(A2) $u_0(\infty) := \lim\limits_{r\to\infty} u_0(r)$ exists, satisfying either (1.1) or (1.2);

(A3) $\left|\displaystyle\int_\varepsilon^\infty r\big\{\beta(u_0(r)) - \beta(u_0(\infty))\big\}\,dr\right| < \infty;$

Hypothesis A3 can be interpreted physically as stating that the perturbation $u_0 - u_0(\infty)$ of the constant state $u_0(\infty)$ has finite mass. We will show in Section 2 that the same holds at every time $t > 0$. The double degeneration of (1.7) with (1.9)—the degeneration of $\beta(f)$ at $f = 0$ and the degeneration of the equation at $\eta = 0$, which coincide—forces us to assume two technical hypotheses in order to prove the result for the remediation process:

(A4) $\quad \sup_{r\geq\varepsilon} u_0(r) \leq 1.$

($H_\beta 3$) $\quad \lim_{s\downarrow 0} \dfrac{s\beta''(s)}{\beta'(s)} = p - 1$ where $0 < p \leq 1$.

Note that in the case of a Freundlich isotherm the condition $H_\beta 3$ is satisfied with $0 < p < 1$, and in the case of a Langmuir isotherm with $p = 1$.

The precise asymtotic statements are:

Theorem A. *Let hypotheses H_β1-2 and A1-3 be satisfied. Further let u be the solution of Problem I with $u_e = 1$ and $u_0(\infty) = 0$ (contamination process), and let f denote the solution of (1.7) and (1.8).*

(a) If ε is fixed, then there exists a constant $C_1 > 0$, depending on β, λ, u_0, and ε, such that

$$\sup_{\varepsilon\leq r<\infty} |u(r,t) - f(r/\sqrt{t})| \leq \frac{C_1}{t^k},$$

for all $t > 0$.

(b) If u_0 is constant, and therefore identically equal to $u_0(\infty) = 0$, then there exists a constant $C_2 > 0$, depending only on β and λ, such that

$$\sup_{\varepsilon\leq r<\infty} |u(r,t) - f(r/\sqrt{t})| \leq C_2\left(\frac{\varepsilon^2}{t}\right)^k,$$

for all $t > 0$ and $\varepsilon > 0$.

Here the exponent k is given by

$$k = \begin{cases} \lambda/3 & \text{for } \lambda < 1, \\ 1/3 & \text{for } \lambda \geq 1. \end{cases}$$

Theorem B. *Let hypotheses H_β1-3 and A1-4 be satisfied. Further let u be the solution of Problem I with $u_e = 0$ and $u_0(\infty) = 1$ (remediation process), and let f denote the solution of (1.7) and (1.9). Then the conclusions are the same as those of Theorem 4.1.*

The restriction to constant u_0 when ε is varied is a natural one. Since the influence of changes in ε on the solution is very small at a fixed time and far away from the origin, it is necessary for convergence to self-similar solutions that the behaviour at small times and away from the well corresponds to the behaviour of the self-similar solution. In practical terms, this means u_0 has to be constant. Remark that when u_0 is constant, the two limit processes $\varepsilon \downarrow 0$ and $t \to \infty$ are truly equivalent.

2. Weak solutions: existence and uniqueness.

We present here the existence and uniqueness results for weak solutions of Problem I. Most of these results are obtained by a straightforward generalisation of the work of Díaz and Kersner [3], Gilding [8], and Goncerzewicz [9]. In those cases we omit the details and only give the appropriate references. However, special attention has to be given to the flux boundary condition at $r = \varepsilon$.

As usual weak solutions are obtained as limits of approximating positive classical solutions. Since these approximations are used later on in this paper to prove the asymptotic results, we describe the procedure in some detail in the existence proof below.

Theorem 2.1. *Let hypotheses H_β1-2 and A1 be satisfied. Then Problem I has a unique weak solution.*

Proof. To show existence, we slightly alter the initial and boundary conditions in Problem I, ensuring that the corresponding solution remains strictly positive. This is achieved by considering the approximating sequences $\{u_{0n}\}_{n\geq 1}$ and $\{u_{en}\}_{n\geq 1}$, satisfying

$$u_{0n} \in C^\infty([\varepsilon, n]);$$
$$u_{0n} \downarrow u_0 \quad \text{as } n \to \infty, \text{ uniformly on bounded subsets of } [\varepsilon, \infty);$$
$$\frac{1}{n} \leq u_{0n} \leq M := ||u_0||_{L^\infty(\varepsilon,\infty)}; \tag{2.1}$$
$$\sup_{\varepsilon\leq r\leq n} |r{u_{0n}}'(r)| \leq \sup_{\varepsilon\leq r<\infty} |ru_0'(r)|; \tag{2.2}$$
$$u_{0n}(r) = M \quad \text{for } n-1 \leq r \leq n;$$
$${u_{0n}}'(\varepsilon) = 0,$$

and

$$u_{en}(t) = u_e - (u_e - u_{0n}(\varepsilon))e^{-nt}.$$

Note that the compatibility condition

$$u_{en}(0) = u_{0n}(\varepsilon)$$

is satisfied. In Problem I we now replace u_0 by u_{0n} and u_e by u_{en}. This yields the approximate problems

$$\mathrm{I}_n \begin{cases} \beta(u)_t + \dfrac{\lambda - 1}{r} u_r - u_{rr} = 0 & \text{in } S_T^n = (\varepsilon, n) \times (0, T], \\ u_r = \dfrac{\lambda}{\varepsilon}(u - u_{en}(t)) & \text{at } r = \varepsilon,\ t \in (0, T], \\ u = M & \text{at } r = n,\ t \in (0, T], \\ u = u_{0n} & \text{at } t = 0,\ r \in [\varepsilon, n], \end{cases}$$

for $n \geq 1$. Existence and uniqueness of solutions to this problem are classical and can be found in e.g. [11], page 491. The solutions obtained have the regularity $C^\infty(S_T^n) \cap C^{2+\alpha,1+\alpha/2}(\overline{S_T^n})$.

In order to obtain an estimate on the spatial derivative of the solutions u_n, we derive an equation for the flux

$$F_n = \lambda u_n - r u_{nr}.$$

The functions u_n satisfy the equation

$$\beta'(u_n)u_{nt} + \tfrac{1}{r}F_{nr} = 0. \tag{2.3}$$

Differentiating this equation with respect to r yields for F_n the uniformly parabolic equation

$$\beta'(u_n)F_{nt} = F_{nrr} - F_{nr}\{\frac{\lambda+1}{r} + \frac{\beta''(u_n)}{\beta'(u_n)}u_{nr}\},$$

Using (2.3) once more we find the boundary conditions

$$\begin{aligned} F_n &= \lambda u_{en}, \qquad && r = \varepsilon; \\ F_{nr} &= 0, && r = n. \end{aligned}$$

Hypothesis A1 and properties (2.1) and (2.2) of the functions u_{0n} imply that F_n is bounded uniformly in n at $t = 0$. By the maximum principle, the same then holds for F_n on the whole of $\overline{S_T^n}$. Then we have finally

$$\sup_{\overline{S_T^n}} |u_{nr}| \leq L \qquad \text{for all } n.$$

Next we investigate the regularity in time. We first consider the behaviour of u_n at the boundary $r = \varepsilon$.

Lemma 2.2. *There exists a constant c, independent of n, such that*

$$|u_n(\varepsilon, t_2) - u_n(\varepsilon, t_1)| \leq c|t_2 - t_1|^{\frac{1}{4}} \qquad \textit{for all} \qquad t_1,\, t_2 \in [0, T].$$

Proof. Suppose $t_2 > t_1$ and pick $R > \varepsilon$. We multiply equation (Ia) by $u_n(r, t_2) - u_n(r, t_1)$ and integrate over the rectangle $[\varepsilon, R] \times [t_1, t_2]$. By using the bound on F_n we find that there is a constant c, independent of n and R, such that

$$\int_\varepsilon^R \big(\beta(u_n(r, t_2)) - \beta(u_n(r, t_1))\big)\big(u_n(r, t_2) - u_n(r, t_1)\big)\, r dr \leq c(t_2 - t_1).$$

If $\sup\{1/\beta'(s) : s \in (0, M]\} = b$, then this implies

$$\int_\varepsilon^R |u_n(r, t_1) - u_n(r, t_2)|^2\, r dr \leq bc(t_2 - t_1).$$

Now we use the Lipschitz continuity of u_n with respect to r to conclude that for large enough R we have

$$|u_n(r, t_1) - u_n(r, t_2)|^4 \leq 2L\varepsilon^2 bc(t_2 - t_1),$$

for all $r \in [\varepsilon, R]$, and in particular at $r = \varepsilon$. Here L is the Lipschitz constant of u_n. We conclude the proof by remarking that L, b, and c are all independent of n. $\bullet$

The regularity result of Gilding [6] then yields that

$$||u_n||_{C^{0+1,0+1/4}(\overline{S_T^n})} \leq C \quad \text{for all } n.$$

This suffices to apply the Arzela-Ascoli theorem and conclude that there exists a subsequence that converges uniformly on compact subsets of $\overline{S_T^\varepsilon}$. By a familiar argument (see e.g. Oleinik [12], p. 361) the limit function u can be shown to be a weak solution of Problem I. This concludes the proof of the existence.

