CONVERGENCE TO TRAVELLING WAVES IN A REACTION-DIFFUSION SYSTEM ARISING IN CONTAMINANT TRANSPORT

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ABSTRACT. We consider a nonlinear system of parabolic equations describing the transport of contaminant in porous media. For this system travelling wave solutions are known to exist. We prove existence and uniqueness of general solutions of the system and convergence to travelling waves.

1. INTRODUCTION

In this paper we consider the initial value problem

$$\beta(u)_t + v_t + u_x - u_{xx} = 0 \tag{1.1}$$

$$v_t = \mathcal{F}(u, v) \tag{1.2}$$

on $Q = \mathbb{R} \times \mathbb{R}^+$ with initial data

$$(u, v) = (u_0, v_0) \quad \text{at} \quad t = 0. \tag{1.3}$$

Throughout the paper the nonlinearities β and $\mathcal{F}$ satisfy the hypotheses

$$\begin{cases} \beta \in C([0,\infty)) \cap C^2((0,\infty)),\ \beta(0) = 0,\ \beta' \geq b_0 > 0, \\ \text{and } \beta' \text{ and } \beta'' \text{ are bounded on compact sets away from} \\ \text{the origin;} \end{cases} \tag{1.4}$$

$$\begin{cases} \mathcal{F} : [0,\infty) \times [0,\infty) \to \mathbb{R} \text{ is continuous, strictly increas-} \\ \text{ing in its first argument and strictly decreasing in its} \\ \text{second argument, and satisfies} \\ \qquad -L \leq \dfrac{\mathcal{F}(u, v_1) - \mathcal{F}(u, v_2)}{v_1 - v_2} \leq 0 \\ \text{for all } u, v_1, v_2 \in [0,\infty) \text{ such that } v_1 \neq v_2; \end{cases} \tag{1.5}$$

and we assume that the initial data (u_0, v_0) satisfy

$$\begin{cases} \text{There exist } \bar{u}, \bar{v} \in (0,\infty) \text{ such that } u_0, v_0 \in L^\infty(\mathbb{R}) \\ \text{with } 0 \leq u_0 \leq \bar{u},\ 0 \leq v_0 \leq \bar{v}, \text{ and } \mathcal{F}(\bar{u}, \bar{v}) = \mathcal{F}(0,0) = \\ 0. \end{cases} \tag{1.6}$$

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We show that this problem is well-posed and prove the convergence of the solution to a travelling wave under an additional condition on the initial data.

We remark that due to the possible degeneration of β and $\mathcal{F}$ at $u = 0$ the solutions of Problem (1.1-1.3) may lack regularity at points where they vanish.

Equations (1.1-1.2) arise in the modelling of contaminant transport in porous media. A chemical substance A is transported by water flow while undergoing adsorption processes on the surface of the soil particles. If C is a scaled dissolved concentration and S a scaled adsorbed concentration, the one-dimensional law of conservation of mass reads

$$(C+S)_t + qC_x - DC_{xx} = 0.$$

Here q denotes the discharge and D the combined coefficient of mechanical dispersion and molecular diffusion. Both are assumed constant.

Commonly the interaction between dissolved and adsorbed contaminant is modelled in the form of a first-order differential equation (see [29] for a discussion of adsorption kinetics):

$$S_t = kf(C, S). \tag{1.7}$$

We wish to allow for a more general model in which there is a subdivision at the pore scale between different adsorption *sites*. At some sites, which we call of type 1, the process is described by (1.7), while at others, those of type 2, the adsorption kinetics are assumed to be significantly faster than the other processes. For type-2 sites we adopt the *equilibrium assumption*

$$f(C, S_2) \equiv 0.$$

Typically this equation can be solved for S_2 in terms of C to yield an *isotherm* Ψ:

$$f(C, S_2) = 0 \iff S_2 = \Psi(C).$$

We regain equations (1.1-1.2) after substituting $S = S_1 + S_2$ and scaling variables and parameters by

$$\begin{aligned} u &:= C, & \beta(u) &:= C + \Psi(C), & x &:= \frac{D}{q}x, \\ v &:= S_1, & \mathcal{F}(u,v) &:= \frac{Dk_1}{q^2} f(C, S_1), & t &:= \frac{D}{q^2}t. \end{aligned}$$

Although the physical motivation of the multiple-site adsorption assumption is not clear, some experiments with pesticides do indicate a marked difference in performance between one-site and multiple-site models [4, 5, 6] (see also [27, 8]). For this reason we wish to include this case in our analysis.