The uniqueness follows directly from the

Proposition 2.3 (Comparison Principle). *Let u^1 be a subsolution and u^2 be a supersolution of Problem I, with initial values u_0^1 and u_0^2, and boundary conditions at $r = \varepsilon$:*

$$u_r^1 \geq \frac{\lambda}{\varepsilon}(u^1 - u_e^1) \quad \text{and} \quad u_r^2 \leq \frac{\lambda}{\varepsilon}(u^2 - u_e^2).$$

If $u_0^1 \leq u_0^2$ on $[\varepsilon, \infty)$ and $u_e^1 \leq u_e^2$, then $u^1 \leq u^2$ on S_T^ε.

The proof of Proposition 2.3 is a simple extension of the proof in Goncerzewicz [10], and follows the ideas of Díaz and Kersner [3]. This concludes the proof of Theorem 1.1. $\bullet$

We conclude this section with a property of solutions of Problem I that is crucial for the large-time behaviour.

Proposition 2.4. *Let u be a solution of Problem I. Then*

$$\int_\varepsilon^\infty \left\{\beta(u(r,t)) - \beta(u_0(\infty))\right\} rdr = \int_\varepsilon^\infty \left\{\beta(u_0(r)) - \beta(u_0(\infty))\right\} rdr + \lambda t(u_e - u_0(\infty)).$$

This can be interpreted as stating that the only increase of 'mass'—in the case of the model described in the Introduction, this would be mass of contaminant—comes from the injection at the boundary. A proof of this statement for the porous media equation (1.4) can be found in [7]; the proof of Proposition 2.4 follows along the same lines.

3. Limit Profiles.

Equation (Ia) admits self-similar solutions of the form

$$u(r,t) = f(\eta) \quad \text{with} \quad \eta = \frac{r}{\sqrt{t}},$$

satisfying the equation

$$\tfrac{1}{2}\eta^2\{\beta(f)\}' + (\eta f' - \lambda f)' = 0, \tag{3.1}$$

where primes denote differentiation with respect to η. Anticipating on the limit $\varepsilon \downarrow 0$ we seek solutions f of (3.1) on the domain $0 < \eta < \infty$. The Contamination event and the Remediation event introduced in the Introduction correspond to the following boundary conditions on f:

$$\text{Contamination event:} \quad f(0) = 1, \; f(\infty) = 0, \tag{1.8}$$

$$\text{Remediation event:} \quad f(0) = 0, \; f(\infty) = 1. \tag{1.9}$$

For any $a, b \in [0, 1]$ we will denote by $\mathrm{P}(a, b)$ the problem

$$\mathrm{P}(a,b) \begin{cases} \frac{1}{2}\eta^2\{\beta(f)\}' + (\eta f' - \lambda f)' = 0, & 0 < \eta < \infty \\ f(0) = a, \quad f(\infty) = b. \end{cases}$$

This section is devoted to the study of $\mathrm{P}(1,0)$ and $\mathrm{P}(0,1)$. We first prove existence and uniqueness of solutions of $\mathrm{P}(a,b)$ and then enter more deeply into the specific cases $\mathrm{P}(0,1)$ and $\mathrm{P}(1,0)$. Some of the proofs will only be sketched; the reader can find comprehensive and detailed studies of Problem $\mathrm{P}(1,0)$ in [5] and of Problem $\mathrm{P}(0,1)$ in [14].

Because of the degeneration of the equation when $f = 0$, we must again define the notion of a solution of this problem. We introduce the negative flux function

$$F(\eta) = \eta f'(\eta) - \lambda f(\eta), \quad \eta > 0. \tag{3.2}$$

Definition 3.1. *Let a, $b \in [0, 1]$. A function $f : [0, \infty) \to [0, 1]$ is called a solution of the problem $P(a, b)$ if*
(i) F and $\beta(f)$ are absolutely continuous on $[0, \infty)$;
(ii)

$$F' + \frac{1}{2}\eta^2\{\beta(f)\}' = 0 \quad \text{a.e. on } (0, \infty); \tag{3.3}$$

(iii)

$$f(0) = a \quad \text{and} \quad f(\infty) = b. \tag{3.4}$$

We can directly deduce from this definition

Proposition 3.1. *Let f be a solution of problem $P(a,b)$ and let P be the positivity set $\{\eta > 0 : f(\eta) > 0\}$. Then*
(i) $f \in C^1((0,\infty)) \cap C^\infty(P)$;
(ii) f is monotone, and $f' \neq 0$ on P unless $a = b$;
(iii) $F(\eta) \to -\lambda b$ as $\eta \to \infty$.

Proof. (i) The continuity of F and f imply that $f \in C^1((0,\infty))$ and that P is open. Now take any point $\eta_0 \in P$, and a neighbourhood N of η_0 such that f is bounded away from zero on N. Then $F \in C^1(N)$ by equation (3.3), which combined with (3.2) implies that $f \in C^2(N)$. By induction the result follows.

(ii) This follows from a local uniqueness argument as in [1,2].

(iii) From (3.2), (3.3), and the monotonicity of f we find that $\lim_{\eta\to\infty} F(\eta)$ exists. But any limit value other than $-\lambda b$ contradicts the existence of the limit of f. •

We have the following comparison principle.

Proposition 3.2 (Comparison Principle). *Let f_i, for $i = 1, 2$, be solutions of the following problems:*

$$\begin{cases} \frac{1}{2}\eta^2\{\beta_i(f_i)\}' + (\eta f_i' - \lambda f_i)' = 0, & 0 < \eta < \infty \\ f_i(0) = a_i, \quad f_i(\infty) = b_i \end{cases}$$

where the functions β_i satisfy hypotheses H_β1-2, and $a_i, b_i \in [0,1]$. Suppose that
(i) $\beta_1'(s) \le \beta_2'(s)$ for all $s \in [0,1]$;
(ii) $a_1 \le a_2$ and $b_1 \le b_2$.
Then $f_1 \le f_2$ on $[0,\infty)$.

Proof. Denote the positivity sets of the functions f_1 and f_2 by P_1 and P_2. Suppose that the difference $v = f_1 - f_2$ assumes a positive maximum at $\eta_0 \in (0,\infty)$. Then $f_1(\eta_0) > 0$ so that $\eta_0 \in P_1$, which implies that f_1 is twice continuously differentiable in η_0.

- If $\eta_0 \in P_2$ then f_2 also is twice differentiable in η_0, and then the result follows from subtracting the equations for f_1 and f_2 at $\eta = \eta_0$.
- If $\eta_0 \in \mathbf{R} \setminus P_2$ then $f_2'(\eta_0) = 0$, which implies $f_1'(\eta_0) = 0$. From Proposition 3.1 follows that this only is possible when $a_1 = b_1$ and f_1 is constant on P_1. Because $f_1(\eta_0) > 0$ we have $a_1 = b_1 > 0$, and the boundary conditions then imply that f_2 is not monotone. This contradicts Proposition 3.1. •

Corollary 3.3. *For every $a, b \in [0,1]$, a solution of Problem P(a,b) is unique.*

For the proof of the existence of a solution we shall need an a priori estimate of the rate of convergence of solutions to their value at infinity.

Proposition 3.4. *Let f be a solution of Problem P(a,b). Then there exist positive constants η_0, C, and K, such that*

$$|b - f(\eta)| \le C\int_{\frac{1}{\lambda}\eta^\lambda}^{\infty} e^{-K\sigma^{2/\lambda}}\, d\sigma$$

for all $\eta > \eta_0$.

The proof consists of an estimation of $\beta'(t)$ near $t = b$, and the reader can find it in [14]. Note that this implies that

$$\begin{aligned} |b - f(\eta)| &\le C\int_{\frac{1}{\lambda}\eta^\lambda}^{\infty} e^{-K\sigma^{2/\lambda}}\, d\sigma \\ &\le C\int_{\frac{1}{\lambda}\eta^\lambda}^{\infty} \frac{2K}{\lambda}\sigma^{2/\lambda-1} e^{-K\sigma^{2/\lambda}}\, d\sigma \\ &= Ce^{-K\lambda^{-2/\lambda}\eta^2} \end{aligned}$$

if η is large enough.

Proposition 3.5. *For every $a, b \in [0,1]$, Problem P(a,b) has a unique solution.*

Proof. First we introduce a change of variables that we shall also use later on in this section. Set

$$s = \tfrac{1}{\lambda}\eta^\lambda \quad \text{and} \quad g(s) = f(\eta).$$

Then Problem P(a,b) can be written as

$$\mathrm{P}_g(a,b) \begin{cases} g'' + \mu s^\alpha \{\beta(g)\}' = 0 & \text{for} \quad 0 < s < \infty \\ g(0) = a, \quad g(\infty) = b \end{cases}$$

in which this time differentiation is with respect to the independent variable s, and the constants α and μ are given by

$$\alpha = \frac{2}{\lambda} - 1 \quad \text{and} \quad \mu = \tfrac{1}{2}\lambda^\alpha.$$

A solution of $\mathrm{P}_g(a,b)$ is defined in a sense similar to the case of Problem P(a,b), and it can easily be verified that the two problems are equivalent.

If both a and b are positive, then by the Comparison Principle any solution of Problem P(a,b) will take values between a and b. Therefore the problem is non-degenerate and the existence of a solution to the boundary value problem P(a,b) is shown by standard methods. In the rest of this proof we will suppose that $a = 0$ and $b > 0$, and merely assert that the other case, $b = 0$ and $a > 0$, can be handled in a totally analogous way.

We will construct a solution of $\mathrm{P}_g(0,b)$ as the limit as $\varepsilon \downarrow 0$ of solutions of $\mathrm{P}_g(\varepsilon,b)$. For $\varepsilon > 0$ the solution of $\mathrm{P}_g(\varepsilon,b)$ is defined and unique, and by the Comparison Principle the sequence $\{g_\varepsilon\}$ depends monotonically on ε. We will now show that the pointwise limit of this sequence, denoted by g, is a solution of Problem $\mathrm{P}_g(a,b)$. By twice integrating the equation in $\mathrm{P}_g(\varepsilon,b)$ we find the following integral identity for g_ε:

$$g_\varepsilon(s) = b - \mu \int_s^\infty \big[(1+\alpha)\sigma - \alpha s\big]\sigma^{\alpha-1}\big\{\beta(b) - \beta(g_\varepsilon(\sigma))\big\}\,d\sigma \tag{3.5}$$

for all $s \in [0,\infty)$. The finiteness of the integral follows from the exponential convergence proved in Proposition 3.4. Now let ε tend monotonically to zero. Since $g_\varepsilon \downarrow g$, and therefore $(\beta(b) - \beta(g_\varepsilon)) \uparrow (\beta(b) - \beta(g))$ on $[0,\infty)$, we can apply the monotone convergence theorem to the integral in (3.5) to conclude that it converges; the positivity of the left-hand side implies that the limit is finite. This results in the same integral equality, but this time for the limit function g:

$$g(s) = b - \mu \int_s^\infty \big[(1+\alpha)\sigma - \alpha s\big]\sigma^{\alpha-1}\big\{\beta(b) - \beta(g(\sigma))\big\}\,d\sigma \tag{3.6}$$

for all $s \in [0,\infty)$. Starting with (3.6) and differentiating twice we can show that g is a solution to problem $\mathrm{P}_g(0,b)$. This implies that the corresponding function f is a solution of Problem $\mathrm{P}(0,b)$. $\bullet$

We now shall derive some precise information about the behaviour of the solutions of the problems $\mathrm{P}(0,1)$ and $\mathrm{P}(1,0)$. We will consider the two cases separately.

Problem $\mathbf{P}(1,0)$.

From Proposition 3.1 we know that $f' < 0$ on P. Together with the left boundary condition $f(0) = 1$ this implies that $P = [0,L)$ for some $L \in (0,\infty]$. If $1/\beta(s)$ is integrable at $s = 0$, we can define the function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$,

$$\psi(s) = \int_0^s \frac{1}{\beta(\sigma)}\,d\sigma.$$

Proposition 3.6. *Let f be the solution of $P(1,0)$. Then $L < \infty$ if and only if $1/\beta(s)$ is integrable at $s = 0$; in that case, $\lim_{\eta\uparrow L}\{\psi(f(\eta))\}' = -\frac{1}{2}L$.*

The proof of Proposition 3.6 can be found in [5].

Proposition 3.7. *The limit $\lim_{\eta\downarrow 0}\eta^{1-\lambda}f'(\eta)$ exists in $(-\infty,0)$.*

Proof. On the interval $(0,L)$ we may write the differential equation as

$$\frac{f''}{f'} = \frac{\lambda-1}{\eta} - \tfrac{1}{2}\eta\beta'(f(\eta)).$$

Integrating this expression gives

$$\eta_1^{1-\lambda}f'(\eta_1) = \eta_0^{1-\lambda}f'(\eta_0)e^{-\frac{1}{2}\int_{\eta_0}^{\eta_1} y\beta'(f(y))dy} \qquad \text{for} \qquad 0 < \eta_0 < \eta_1 < L,$$

which implies the existence of the limit (fix $\eta_1 > 0$ and let $\eta_0 \downarrow 0$). $\bullet$

Problem $\mathbf{P}(0,1)$.

By Proposition 3.1 the solution of Problem $\mathrm{P}(0,1)$ will be strictly increasing on its positivity set P. But, in contrast to the preceding case, there can be no free boundary:

Proposition 3.8. *Let f be the solution of Problem $P(0,1)$. Then $f > 0$ on $(0,\infty)$.*

Proof. We will use the formulation of the problem in terms of g, as introduced in the proof of Proposition 3.5 (Problem $\mathrm{P}_g(0,1)$). The monotonicity of f on P implies the monotonicity of g on its positivity set P_g, and using the equation for g this in turn implies that $g'' < 0$ on P_g. Therefore, if $\ell = \inf P_g > 0$, then $g'(\ell+) > 0$. On the other hand, $g \equiv 0$

on $(0,\ell)$, and $g'(\ell-)=0$. We find a contradiction with the differentiability of f and g on $(0,\infty)$. •

Equation (3.1) has two elements that distinguish it from a run-of-the-mill ordinary differential equation: the nonlinearity β and the degeneration of the equation at $\eta=0$. We encounter these two elements when describing the behaviour of solutions. For Problem P$(1,0)$, by Proposition 3.7 the behaviour of f near $\eta=0$ can be characterised in terms of λ, which can intuitively be understood by writing the equation in the following form:

$$\tfrac{1}{2}\eta\beta(f)'+f''-\frac{\lambda-1}{\eta}f'=0.$$

At $\eta=L$, the degeneration of β becomes predominant, and therefore it is not surprising to find (by Proposition 3.6) that properties of the solution are expressed in terms of ψ, which is derived from β.

In the case of Problem P$(1,0)$, the boundary condition $f(0)=0$ implies that both degenerations take place at $\eta=0$. We therefore see a tug-of-war between λ and β over which should prevail. The overall picture is the following: there is a critical value of λ, denoted by λ_c, such that

- If $\lambda<\lambda_c$, then $f(\eta)\sim\eta^\lambda$ as $\eta\to0$;
- If $\lambda>\lambda_c$, then $f(\eta)\sim\eta^{\lambda_c}$ as $\eta\to0$.

The value of λ_c is uniquely determined by β. We shall next make these claims precise.

In order to be able to make definite statements we must assume the extra hypothesis on β

$$(\mathrm{H}_\beta 3)\qquad \lim_{f\downarrow0}\frac{f\beta''(f)}{\beta'(f)}=p-1$$

for some constant $p\in(0,1]$. This condition expresses that for small data β behaves essentially as a power with exponent p.

For a nonlinearity β in the form of $\beta(f)=cf^p$, the equation has certain scaling properties that allow us to transform it into an autonomous one, and then apply a phase plane analysis. The details of this analysis can be found in Appendix A, and here we shall merely state the results.

Proposition 3.9. *Let f be the solution of P$(0,1)$, where $\beta(f)=cf^p$ for some $p\in(0,1)$ and $c>0$, and let μ be given, as in Proposition 3.5, by $\mu=\frac{1}{2}\lambda^{\frac{2}{\lambda}-1}$.*

(a) If $\lambda<2/(1-p)$, then the limit

$$\lim_{\eta\downarrow0}\frac{f(\eta)}{\eta^\lambda}$$

exists and is strictly positive.