Other examples of systems of the form (1.1-1.2) arise when modelling the propagation of impulses in nerve axons [12, 13, 14, 19], chemical reactions in catalyst particles [1, 17, 16, 15], and solid combustion ([10] and [2], Chapter 4).

A travelling wave (U, V, c) is a solution of system (1.1-1.2) that can be written as a function of the travelling wave coordinate $\eta = x - ct$ for some $c \in \mathbb{R}$:

$$(U, V)(x, t) = (f, g)(x - ct) = (f, g)(\eta). \tag{1.8}$$

The constant c is called the wave speed. Van Duijn and Knabner [28, 29] extensively studied the existence and some qualitative properties of travelling wave solutions of (1.1-1.2). One of their theorems is cited below (Proposition 1.1) and the interested reader can find more details in [28].

We seek bounded travelling waves that tend to limit values $(\bar{u}, \bar{v})$ and $(0, 0)$ at plus and minus infinity:

(1.9) $(f, g) \to (\bar{u}, \bar{v})$ as $\eta \to -\infty$ (upstream concentration)

(1.10) $(f, g) \to (0, 0)$ as $\eta \to \infty$ (downstream concentration)

A necessary condition for convergence at infinity is the condition

$$\mathcal{F}(0, 0) = \mathcal{F}(\bar{u}, \bar{v}) = 0,$$

which describes the assumption that $(0, 0)$ and $(\bar{u}, \bar{v})$ are states of chemical equilibrium.

Proposition 1.1. [28] *Let $\mathcal{F} \in \mathrm{Lip}((0, \bar{u}] \times (0, \bar{v}])$, and suppose that there exists a strictly increasing function $\psi \in C([0, \bar{u}]) \cap \mathrm{Lip}((0, \bar{u}])$ such that*

$$\mathcal{F}(u, v) \underset{\geq}{\overset{\leq}{=}} 0 \iff v \underset{\leq}{\overset{\geq}{=}} \psi(u)$$

for all $(u, v) \in (0, \bar{u}) \times (0, \bar{v})$. If β and ψ satisfy

$$\frac{\beta(u) + \psi(u)}{u} > \frac{\beta(\bar{u}) + \bar{v}}{\bar{u}} \quad \textit{for all} \quad 0 < u < \bar{u}, \tag{1.11}$$

then there exists a travelling wave solution (U, V, c) of system (1.1), (1.2) which is unique up to translation. The functions f and g defined by (1.8) have the same support and on this support they are strictly decreasing.

Note that the regularity restrictions on $\mathcal{F}$ are slightly more restrictive than those of (1.5).

Condition (1.11) plays a role similar to the convexity condition that arises in the investigation of travelling wave solutions of scalar conservation laws.

Remark 1.2 A travelling wave (f, g) satisfies the system of ordinary differential equations

$$\begin{cases} -c\beta(f)' - cg' + f' - f'' = 0 \\ -cg' = \mathcal{F}(f, g). \end{cases}$$

If β and $\mathcal{F}$ satisfy the regularity assumptions (1.4) and (1.5), then $f, g \in C^1(\mathbb{R})$. We use this regularity in Section 5. If we suppose that f and g satisfy (1.9-1.10), then the wave speed can be calculated by integrating the first equation and taking limits:

$$c = \frac{\bar{u}}{\beta(\bar{u}) + \bar{v}}$$

•

We shall refer to equations (1.1-1.2) on Q together with the initial conditions (1.3) as Problem P. We shall be interested in solutions in the following sense:

Definition 1.3. *A solution of Problem P is a pair of functions (u, v) such that*

1. $u \in C(Q) \cap L^\infty(Q)$ *with* $u_x \in L^2_{loc}(Q)$*;*
2. $v \in C([0,\infty); L^\infty(\mathbb{R}))$*;*
3. u *and* v *satisfy the equations*

$$\int_{\mathbb{R}} (\beta(u(t)) + v(t))\varphi(t) - \int_{Q_t} \Big((\beta(u) + v)\varphi_t + u(\varphi_x + \varphi_{xx}) \Big) = \int_{\mathbb{R}} (\beta(u_0) + v_0)\varphi(0) \tag{1.12}$$

and

$$\int_{\mathbb{R}} v(t)\varphi(t) - \int_{Q_t} v\varphi_t = \int_{\mathbb{R}} v_0\varphi(0) + \int_{Q_t} \mathcal{F}(u, v)\varphi \tag{1.13}$$

for all $t > 0$ and $\varphi \in C^{2,1}_c(\overline{Q})$.