(b) If $\lambda=2/(1-p)$, then

$$\lim_{\eta\downarrow0}\frac{f^{1-p}(\eta)}{\eta^2\log\eta}=-2c\mu p\lambda^{1-p}.$$

(c) If $\lambda > 2/(1-p)$, then

$$\lim_{\eta \downarrow 0} \frac{f(\eta)}{\eta^{\frac{2}{1-p}}} = \frac{A}{\lambda^k},$$

in which $k = 2/(\lambda(1-p))$ and

$$A = (\frac{c\mu p}{1-k})^{\frac{1}{1-p}}.$$

For more general nonlinearities β the analysis is more involved, and the results less precise. We conclude with

Proposition 3.10. *Let f be the solution of Problem $P(0,1)$, where β satisfies not only $(H_\beta 1)$ and $(H_\beta 2)$ but also $(H_\beta 3)$ for some $p \in (0,1]$. Then*

(a) If $\lambda < \dfrac{2}{1-p}$ then $\lim_{\eta \downarrow 0} f(\eta)\eta^{-\lambda}$ exists (as a positive number);

(b) If $\lambda \geq \dfrac{2}{1-p}$ then $\lim_{\eta \downarrow 0} \eta^2 \beta'(f(\eta)) = 2\left(\lambda - \dfrac{2}{1-p}\right)$.

In this statement, the number $2/(1-p)$ should be replaced by ∞ when $p = 1$.

Proof. We shall need the following auxillary function.

Definition. *Let φ be a real-valued function defined on an interval $(0, \delta)$ for some $\delta > 0$. Then the index $\nu(\varphi)$ is defined as*

$$\nu(\varphi) = \lim_{x \downarrow 0} \frac{x\varphi'(x)}{\varphi(x)},$$

provided that the limit exists.

If φ is a power of its argument, $\nu(\varphi)$ simply is the exponent. One can derive some properties of ν which extend this correspondance: if φ and ψ are such that $\nu(\varphi)$ and $\nu(\psi)$ are defined, then

$$\begin{aligned}
\nu(c\varphi) &= \nu(\varphi) \quad \text{for all } c \neq 0 \\
\nu(\varphi\psi) &= \nu(\varphi) + \nu(\psi) \quad \text{provided} \quad \psi(0) = 0 \\
\nu(\varphi^\alpha) &= \alpha\nu(\varphi) \quad \text{for all } \alpha \in \mathbf{R} \\
\nu(\varphi \circ \psi) &= \nu(\varphi)\nu(\psi) \\
\varphi \in L^1(0,\delta) &\implies \nu(\varphi) \geq -1 \\
\varphi \notin L^1(0,\delta) &\implies \nu(\varphi) \leq -1
\end{aligned}$$

Besides, by de l'Hôpital's rule, the existence of $\nu(\varphi')$ implies that $\nu(\varphi)$ exists and that

$$\nu(\varphi) = 1 + \nu(\varphi'). \tag{3.7}$$

Note that with this notation assumption ($\mathrm{H}_\beta 3$) can be written as $\nu(\beta') = p - 1$.

We return to the proof of the theorem. If we write

$$\gamma(s) = \frac{sg'(s)}{g(s)} \quad \text{and} \quad \delta(s) = \mu s^{\alpha+1}\beta'(g(s)),$$

then γ and δ satisfy the system

$$\begin{cases} s\gamma' = \gamma(1-\gamma-\delta) \\ s\delta' = \delta(\alpha+1+\zeta(s)\gamma) \end{cases}$$

where

$$\zeta(s) = \frac{g(s)\beta''(g(s))}{\beta'(g(s))}.$$

By assumption $\zeta(s) \to p-1$ as $s \downarrow 0$. After a simple transformation (such as $\sigma = \log s$) this system is asymptotically autonomous in the sense of Thieme [16]. We wish to apply a theorem of Poincare-Bendixson type (Theorem 1.6 of the same reference) to conclude that (γ, δ) tends to an equilibrium of the 'limit' system

$$\begin{cases} s\gamma' = \gamma(1-\gamma-\delta) \\ s\delta' = \delta(\alpha+1+(p-1)\gamma), \end{cases} \tag{3.8}$$

as $s \downarrow 0$. The only condition thereto still to be verified is that the orbit under consideration is bounded as $s \downarrow 0$.

Indeed, first remark that from the concaveness of g it follows that

$$\left(\frac{g}{g'}\right)' = 1 - \frac{g''g}{g'^2} \geq 1$$

and therefore

$$g(s) \geq sg'(s) \quad \text{for all} \quad s > 0,$$

which implies $0 \leq \gamma \leq 1$ for all $s > 0$. Since δ is positive, the orbit (γ, δ) can only be unbounded in the positive δ-direction. To show that this is not the case, note that since $\alpha + 1 > 0$ and $\zeta(s) \to p-1$ when $s \downarrow 0$, there exists an $\varepsilon > 0$ such that when $\gamma < \varepsilon$, δ' is positive. On the other hand, when $\gamma > \varepsilon$ we can write

$$\frac{\delta'}{\gamma'} = \frac{\delta(\alpha+1+\zeta(s)\gamma)}{\gamma(1-\gamma-\delta)}$$

and the right-hand side of this expression is bounded from above and below when δ is large. The two observations combined imply that δ is bounded from above. As a result we can apply Thieme's theorem to conclude that the orbit (γ, δ) tends to an equilibrium of the limit system (3.8).

This system has the equilibria $e_0 = (0,0)$ and $e_1 = (1,0)$, and if $\alpha + p < 0$ also

$$e_2 = \left(\frac{\alpha+1}{1-p}, -\frac{\alpha+p}{1-p}\right).$$

Of these the first, $(0,0)$, can be quickly ruled out: by definition $\nu(g) = \lim_{s\downarrow 0}\gamma(s)$, and by writing the equation for g in $\mathrm{P}_g(a,b)$ as

$$\frac{sg''(s)}{g'(s)} = -\delta(s)$$

we see that $\nu(g') = \lim_{s\downarrow 0}\delta(s)$. Consequently $(\gamma,\delta)\to(0,0)$ implies on one hand $\nu(g') = 0$ and on the other $\nu(g) = 0$; this is incompatible due to (3.7). For the other two equilibria, we distinguish between three cases:

- when $\alpha + p < 0$, the equilibrium $(1,0)$ is instable (in backward time) and is therefore not admissible; it follows that $(\gamma,\delta)\to e_2$ as $s\downarrow 0$, and more specifically $\delta(s)\to -(\alpha+p)/(1-p)$;
- when $\alpha + p = 0$, $e_1 = e_2$ and therefore $\delta(s)\to 0$ as $s\downarrow 0$;
- when $\alpha + p > 0$, e_1 is the only admissible equilibrium and therefore $\nu(g) = 1$; we conclude from
$$\nu\big(s^\alpha\beta'(g(s))g'(s)\big) = \alpha + p - 1 > -1,$$
that g'' is integrable; as a result, $g'(0+)$ is finite.

We can rearrange this information in the following form:

- When $\alpha + p > 0$, $\lim_{s\downarrow 0} g'(s)$ is finite;
- When $\alpha + p \le 0$, $\mu s^{\alpha+1}\beta'(g(s)) \to \dfrac{\alpha+p}{1-p}$.

In terms of λ, f, and η, this is the statement of the theorem. •

4. The Main Result.

We shall prove the following theorems.

Theorem 4.1. *Let u be the solution of Problem I, $u_e = 1$, and $u_0(\infty) = 0$. Let f denote the self-similar solution of Problem $P(1,0)$, obtained in the preceding section.*

(a) If ε is fixed, then there exists a constant $C_1 > 0$, depending on β, λ, u_0, and ε, such that

$$\sup_{\varepsilon\le r<\infty} |u(r,t) - f(r/\sqrt{t})| \le \frac{C_1}{t^k},$$

for all $t > 0$.

(b) If u_0 is constant, and therefore identically equal to $u_0(\infty) = 0$, then there exists a constant $C_2 > 0$, depending only on β and λ, such that

$$\sup_{\varepsilon\le r<\infty} |u(r,t) - f(r/\sqrt{t})| \le C_2\left(\frac{\varepsilon^2}{t}\right)^k,$$

for all $t > 0$ and $\varepsilon > 0$.

Here the exponent k is given by

$$k = \begin{cases} \lambda/3 & \text{for } \lambda < 1, \\ 1/3 & \text{for } \lambda \geq 1. \end{cases}$$

Theorem 4.2. *Let u be the solution of Problem I, $u_e = 0$, and $u_0(\infty) = 1$. Let f denote the self-similar solution of Problem P$(0,1)$. Then the conclusions are the same as those of Theorem 4.1.*

We shall discuss the proof in full for Theorem 4.1, and merely comment on the differences with Theorem 4.2.

Proof of Theorem 4.1. We first shall explain the underlying idea. If we scale the domain of u as to render it independent of ε by introducing the new variables

$$\xi = \frac{r}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon^2}, \qquad \text{and} \qquad w(\xi, \tau) = u(r, t),$$

we find that w solves the problem

$$(\mathrm{I}') \begin{cases} \beta(w)_\tau + \dfrac{\lambda - 1}{\xi} w_\xi - w_{\xi\xi} = 0 & 1 < \xi < \infty, \ \tau > 0 \\ w_\xi = \lambda(w - u_e) & \xi = 1, \ \tau > 0 \\ w(\xi, 0) = u_0(\varepsilon\xi), & \end{cases}$$

which has already been shown to have a unique solution. The proof of Theorems 4.1 and 4.2 consists of an estimate of the convergence of solutions of (I') to the self-similar solutions derived in the previous section as $\tau \to \infty$. The statements of the theorems then follow from this estimate in the following way. If ε is constant, then the large-time behaviour of the solution of (I') is immediately translated into the large-time behaviour of the solution u of Problem I, since the estimate is a uniform one. If ε is varied, then with the extra assumption that u_0 is constant we can deduce that solutions u of Problem I for different values of ε are in fact identical when expressed as functions of the variables ξ and τ. Therefore an estimate that is valid for w when $\tau \to \infty$ translates into an estimate for u when either $t \to \infty$ or $\varepsilon \downarrow 0$, or both.

In order to compare the solution of the original problem with the self-similar solution we shall reformulate the problem in self-similar variables. If u is the solution of Problem I, then define z by $z(\eta, \tau) = w(\xi, \tau) = u(r, t)$, where the independent variables are again linked through the relations

$$\eta = \frac{r}{\sqrt{t}} = \frac{\xi}{\sqrt{\tau}} \qquad \text{and} \qquad \tau = \frac{t}{\varepsilon^2}.$$

The function z satisfies the equation

$$\tau\beta(z)_\tau - \tfrac{1}{2}\eta\beta(z)_\eta + \frac{\lambda-1}{\eta}z_\eta - z_{\eta\eta} = 0 \quad \text{for} \quad \eta > 1/\sqrt{\tau},\ \tau > 0 \tag{4.1}$$

and the boundary condition

$$z_\eta = \lambda\sqrt{\tau}(z-u_e) \quad \text{for} \quad \eta = 1/\sqrt{\tau},\ \tau > 0.$$

The first step consists of an integral estimate, derived from the differential equation.

Proposition 4.3. *Let the integral function* $\Phi : \mathbf{R}^+ \to \mathbf{R}$ *be defined by*

$$\Phi(\tau) = \int_{1/\sqrt{\tau}}^\infty \eta\big\{\beta(z(\eta,\tau)) - \beta(f(\eta))\big\}\,d\eta, \quad \tau > 0,$$

and suppose that either ε *is fixed or* u_0 *is constant. Then there is a constant* χ *such that*

$$\Phi(\tau) \le \frac{\chi}{\tau} \quad \textit{for all } \tau > 0. \tag{4.2}$$

If u_0 *is constant, then* χ *does not depend on* ε.

Proof. By integrating (3.1) and using boundary conditions (1.8) and Proposition 3.4 we find that

$$\int_{1/\sqrt{\tau}}^\infty \beta(f(\eta))\,\eta d\eta = \lambda - \int_0^{1/\sqrt{\tau}} \beta(f(\eta))\,\eta d\eta$$

The conservation of mass (Proposition 2.4) reads in the η, τ coordinates

$$\int_{1/\sqrt{\tau}}^\infty \beta(z)\,\eta d\eta = \frac{1}{\varepsilon^2\tau}\int_\varepsilon^\infty \beta(u_0(r))\,rdr + \lambda.$$

By combining these two we find that Φ is well-defined and that

$$\tau\Phi(\tau) = \frac{1}{\varepsilon^2}\int_\varepsilon^\infty \beta(u_0(r))\,rdr + \tau\int_0^{1/\sqrt{\tau}} \beta(f(\eta))\,\eta d\eta. \tag{4.3}$$

The second term in (4.3) is bounded by $\beta(1)/2$. When ε is constant, the result follows immediately; when ε varies, but u_0 is constant and therefore equal to 0, the first term on the right-hand side vanishes and the remainder is bounded independent of ε. •

Part of the interest of this integral estimate lies in the fact that z and f are ordered, and that therefore the argument of the integral is positive. Indeed, if v is the self-similar solution of equation (Ia) corresponding to f, i.e. $v(r,t) = f(r/\sqrt{t})$, we can integrate equation (Ia) from 0 to ε to obtain

$$\int_0^\varepsilon \beta(v)_t\,rdr + \big[\lambda v - rv_r\big]_0^\varepsilon = 0.$$

Now $v_t(r,t) = -\frac{1}{2}rt^{3/2}f'(r/\sqrt{t}) > 0$ for all r and t and therefore we have $\lambda v(\varepsilon,t) - \varepsilon v_r(\varepsilon,t) \leq \lambda$. By the Comparison Principle (Proposition 2.2) we then find that u lies above v on the whole of S_T^ε, which implies the same for z and f (on the appropriate domain).

We shall want to use a Lemma that has been applied in several forms by several authors. For an idea of the proof we refer to [13].

Lemma 4.4. *Let ϕ be a non-negative continuous function on $[0,\infty)$ with lower Lipschitz constant L, i.e.*

$$\frac{\phi(x)-\phi(y)}{x-y} \geq -L \quad \text{for all } x,y \in [0,\infty),\, x \neq y.$$

Let $x_0 > 0$. If $\int_{x_0}^\infty x\phi(x)\,dx = \alpha$, then

$$\sup_{x_0 \leq x < \infty} \phi(x) \leq \sqrt[3]{6L^2\alpha}.$$

We shall not apply this lemma directly to Φ, but to the integral

$$\int_{1/\sqrt{\tau}}^\infty \eta\{z(\eta,\tau) - f(\eta)\}\,d\eta.$$

For this integral we obtain an estimate similar to (4.2) by pointing out that because β is concave and strictly increasing on its domain, we can find $b > 0$ such that the function $s \mapsto \beta(s) - bs$ is non-decreasing for $0 \leq s \leq 1$. This implies that $\beta(z(\eta,\tau)) - \beta(f(\eta)) \geq b(z(\eta,\tau) - f(\eta))$ and thus

$$\int_{1/\sqrt{\tau}}^\infty \eta\{z(\eta,\tau) - f(\eta)\}\,d\eta \leq \frac{\chi}{b\tau}. \tag{4.4}$$

The function ϕ as mentioned in the Lemma shall therefore be given by $\phi(\eta,\tau) = z(\eta,\tau) - f(\eta)$.

In order to apply Lemma 4.4, we shall need some information about the lower Lipschitz constant L of ϕ with respect to the variable η. From Proposition 3.1 we know that f decreases on the whole of $\mathbf{R}$, so the lower Lipschitz constant of ϕ only depends on z. We have

Proposition 4.5. *If $0 < \lambda < 1$ then there exist positive constants ℓ and m independent of η and τ such that*

$$z_\eta(\eta,\tau) \geq -\ell\tau^{\frac{1-\lambda}{2}} - m \quad \text{for all } \tau > 0 \text{ and } \eta > 1/\sqrt{\tau};$$

if $\lambda \geq 1$ then there exists a constant m independent of η and τ such that

$$z_\eta(\eta,\tau) \geq -m \quad \text{for all } \tau > 0 \text{ and } \eta > 1/\sqrt{\tau}.$$

If, for the moment, we consider this proposition proved, the conclusions of Theorem 4.1 follow by combining (4.4) and Proposition 4.5 and applying Lemma 4.4.

Proof of Proposition 4.5. Let $z_n(\eta,\tau) = u_n(r,t)$ where u_n denotes the regularised solution constructed in Section 2. The domain of definition of z_n is

$$D^n = \left\{ (\eta,\tau) : \frac{1}{\sqrt{\tau}} < \eta < \frac{n}{\varepsilon\sqrt{\tau}} \right\}$$

which is drawn in Fig. 4.1.

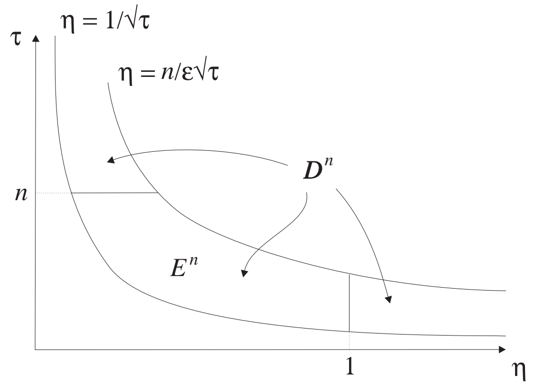

Fig. 4.1. The domains D^n and E^n.

The first part of the proof is the following lemma.