Sub- and supersolutions satisfy parts 1 and 2, as well as (1.12) and (1.13) with the equality signs replaced by $\leq$ for subsolutions and $\geq$ for supersolutions.

Our first theorems concern the well-posedness of Problem P.

Theorem 1.4. *Let (u_0, v_0) satisfy hypothesis (1.6). Then there exists a solution of Problem P with initial data (u_0, v_0).*

For non-degenerate (i.e. Lipschitz continuous) functions β the uniqueness of solutions of (1.1-1.2) on bounded domains has been shown by Knabner [20]. Recently this result has been extended to non-linearities β that are not Lipschitz continuous and can even be maximal monotone graphs [21]. We use here a result due to Otto [23, 24] on bounded domains for single degenerate parabolic equations to prove the uniqueness of solutions of (1.1-1.2) on an unbounded domain:

Theorem 1.5. *Let (u_1, v_1) be a subsolution and (u_2, v_2) a supersolution of Problem P with corresponding initial data (u_{10}, v_{10}) and (u_{20}, v_{20}), such that*

$$[\beta(u_{10}) - \beta(u_{20})]_+ + [v_{10} - v_{20}]_+ \in L^1(\mathbb{R}).$$

Then

$$\int_{\mathbb{R}} \Big([\beta(u_1(t)) - \beta(u_2(t))]_+ + [v_1(t) - v_2(t)]_+ \Big) \le \int_{\mathbb{R}} \Big([\beta(u_{10}) - \beta(u_{20})]_+ + [v_{10} - v_{20}]_+ \Big) \tag{1.14}$$

for all $t \ge 0$.

Knabner and Otto [21] have obtained a result that generalizes Theorem 1.5 in the case of higher space dimensions but on a bounded domain.

Remark 1.6 An immediate consequence of (1.14) is the property of contraction in $L^1(\mathbb{R})$:

$$\int_{\mathbb{R}} \Big(|\beta(u_1(t)) - \beta(u_2(t))| + |v_1(t) - v_2(t)| \Big) \le \int_{\mathbb{R}} \Big(|\beta(u_{10}) - \beta(u_{20})| + |v_{10} - v_{20}| \Big),$$

for all $t \ge 0$. In physical terms, the quantity $\beta(u) + v$ corresponds to the total amount of chemical contaminant that is present in a unit volume. Theorem 1.5 therefore states that no mass is created, and we shall prove in Corollary 2.3 that no mass disappears either: if

$$\int_{\mathbb{R}} \Big(|\beta(u_{10}) - \beta(u_{20})| + |v_{10} - v_{20}| \Big) < \infty,$$

then

$$\int_{\mathbb{R}} \Big(\beta(u_1) - \beta(u_2) + v_1 - v_2 \Big) \quad \text{is constant in time.} \tag{1.15}$$

•

Our main result concerns the stability of travelling waves. Using the technique of Osher and Ralston [22] we prove the following result (see also [3, 18, 30, 9]):

Theorem 1.7. *Let the conditions of Proposition 1.1 be satisfied, and denote the travelling wave given by Proposition 1.1 by $(U, V)(x, t) = (f, g)(x - ct)$. If (u_0, v_0) satisfies*

$$\int_{\mathbb{R}} \Big(|\beta(u_0) - \beta(f)| + |v_0 - g| \Big) < \infty, \tag{1.16}$$

then there exists a translation of (U, V), again denoted (U, V), such that

$$\int_{\mathbb{R}} \Big(|\beta(u(t)) - \beta(U(t))| + |v(t) - V(t)| \Big) \to 0 \quad as \quad t \to \infty,$$

where $(u(t), v(t))$ is the solution given by Theorem 1.4.

Remark 1.8 An important question in the study of convergence of general solutions towards travelling waves is the following: given the initial data (u_0, v_0), to which translation of the travelling wave (U, V) does the solution converge? Is it possible to characterize the limit in terms of the initial data? In the course of the proof of Theorem 1.7 it is shown that it follows from hypothesis (1.16) that there is exactly one translation of (U, V) that satisfies

$$\int_{\mathbb{R}} \Big(\beta(u_0) - \beta(f) + v_0 - g \Big) = 0, \tag{1.17}$$

and therefore by (1.15),

$$\int_{\mathbb{R}} \Big(\beta(u(t)) - \beta(U(t)) + v(t) - V(t) \Big) = 0 \tag{1.18}$$

for all time t. In view of (1.18), condition (1.17) characterizes in a unique way the travelling wave to which the solution converges. •

Example An expression for $\mathcal{F}$ that is commonly used in soil science is

$$\mathcal{F}(u, v) = k(\psi(u) - v), \quad k > 0$$

where the isotherm ψ is increasing and concave and satisfies $\psi(0) = 0$. Similarly, the function $\beta(u)$ is often taken either equal to u or equal to $u + \varphi(u)$, where φ is a (possibly different) concave increasing isotherm (see [29]). Both β and $\mathcal{F}$ are assumed smooth away from the origin. For such nonlinearities β and $\mathcal{F}$, all necessary conditions are satisfied.

For given $(\bar{u}, \bar{v})$ such that $\mathcal{F}(\bar{u}, \bar{v}) = \mathcal{F}(0, 0) = 0$ there exists a travelling wave solution (f, g) with values $(0, 0)$ at plus infinity and $(\bar{u}, \bar{v})$ at minus infinity. This follows from Proposition 1.1. By Theorem 1.7 this travelling wave os stable with respect to perturbations. •

Organisation of this paper. In Sections 2 and 3 we discuss existence and uniqueness of solutions (Theorems 1.5 and 1.4). In Section 4 we introduce a semigroup operator which we use in Section 5 to prove the convergence to travelling waves (Theorem 1.7).

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2. Uniqueness and related properties

Define for every $\lambda > 0$ the weight function

$$\omega_\lambda(x) = e^{-\lambda\sqrt{1+x^2}} \quad \text{for} \quad x \in \mathbb{R}.$$

Theorem 1.5 is a consequence of the following more general result:

Theorem 2.1. *Let (u_1, v_1) be a subsolution and (u_2, v_2) a supersolution of Problem P with corresponding initial data (u_{10}, v_{10}) and (u_{20}, v_{20}). Then there exists a constant $C > 0$ that does not depend on λ such that*

$$\begin{aligned}&\int_{\mathbb{R}} \omega_\lambda\Big([\beta(u_1(t)) - \beta(u_2(t))]_+ + [v_1(t) - v_2(t)]_+\Big)\\ &\qquad \le e^{C(\lambda+\lambda^2)t} \int_{\mathbb{R}} \omega_\lambda\Big([\beta(u_{10}) - \beta(u_{20})]_+ + [v_{10} - v_{20}]_+\Big)\end{aligned} \tag{2.1}$$

for all $0 \le t \le T$.

Theorem 2.1 simultaneously provides three distinct results:

1. A *comparison principle:*

   if $u_{10} \ge u_{20}$ and $v_{10} \ge v_{20}$, then $u_1 \ge u_2$ and $v_1 \ge v_2$ on Q_T;
2. A *contraction in* $L^1(\mathbb{R})$: if $\beta(u_{10}) - \beta(u_{20}) \in L^1(\mathbb{R})$ and $v_{10} - v_{20} \in L^1(\mathbb{R})$ then

   $$\begin{aligned}&\|\beta(u_1(t)) - \beta(u_2(t))\|_{L^1(\mathbb{R})} + \|v_1(t) - v_2(t)\|_{L^1(\mathbb{R})}\\ &\qquad \le \|\beta(u_{10}) - \beta(u_{20})\|_{L^1(\mathbb{R})} + \|v_{10} - v_{20}\|_{L^1(\mathbb{R})},\end{aligned} \tag{2.2}$$

   for all $0 \le t \le T$. This follows from the limit $\lambda \to 0$ in (2.1);
3. *Continuous dependence on initial data* in a weighted space $L^1_{\omega_\lambda}(\mathbb{R})$ induced by the norm

   $$\|u\|_{L^1_{\omega_\lambda}(\mathbb{R})} = \int_{\mathbb{R}} \omega_\lambda\, |u|$$

   of the form

   $$\begin{aligned}&\|\beta(u_1(t)) - \beta(u_2(t))\|_{L^1_{\omega_\lambda}(\mathbb{R})} + \|v_1(t) - v_2(t)\|_{L^1_{\omega_\lambda}(\mathbb{R})}\\ &\qquad \le e^{C(\lambda+\lambda^2)t}\Big(\|\beta(u_{10}) - \beta(u_{20})\|_{L^1_{\omega_\lambda}(\mathbb{R})} + \|v_{10} - v_{20}\|_{L^1_{\omega_\lambda}(\mathbb{R})}\Big),\end{aligned}$$

   for all $0 \le t \le T$.

While the contraction in $L^1(\mathbb{R})$ only has a sensible definition for initial data with difference in $L^1(\mathbb{R})^2$, the continuous dependence result in $L^1_{\omega_\lambda}(\mathbb{R})^2$ holds for all bounded initial data.