Lemma 4.6. *There exists a constant C, depending on β, λ and u_0, such that*

$$|z_{n\eta}| \le \frac{C}{\eta} \quad \text{on } D^n.$$

The constant C does not depend on n, and if u_0 is constant, then it does not depend on ε, either.

Proof of Lemma 4.6. Note that if (η,τ) and (r,t) are linked by the relations $\eta = r/\sqrt{t}$ and $\tau = t/\varepsilon^2$, then

$$\lambda u_n(r,t) - r u_{nr}(r,t) = \lambda z_n(\eta,\tau) - \eta z_{n\eta}(\eta,\tau)$$

on the relevant domains. In Section 2 it was shown that $\sup_{r,t} |\lambda u_n - r u_{nr}| \le C'$ where C' does not depend on n. Since z_n is bounded by 1, this yields

$$|z_{n\eta}| \le \frac{C' + \lambda}{\eta} \quad \text{on } D^n.$$

If u_0 is constant, then the same remark applies as at the beginning of the proof of Theorem 4.1, from which we then conclude that C' is also independent of ε. •

The remainder of the proof is based on the application of the maximum principle for parabolic equations to certain flux-type functions, depending on the value of λ. We shall split up into two cases.

CASE I. $0 < \lambda \leq 2$. In order to reduce the domain of definition of z_n to a bounded domain, we cut it off at $\eta = 1$ and at $\tau = n$:

$$E^n = D^n \cap \{\eta < 1\} \cap \{\tau < n\} \qquad \text{(see Fig. 4.1).}$$

Take n to be fixed. On E^n we define the following flux-type function:

$$F_n = F_n(\eta, \tau) = \eta^{1-\lambda}\left[z_{n\eta}(\eta,\tau) + \tfrac{1}{2}\eta\beta(z_n(\eta,\tau))\right]. \tag{4.5}$$

By differentiating (4.5) and substituting (4.1), we find the equation (4.6). For clarity, we shall omit the variable pair (η, τ), and simply write β' and β'' for $\beta'(z_n)$ and $\beta''(z_n)$, wherever this does not present any ambiguity:

$$\tau\beta' F_{nt} - F_{n\eta\eta} - bF_{n\eta} - cF_n = d \quad \text{on } E^n \tag{4.6}$$

where the coefficients b, c, and d are given by

$$b(\eta,\tau) = -\frac{1-\lambda}{\eta} + \tfrac{1}{2}\eta\beta' - \frac{\beta''}{\beta'}z_{n\eta}, \qquad c(\eta,\tau) = (\frac{\lambda}{2} - 1)\{\beta' - \frac{\beta''}{\beta'}\beta(z_n)\},$$
$$d(\eta,\tau) = (\frac{\lambda}{2} - 1)\eta^{2-\lambda}\frac{\beta''}{\beta'}\beta(z_n)^2.$$

Due to the regularisation, the coefficients of (4.6), β', b, c, and d, are all smooth and bounded on E^n. Note that $c \leq 0$ and $d \geq 0$, and that therefore F_n is a supersolution for the equation

$$\tau\beta' G_\tau - G_{\eta\eta} - bG_\eta - cG = 0 \quad \text{on } E^n.$$

By the maximum principle (see e.g. [15]), a non-positive minimum of F_n on E^n must be assumed on its parabolic boundary, i.e. $\gamma_1 \cup \gamma_2$. Let us investigate these boundary curves.

On γ_1, given by $\eta = 1/\sqrt{\tau}$, we have

$$\begin{aligned} F_n(1/\sqrt{\tau}, \tau) &= \tau^{(\lambda-1)/2}\{\lambda\sqrt{\tau}(z_n(1/\sqrt{\tau},\tau) - 1) + \tfrac{1}{2}\tau^{-\frac{1}{2}}\beta(z_n)\} \\ &\geq -\lambda\tau^{\lambda/2}(1 - f(1/\sqrt{\tau})) \\ &\geq A, \end{aligned}$$

in which A is the (negative) limit value given by Proposition 3.7. On γ_2 we have $F_n = z_{n\eta}(1,\tau) + \frac{1}{2}\beta(z_n(\eta,1)) \geq -\chi$ by Lemma 4.6. This implies that $F_n \geq \min\{A, -\chi\}$ on E^n, and therefore by the boundedness of $\eta\beta(z_n)$ we have $z_{n\eta} \geq -\ell\eta^{1-\lambda} - C'$ on E^n for some constants ℓ and C'. If we then let n tend to infinity we find that $z_\eta \geq -\ell\eta^{1-\lambda} - C'$ for all $1/\sqrt{\tau} < \eta < 1$. When we combine this with Lemma 4.6 we obtain the required result.

CASE II. The case of $\lambda > 2$ demands a different flux function:

$$F_n = F_n(\eta, \tau) = \eta^{1-\lambda}\big[z_{n\eta}(\eta, \tau) - \alpha\eta(\beta(1) - \beta(z_n(\eta, \tau)))\big],$$

where α is a non-negative constant, and the corresponding equation satisfied by F_n on E^n is

$$\begin{aligned}\tau\beta' F_{nt} - F_{n\eta\eta} - bF_{n\eta} = c\big\{F_n + \alpha\eta^{2-\lambda}[\beta(1) - \beta(z_n)]\big\}\\ + d\big\{F_n + \alpha\frac{2\alpha - 1}{2\alpha - \frac{\lambda}{2}}\eta^{2-\lambda}[\beta(1) - \beta(z_n)]\big\},\end{aligned} \tag{4.8}$$

in which the coefficients b, c and d are given by

$$b(\eta, \tau) = -\frac{1-\lambda}{\eta} + \tfrac{1}{2}\eta\beta' - \frac{\beta''}{\beta'}z_{n\eta}, \qquad c(\eta, \tau) = \alpha(\lambda - 2)\frac{\beta''}{\beta'}\big[\beta(1) - \beta(z_n)\big],$$
$$d(\eta, \tau) = (-2\alpha + \frac{\lambda}{2})\beta'.$$

Now define the function ω by

$$\omega(F, \eta, \tau) \stackrel{\text{def}}{=} c\big\{F + \alpha\eta^{2-\lambda}[\beta(1) - \beta(z_n)]\big\} + d\big\{F + \alpha\frac{2\alpha - 1}{2\alpha - \frac{\lambda}{2}}\eta^{2-\lambda}[\beta(1) - \beta(z_n)]\big\},$$

and remark that by choosing $\alpha > \lambda/4$ we shall have $c < 0$ and $d < 0$. We claim that the function $\eta \mapsto \eta^{2-\lambda}(\beta(1) - \beta(z_n))$ is bounded on $\mathbf{R}^+$ by a constant independent of τ and n: on one hand,

$$\begin{aligned}0 \le \eta^{2-\lambda}\big[\beta(1) - \beta(z_n)\big] &\le \eta^{2-\lambda}\big[\beta(1) - \beta(f(\eta))\big]\\ &\le 2\beta'(1)\frac{A}{\lambda}\eta^2,\end{aligned}$$

if η is small enough, in which A is again the limit value given by Proposition 3.7. On the other hand,

$$0 \le \eta^{2-\lambda}\big[\beta(1) - \beta(z_n)\big] \le \beta(1)\eta^{2-\lambda}.$$

The combination of the first for small η and the second for large η yields the uniform bound. Therefore, by choosing $\alpha > \lambda/4$ and $F_0 \in \mathbf{R}$, $F_0 < A$ negative and large enough, $\omega(F_0, \eta, \tau)$ can be made positive for all η and τ. This implies that the constant F_0 is a subsolution for equation (4.8), and by following the same line of reasoning as for case I, we can conclude that $F_n \ge F_0$ on E^n, for all n. The required result is then obtained in a similar fashion. •

This concludes the proof of Theorem 4.1.