Remark 2.2 If the monotonicity assumptions on $\mathcal{F}$ are omitted, and $\mathcal{F}$ is assumed to be one-sided Lipschitz continuous:

$$\mathcal{F}_u \ge -L \quad \text{and} \quad \mathcal{F}_v \le L, \tag{2.3}$$

then the Comparison Principle ceases to hold, and the property of contraction in L^1 relaxes into a bound in L^1 that grows exponentially in time. If even conditions (2.3) are violated, then non-uniqueness can occur. Proofs of these statements and counterexamples can be found in [25]. •

Proof of Theorem 2.1. We use an extension of a theorem due to Otto [23, 24] that gives an estimate for single equations. Although strictly speaking the hypotheses of [24] exclude application of these results to the situation at hand, it is readily verified that the main result is valid nonetheless. We will return to this question later and in the Appendix.

The functions u_1 and u_2 given by the hypotheses of Theorem 2.1 both satisfy an equation of the form

$$\beta(u)_t + u_x - u_{xx} = g$$

on a domain $Q_T^{a,b} = (a,b) \times (0,T)$ for any $a < b$, with (different) non-homogeneous Dirichlet boundary data on $\{a,b\} \times (0,T)$. The right-hand side is given by $g_i = -\mathcal{F}(u_i, v_i) \in L^\infty(Q_T^{a,b})$. Set $\tilde{u} = u_1 - u_2$, $\tilde{\beta} = \beta(u_1) - \beta(u_2)$, and $\tilde{\mathcal{F}} = \mathcal{F}(u_1, v_1) - \mathcal{F}(u_2, v_2)$, and define H to be the Heaviside function with $H(0) = 0$. We use the inequality (see (A.1))

$$-\int_{Q_T^{a,b}} \tilde{\beta}_+ \gamma_t + \int_{Q_T^{a,b}} H(\tilde{u})(\tilde{u}_x - \tilde{u})\gamma_x - \int_{Q_T^{a,b}} H(\tilde{u})\tilde{\mathcal{F}}\gamma \le \int_a^b \tilde{\beta}(0)_+\gamma(0),$$

for every non-negative $\gamma \in C_c^\infty((a,b) \times (-\infty, T))$.

Let χ be any non-negative element of $C_c^\infty(\mathbb{R})$, and choose $a, b \in \mathbb{R}$ such that $\operatorname{supp}\chi \subset (a,b)$. When we apply this result to the test function $\gamma(x,\tau) = H(t-\tau)\chi(x)$ we obtain

(2.4)
$$\int_{\mathbb{R}} \chi\tilde{\beta}_+(t) - \int_{\mathbb{R}} \chi\tilde{\beta}_+(0) \le -\int_{Q_t} \chi_x H(\tilde{u})(\tilde{u}_x - \tilde{u}) - \int_{Q_t} \chi\tilde{\mathcal{F}}H(\tilde{u}).$$

Here $Q_t = \mathbb{R} \times (0, t]$.

Before we continue we briefly remark on the hypotheses of Otto's theorem. First, the function γ given above clearly does not satisfy the regularity assumption $\gamma \in C_c^\infty((a,b) \times (-\infty,T))$. However, since u_1, u_2, and β are continuous functions, this problem can be remedied by a regularisation argument. Second, the theorem proved in [24] does not include zero-order terms that depend explicitly on x and t. It is readily verified that the proof holds unchanged for zero-order terms that are in L^∞. Third, Otto demands Dirichlet data that are constant in time. For test functions with support inside $(a,b) \times (0,T)$ this restriction is not essential either, but this remark merits some additional explanation. We provide this in the Appendix.

We now continue the proof. Besides the notation introduced above we set $\tilde{v} = v_1 - v_2$. Now since

$$-\int_{Q_t} \chi_x H(\tilde{u})\tilde{u}_x = -\int_{Q_t} \chi_x (\tilde{u}_+)_x = \int_{Q_t} \chi_{xx}\tilde{u}_+,$$

we obtain from (2.4)

$$\int_{\mathbb{R}} \chi\tilde{\beta}_+(t) - \int_{\mathbb{R}} \chi\tilde{\beta}_+(0) \le \int_{Q_t} \tilde{u}_+(\chi_x + \chi_{xx}) - \int_{Q_t} \chi\tilde{\mathcal{F}}H(\tilde{u}).$$

Now let χ converge to the function ω_λ, for instance by setting $\chi = \omega_\lambda\eta_n$ where η_n is a cut-off function. By observing that there exists a constant $c > 0$ independent of λ such that

$$|w_{\lambda x}|\,, |w_{\lambda xx}| \le c(\lambda + \lambda^2)\omega_\lambda \quad \text{on} \quad \mathbb{R},$$

we find that

(2.5)
$$\int_{\mathbb{R}} \omega_\lambda\tilde{\beta}_+(t) - \int_{\mathbb{R}} \omega_\lambda\tilde{\beta}_+(0) \le 2c(\lambda + \lambda^2)\int_{Q_t} \omega_\lambda\tilde{u}_+ - \int_{Q_t} \omega_\lambda\tilde{\mathcal{F}}H(\tilde{u}).$$

Next we consider equation (1.13). From the continuity of $\mathcal{F}$ and the boundedness of u_i and v_i we deduce that $\tilde{v} = v_1 - v_2 \in W^{1,\infty}(0, T; L^\infty(\mathbb{R}))$ and that $\tilde{v}_t = \tilde{\mathcal{F}}$ a.e. in Q. Since $s \mapsto s_+$ is a Lipschitz continuous mapping we also have

$$(\tilde{v}_+)_t = H(v)\tilde{\mathcal{F}}$$

and by integrating this against the weight function ω_λ we find

$$\int_{\mathbb{R}} \omega_\lambda\tilde{v}_+(t) - \int_{\mathbb{R}} \omega_\lambda\tilde{v}_+(0) \le \int_{Q_t} \omega_\lambda\tilde{\mathcal{F}}H(\tilde{v}).$$

By adding this inequality to (2.5),

$$\begin{aligned}(2.6)\quad \int_{\mathbb{R}} \omega_\lambda(\tilde{\beta}_+(t) + \tilde{v}_+(t)) - \int_{\mathbb{R}} \omega_\lambda(\tilde{\beta}_+(0) + \tilde{v}_+(0)) \le\\ \le \int_{Q_t} \omega_\lambda\tilde{\mathcal{F}}(H(\tilde{v}) - H(\tilde{u})) + 2c(\lambda + \lambda^2)\int_{Q_t} \omega_\lambda\tilde{u}_+.\end{aligned}$$

Set $h = H(\tilde{v}) - H(\tilde{u})$. Clearly h takes values in the set $\{-1, 0, 1\}$. At points $(x, t) \in Q_t$ where $h(x, t) = 1$ we have $v_1 > v_2$ and $u_1 \le u_2$; therefore, by hypothesis (1.5),

$$\mathcal{F}(u_1, v_1) \le \mathcal{F}(u_2, v_2)$$

and $\tilde{\mathcal{F}}h \le 0$. Similarly, where $h = -1$ we also have $\tilde{\mathcal{F}}h \le 0$. This implies that the first term on the right-hand side in (2.6) is non-positive. Since $\beta' \ge b_0 > 0$, we can estimate the last term in (2.6) by

$$\frac{2c}{b_0}(\lambda + \lambda^2)\int_{Q_t} \omega_\lambda\tilde{\beta}_+.$$

Thus we can use Gronwall's Lemma to conclude that

$$\int_{\mathbb{R}} \omega_\lambda(\tilde\beta_+(t)+\tilde v_+(t)) \le e^{C(\lambda+\lambda^2)t}\int_{\mathbb{R}} \omega_\lambda(\tilde\beta_+(0)+\tilde v_+(0)),$$

where $C = 2c/b_0$, as asserted. This concludes the proof of Theorem 2.1. $\bullet$

Corollary 2.3 (Conservation of mass). *If*

$$\int_{\mathbb{R}} \Big(|\beta(u_{10})-\beta(u_{20})| + |v_{10}-v_{20}| \Big) < \infty,$$

then the integral

$$\int_{\mathbb{R}} \Big(\beta(u_1)-\beta(u_2)+v_1-v_2\Big)$$

is constant in time.

Proof. Choose $\chi \in C_c^\infty(\mathbb{R})$ and set $\varphi(x,t) = \chi(x)$. By subtracting equations (1.12) for u_1 and u_2 for this test function φ we obtain

$$\int_{\mathbb{R}} \chi\Big(\beta(u_1)-\beta(u_2)+v_1-v_2\Big)(t) = \int_{\mathbb{R}} \chi\Big(\beta(u_{10})-\beta(u_{20})+v_{10}-v_{20}\Big) + \int_0^t\int_{\mathbb{R}} (u_1-u_2)(\chi_x+\chi_{xx})$$

Since $\beta' \ge b_0 > 0$, the function $u_1-u_2 \in L^1(\mathbb{R}\times(0,t))$; the result then follows from letting χ converge towards the function 1. $\bullet$

For the proof of existence of a solution of the Cauchy Problem P in the next section we shall need a comparison theorem for the Cauchy-Dirichlet Problem on bounded sets. We state here the definition of a solution and the comparison theorem. The proof of this theorem is a slight perturbation of that of Theorem 1.5.