The proof of Theorem 4.2 follows the same lines, with some alterations. First, the ordering of the solution u and the self-similar solution f is inverted, which gives rise to the definition

$$\Phi(\tau) = \int_{1/\sqrt{\tau}}^{\infty} \eta\{\beta(f(\eta)) - \beta(z(\eta,\tau))\}\, d\eta, \quad \tau > 0.$$

The assertions of Proposition 4.3, however, hold unchanged, as does their proof. Second, if we denote $f(\eta) - z(\eta,\tau)$ by $\phi(\eta,\tau)$, the application of Lemma 4.4 requires an estimate of ϕ_η from below. From Proposition 3.8 we know that f is strictly increasing on $\mathbf{R}^+$. For an upper bound on z_η, we have the same situation as in the previous case:

Proposition 4.7. *Let z be the solution of Problem I with $u_e = 0$ and $u_0(\infty) = 1$, transported to the η,τ-plane. If $0 < \lambda < 1$ then there exist positive constants ℓ and m independent of η and τ such that*

$$z_\eta(\eta,\tau) \le \ell\tau^{\frac{1-\lambda}{2}} + m \quad \text{for all } \tau > 0 \text{ and } \eta > 1/\sqrt{\tau};$$

if $\lambda \ge 1$ then there exists a constant m independent of η and τ such that

$$z_\eta(\eta,\tau) \le m. \quad \text{for all } \tau > 0 \text{ and } \eta > 1/\sqrt{\tau}.$$

The proof of Proposition 4.7 follows the same lines as that of Proposition 4.5, and we shall only mention the flux function that is used:

$$F_n = \eta^{1-\gamma}\left[z_{n\eta} + \tfrac{1}{2}\eta\beta(z_n)\right]$$

where $\gamma = \min\{\lambda, 2\}$. The result is then again reached by combination of Propositions 4.3 and 4.7 and Lemma 4.4. •

Appendix A. The phase plane analysis of Problem $\mathrm{P}_g(0,1)$ when $\beta(f) = f^p$.†

If we assume that the nonlinearity β has the form of a power, i.e. $\beta(f) = f^p$ for some $0 < p < 1$, then Problem $\mathrm{P}_g(0,1)$

$$\mathrm{P}_g(0,1) \begin{cases} g'' + \mu s^\alpha\{\beta(g)\}' = 0 & \text{for} \quad 0 < s < \infty \\ g(0) = 0, \quad g(\infty) = 1 \end{cases}$$

can be scaled in such a way that we obtain an autonomous equation, which can then be studied in the phase plane. In this way we obtain existence and uniqueness of a solution, as well as a precise characterisation of the behaviour of g (and therefore also of f) near $s = 0$ (which corresponds to $\eta = 0$).

Set $g(s) = \hat{u}(s)s^k$ for some $k > 0$ which we shall determine later. We find the following equation for $\hat{u}$:

$$\hat{u}''s^2 + \hat{u}'s\left\{2k + \mu p s^{\alpha+k(p-1)+1}\hat{u}^{p-1}\right\} + k(k-1)\hat{u} + \mu k p s^{\alpha+k(p-1)+1}\hat{u}^p = 0.$$

† This appendix is an excerpt of [14].

By taking $k = \frac{\alpha+1}{1-p}$, so that $\alpha + k(p-1) + 1 = 0$, and writing $t = \log s$, $u(t) = \hat{u}(s)$, we obtain the following (autonomous) form:

$$u'' + u'\left\{2k - 1 + \mu p u^{p-1}\right\} + k(k-1)u + \mu k p u^p = 0,$$

or, equivalently,

$$(\mathrm{E}_u)\begin{cases} u' = v \\ v' = -v(c + du^{p-1}) - u(a + bu^{p-1}) \end{cases}$$

in which

$$\begin{aligned} a &= k(k-1) \qquad & c &= 2k-1 \\ b &= \mu k p \qquad & d &= \mu p. \end{aligned}$$

Of course, we are looking for a solution (u, v) of (E_u) that gives us a solution to our original problem. The boundary conditions that are necessary to ensure this will depend on the signs of a and c, so from here on we shall split up into three cases:

(a) $\frac{1}{2} < k < 1$: $a < 0$, $c > 0$
(b) $0 < k \leq \frac{1}{2}$: $a < 0$, $c \leq 0$
(c) $k \geq 1$: $a \geq 0$, $c > 0$.

Case (a). $\frac{1}{2} < k < 1$: $a < 0$, $c > 0$.

Let us start by drawing the isoclines. We have

$$\Gamma_u = \{v = 0\}$$
$$\Gamma_v = \{v = -u\frac{au^{1-p} + b}{cu^{1-p} + d}\}$$

so the equilibrium points are

$$O = (0,0) \quad \text{and} \quad A((-b/a)^{\frac{1}{1-p}}, 0).$$

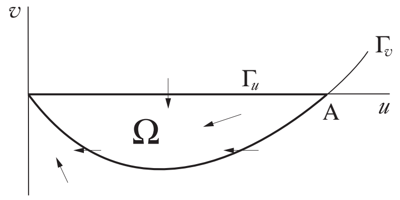

Fig. A.1. The isoclines and the set Ω.

The point A is a saddle point; therefore there are at least two orbits leaving A, and one of them will be in the direction

$$\left(-2, -k + \sqrt{k^2 + 4k(1-k)(1-p)}\right).$$

This orbit enters the region

$$\Omega = \{(u, v) \colon -u\frac{au^{1-p} + b}{cu^{1-p} + d} < v < 0\}.$$

In Ω, $u' < 0$ and $v' < 0$, so the orbit will leave Ω in finite time. But it will stay within the set

$$\Omega' = \text{conv } \Omega \cup \{(-\frac{v_0}{k}, v_0)\}$$

in which $v_0 = \min_{(u,v)\in\Gamma_v} v$, which is invariant because the line $\{v = -ku\}$ is itself an orbit.

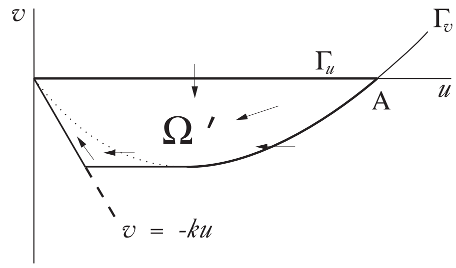

Fig. A.2. The set Ω'.

As $u' < 0$ throughout this trajectory, it will tend to $(0, 0)$ as the only available equilibrium point.

So we have found an orbit, starting at $t = -\infty$ in A, and ending, at $t = \infty$, in $(0, 0)$. The next question is to verify whether this u leads to a solution of $\mathrm{P}_g(0, 1)$. Define $g(s) = u(\log s)s^k$. By retracing these steps one can show that g satisfies the equation $g'' + \mu s^\alpha(\beta(g))' = 0$, and $g(0) = 0$; thus we are left to verify that $g(\infty) = 1$.

In fact, this will probably not be the case. As equation (E_u) is autonomous, we can 'shift' u in time: if $u(t)$ is a solution, then $u(t + t_0)$ will be one too. Call this solution $\bar{u}(t)$. Then

$$\begin{aligned}\bar{g}(s) &= \bar{u}(\log s)s^k = u(\log s + t_0)s^k \\ &= u(\log(se^{t_0}))s^k,\end{aligned}$$

and so

$$\begin{aligned}\lim_{s\to\infty} \bar{g}(s) &= \lim_{s\to\infty} u(\log s)(se^{-t_0})^k \\ &= e^{t_0 k} \lim_{s\to\infty} g(s).\end{aligned}$$

So if g tends to a positive limit, then by choosing t_0 we can make $\bar{g}$ tend to any positive limit, in particular 1.

Now we will show that g tends to a limit. Note first that whenever $(u, v) \in \Omega' \backslash \Omega$,

$$-ku \le v \le -u\frac{au^{1-p} + b}{cu^{1-p} + d} \tag{A.1}$$

and so

$$0 \ge -\frac{1}{ku} \ge \frac{dt}{du} \ge -\frac{1}{u}\frac{cu^{1-p} + d}{au^{1-p} + b}. \tag{A.2}$$

The statement that g tends to a limit is equivalent to

$$\int_{s_0}^{\infty} \frac{d}{ds}(\log g(s))\, ds < \infty \qquad \text{for some } s_0 > 0.$$

Well,

$$\begin{aligned}\int_{s_0}^{\infty} \frac{d}{ds}(\log g(s))\, ds &= \int_{s_0}^{\infty} \Big(\frac{v(\log s)}{u(\log s)} + k\Big)\, \frac{ds}{s} \\ &= \int_{t_0}^{\infty} \Big(\frac{v(t)}{u(t)} + k\Big)\, dt\end{aligned}$$

where $t_0 = \log s_0$; using $(A.1)$ and $(A.2)$ we find that

$$\begin{aligned}\int_{s_0}^{\infty} \frac{d}{ds}(\log g(s))\, ds &= \int_{u(t_0)}^{0} \Big(\frac{v}{u} + k\Big)\, \frac{dt}{du} du \\ &\le \int_{0}^{u(t_0)} \Big(k - \frac{au^{1-p} + b}{cu^{1-p} + d}\Big)\, \frac{1}{u}\, \frac{cu^{1-p} + d}{au^{1-p} + b}\, du \\ &= \int_{0}^{u(t_0)} \frac{1}{(k-1)u + pu^p}\, du < \infty.\end{aligned}$$

So as $s \to \infty$, $g(s)$ tends to a limit that we can choose to be equal to 1. This means we have a solution to Problem $\mathrm{P}_g(0, 1)$.

Case (b). $0 < k \le \frac{1}{2}$: $a < 0$, $c \le 0$.