Let $(a,b) \subset \mathbb{R}$ be a bounded open interval, and let $T > 0$. We consider the problem

$$\beta(u)_t + v_t + u_x - u_{xx} = 0$$
$$v_t = \mathcal{F}(u,v)$$

on $Q_T^{a,b} = (a,b)\times(0,T]$ with the boundary condition

$$u = 0 \quad \text{on} \quad \{a,b\}\times(0,T]$$

and the initial condition

$$(u,v) = (u_0,v_0)$$

at $t=0$. We call this Problem *CD*.

Definition 2.4. *A solution of the Cauchy-Dirichlet Problem CD is a pair of functions* (u,v) *such that*

1. *$u \in C(Q_T^{a,b}) \cap L^2(0,T;H^1(a,b))$;*
2. *$v \in C([0,T];L^\infty(a,b))$;*
3. *u and v satisfy the equations*

$$-\int_{Q_T^{a,b}} \Big((\beta(u)+v)\varphi_t + u(\varphi_{xx}+\varphi_x)\Big) = \int_a^b (\beta(u_0)+v_0)\varphi(0) \tag{2.7}$$

and

$$-\int_{Q_T^{a,b}} v\varphi_t = \int_a^b v_0\varphi(0) + \int_{Q_T^{a,b}} \mathcal{F}(u,v)\varphi \tag{2.8}$$

for every $\varphi \in C^{2,1}(\overline{Q_T^{a,b}})$ such that $\varphi(T)=0$ and $\varphi=0$ on $\{a,b\}\times(0,T]$, and the boundary condition

$$u = 0 \quad on \quad \{a,b\}\times(0,T]. \tag{2.9}$$

Sub- and supersolutions of Problem CD satisfy parts 1 and 2, as well as (2.7), (2.8), and (2.9), with the equality signs replaced by $\le$ for subsolutions and $\ge$ for supersolutions.

Note that we do not incorporate the boundary condition (2.9) into the function space in part 1. This allows us to consider sub- and supersolutions with non-zero boundary values.

Theorem 2.5. *Let (u_1,v_1) be a subsolution and (u_2,v_2) a supersolution of Problem CD with corresponding initial data (u_{10},v_{10}) and (u_{20},v_{20}). Then*

$$\int_a^b \Big([\beta(u_1(t))-\beta(u_2(t))]_+ + [v_1(t)-v_2(t)]_+\Big) \le \int_a^b \Big([\beta(u_{10})-\beta(u_{20})]_+ + [v_{10}-v_{20}]_+\Big) \tag{2.10}$$

for all $0\le t\le T$.

Proof. The proof follows the same lines as the proof of Theorem 1.5. A non-obvious difference is that we apply the theorem of Otto to the test function $\gamma(x,\tau)=H(t-\tau)$, a function that is constant in space. The restriction to test functions with compact support does not hold for the case of zero Dirichlet data. •

3. Existence (proof of Theorem 1.4)

We prove Theorem 1.4 by approximation on a finite domain. Fix $T>0$ and define for $n\in\mathbb{N}$ the spatial domain $\Omega_n=(-n,n)$ and set

$Q_T^n = \Omega_n \times (0,T]$. We consider the approximate problem P_n

$$\begin{aligned} &\beta(u)_t + v_t + u_x - u_{xx} = 0 && x \in \Omega_n,\ 0 < t \le T \\ &v_t = \mathcal{F}(u,v) && x \in \Omega_n,\ 0 < t \le T \\ &u = 0 && x \in \{-n,n\},\ 0 < t \le T \\ &(u,v) = (u_0,v_0) && \text{at} \quad t = 0. \end{aligned}$$

Solutions to this problem are defined as in Definition 2.4.