When $0 < k \le \frac{1}{2}$ the phase plane is slightly different:

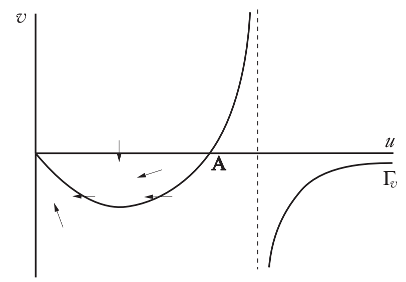

Fig. A.3. The phase plane for $0 < k \le \frac{1}{2}$.

But it is easily seen that all the observations in the previous case still hold—which gives us existence of a solution in this case, too.

Case (c). $k \ge 1$: $a \ge 0$, $c > 0$.

When $k \ge 1$, however, the situation becomes a trifle trickier. The phase plane looks like

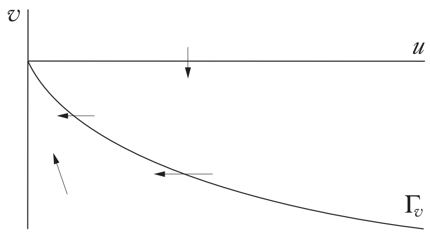

Fig. A.4. The phase plane when $k \geq 1$.

and so there is no equilibrium point A to leave from. It turns out to be useful to regard the set of points (u, v) in which (u', v') is parallel to (u, v). It consists of the line $\{(u, v)\colon v = -ku\}$ and the curve

$$\Gamma_1 = \{(u, v)\colon v = (1 - k - pu^{1-p})\, u\}.$$

Define also the set S, destined to be a set to 'shoot' from, as

$$S = \big\{(u, v)\colon u = 1,\ (1 - k - pu^{1-p})\, u \leq v \leq -u\frac{au^{1-p} + b}{cu^{1-p} + d}\big\}$$

and the set

$$\Omega'' = \big\{(u, v)\colon u \geq 1,\ (1 - k - pu^{1-p})\, u \leq v \leq -u\frac{au^{1-p} + b}{cu^{1-p} + d}\big\} :$$

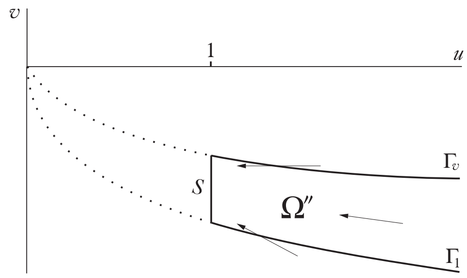

Fig. A.5. The sets Ω'' and S.

We now shoot back in time, starting on S. If $s \in S$, then denote the orbit that passes through s as $\gamma(s)$. In the usual way we define the sets

$$S^+ = \{s \in S\colon \gamma(s) \text{ leaves } \Omega'' \text{ by } \Gamma_v \text{ (in backward time)}\}$$
$$S^- = \{s \in S\colon \gamma(s) \text{ leaves } \Omega'' \text{ by } \Gamma_1 \text{ (in backward time)}\}.$$

S^+ and S^- do not necessarily fill S, but they are not empty—this can be seen by taking any orbit through a point on Γ_v or Γ_1. Because of the continuous dependence on initial data, they both are open in S; we conclude that there must be a point on S that is not an element of either subset. Necessarily, the orbit through this point will lie between Γ_v and Γ_1 for all time.

Call the thus obtained solution u, and define again a candidate solution to our problem $\mathrm{P}_g(0,1)$ by $g(s) = u(\log s)s^k$. In the same way as in the case of $0 < k < 1$, we can show that $g(\infty)$ can be chosen to be 1.

But does $g(s)$ tend to 0 as $s \downarrow 0$? Because the orbit lies above Γ_1,

$$\liminf_{t \to -\infty} \frac{v(t)}{u(t)} \geq 1 - k;$$

and so, for s small,

$$\begin{aligned}
\frac{d}{ds} \log g(s) &= \frac{d}{ds} \left\{ \log(u(\log s)) + k \log s \right\} \\
&= \frac{v(\log s)}{u(\log s)} \frac{1}{s} + \frac{k}{s} \\
&\geq \frac{1}{2s}
\end{aligned}$$

which isn't integrable, which implies that as $s \downarrow 0$, $\log g(s) \to -\infty$ and thus $g(s) \downarrow 0$.

Behaviour near zero.

When $k > 1$, or, equivalently, $\lambda < 2/(1-p)$, we can write the equation in the form

$$\frac{f''}{f'} + \frac{1-\lambda}{\eta} = -\frac{1}{2}\eta\beta'(f) = -\frac{1}{2}p\eta f^{p-1},$$

which we integrate from η_0 to η to obtain

$$f'(\eta)\eta^{1-\lambda} = f'(\eta_0)\eta_0^{1-\lambda} e^{-\frac{1}{2}\int_{\eta_0}^{\eta} p y f^{p-1}(y) dy}. \tag{A.3}$$

Now if $y f^{p-1}(y)$ is not integrable down to zero, then as $\eta \to 0$ we will have $f'(\eta)\eta^{1-\lambda} \to \infty$. For η small enough we will have $f'(\eta) \geq \lambda\eta^{\lambda-1}$, and by integration $f(\eta) \geq \eta^\lambda$. But that implies that

$$\eta f^{p-1}(\eta) \leq \eta^{1+\lambda(p-1)}$$

which is integrable since we assumed $\lambda < 2/(1-p)$. Therefore $\eta f^{p-1}(\eta)$ must be integrable, and

$$\lim_{\eta \downarrow 0} f'(\eta)\eta^{1-\lambda} \quad \text{exists.}$$

This in turn implies by de l'Hôpital's rule that

$$\lim_{\eta \downarrow 0} f(\eta)\eta^{-\lambda} \quad \text{exists.}$$

In other words, there exists a constant $c \geq 0$ such that $f(\eta) \sim c\eta^\lambda$. By remarking that the right-hand side in $(A.3)$ is strictly positive for all η, we find that the constant c must be strictly positive.

However, when $0 < k < 1$, or equivalently $\lambda > 2/(1-p)$, we have already seen (in cases (a) and (b)) that $g(s)$ tends to zero as As^k, or more precisely,

$$\lim_{s\downarrow 0} g(s)s^{-k} = A$$

in which

$$A = \Big(-\frac{b}{a}\Big)^{\frac{1}{1-p}} = \Big(\frac{\mu p}{1-k}\Big)^{\frac{1}{1-p}}.$$

When we transform this back in terms of f, we obtain

$$\lim_{\eta\downarrow 0} f(\eta)\eta^{-\frac{2}{1-p}} = \frac{A}{\lambda^k}.$$

This leaves us with the case of $k = 1$. Since the orbit lies between Γ_v and Γ_1 for all time s,

$$-\mu p u^p \le \frac{du}{ds} \le -\mu p u^p \frac{1}{1 + \mu p u^{p-1}},$$

which yields, by dividing by u^p and integrating over $[0, s]$,

$$-\mu p s \le \frac{1}{1-p}\{u^{1-p}(s) - u^{1-p}(0)\} \le -\mu p \int_0^s \frac{d\sigma}{1 + \mu p u^{p-1}(\sigma)}.$$

Therefore $u^{1-p}(s)/s \to -\mu p(1-p)$ as $s \to -\infty$. When we translate this back into terms of η and $f(\eta)$, we find that

$$\lim_{\eta\downarrow 0} \frac{f^{1-p}(\eta)}{\eta^2(\lambda\log\eta - \log\lambda)} = -\mu p(1-p)\lambda^{1-p}$$

from which we can conclude that

$$\lim_{\eta\downarrow 0} \frac{f^{1-p}(\eta)}{\eta^2\log\eta} = -\mu p(1-p)\lambda^{2-p}.$$

Let us recapitulate this in the

Theorem A.1. *Let f be the solution to Problem $P(0,1)$.*

(a) If $\lambda < 2/(1-p)$, then the limit

$$\lim_{\eta\downarrow 0} \frac{f(\eta)}{\eta^\lambda}$$

exists and is strictly positive.

(b) If $\lambda = 2/(1-p)$, then

$$\lim_{\eta\downarrow 0} \frac{f^{1-p}(\eta)}{\eta^2\log\eta} = -\mu p(1-p)\lambda^{2-p}.$$

(c) If $\lambda > 2/(1-p)$, then

$$\lim_{\eta \downarrow 0} \frac{f(\eta)}{\eta^{\frac{2}{1-p}}} = \frac{A}{\lambda^k},$$

in which $k = 2/(\lambda(1-p))$ and

$$A = \left(\frac{\mu p}{1-k}\right)^{\frac{1}{1-p}}.$$

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