We prove the existence of a solution to Problem P_n by the Schauder fixed point theorem. Define the convex set

$$X = \{u \in L^2(Q_T^n) : 0 \le u(x,t) \le \bar{u} \text{ on } Q_T^n\}.$$

We introduce the operator $\mathcal{T}_1 : X \to L^2(Q_T^n)$, where $\mathcal{T}_1 u$ is defined as the solution $\hat{v}$ of the ordinary differential equation

$$\hat{v}_t = \mathcal{F}(u,\hat{v}) \quad \text{on} \quad Q_T^n \tag{3.1}$$

with initial condition $\hat{v}(\cdot,0) = v_0$. By hypotheses (1.5) and (1.6) the constant $\bar{v}$ is an upper bound for $\hat{v}$: if for any $x \in Q_T^n$, $0 \le \tau \le T$ we have $\hat{v}(x,\tau) = \bar{v}$, then

$$\hat{v}_t(x,\tau) = \mathcal{F}(u(x,\tau),\bar{v}) \le \mathcal{F}(\bar{u},\bar{v}) = 0.$$

For $\hat{v}_t(x,\tau) < 0$ we directly find a contradiction if we assume that (x,τ) is the first occurrence of $v = \bar{v}$. Since $\mathcal{F}$ is one-sided Lipschitz continuous in its second argument, solutions of this ODE are locally unique, and therefore if $\hat{v}_t(x,\tau) = 0$ then $\hat{v}(x,t) = \bar{v}$ for every $0 \le t \le T$, and in particular v is bounded from above by $\bar{v}$. By a similar argument the constant 0 is a lower bound for $\hat{v}$. As a result, $\mathcal{T}_1 u$ belongs to the set

$$Y = \{v \in L^2(Q_T^n) : 0 \le v(x,t) \le \bar{v} \text{ on } Q_T^n\}.$$

The operator $\mathcal{T}_2 : Y \to L^2(Q_T^n)$ is defined in the following way: $\hat{u} = \mathcal{T}_2 v$ is the solution of

$$\beta(\hat{u})_t + \hat{u}_x - \hat{u}_{xx} = -\mathcal{F}(\hat{u},v) \quad \text{on} \quad Q_T^n, \tag{3.2}$$

subject to the boundary condition $\hat{u} = 0$ and the initial condition $\hat{u} = u_0$. The constants $u = 0$ and $u = \bar{u}$ are sub- and supersolutions: for $\bar{u}$ this follows from

$$\beta(\bar{u})_t = -\mathcal{F}(\bar{u},\bar{v}) \ge -\mathcal{F}(\bar{u},v)$$

since $v \le \bar{v}$ on Q_T^n, and for $u = 0$ the argument is similar. As a result the composite operator $\mathcal{T} = \mathcal{T}_2 \circ \mathcal{T}_1$ maps X into X.

Next we prove that the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ are continuous in the L^2-norm. For the length of this proof, let $\|\cdot\|$ denote the norm of $L^2(Q_T^n)$. We first consider $\mathcal{T}_1$. Let $u^m, u \in X$, $\|u^m - u\| \to 0$ as $m \to \infty$, and set $v^m = \mathcal{T}_1 u^m$, $v = \mathcal{T}_1 u$. We need to prove that $\|v^m - v\| \to 0$. By multiplying the difference of equations (3.1) with $w^m = v^m - v$ we find

$$\sup_{0\le t\le T} \tfrac{1}{2}\int_{\Omega_n} (w^m)^2 \le \sup_{0\le t\le T}\int_0^t\int_{\Omega_n} (\mathcal{F}(u^m,v^m)-\mathcal{F}(u^m,v))w^m + \|\mathcal{F}(u^m,v)-\mathcal{F}(u,v)\|\,\|w^m\|. \tag{3.3}$$

The first term on the right-hand side is non-positive by hypothesis (1.5) and we find the estimate

$$\|w^m\| \le C\,\|\mathcal{F}(u^m,v)-\mathcal{F}(u,v)\|.$$

Since $\mathcal{F}$ is continuous, there is a subsequence along which $\mathcal{F}(u^m,v) - \mathcal{F}(u,v) \to 0$ almost everywhere in Q_T^n, and consequently $\|w^m\| \to 0$ along that sequence. By the uniqueness of the limit the whole sequence converges to zero.

For the continuity of $\mathcal{T}_2$ we take analogously a sequence $v^m, v \in Y$, such that $\|v^m - v\| \to 0$, and define in the same way $u^m = \mathcal{T}_2 v^m$ and $u = \mathcal{T}_2 v$, and $z^m = u^m - u$. By the results of Sacks [26] the sequence $\{u^m\}$ is equicontinuous and we can therefore extract a subsequence along which u^m and v^m converge pointwise almost everywhere in Q_T^n. We can then pass to the limit in the equation

$$-\int_0^T\int_{\Omega_n}\Big(\beta(u^m)\varphi_t + u^m(\varphi_x+\varphi_{xx})\Big) = \int_{\Omega_n}\beta(u_0)\varphi(0) + \int_0^T\int_{\Omega_n}\mathcal{F}(u^m,v^m)\varphi$$

for every $\varphi \in C_c^\infty([0,T]\times\Omega_n)$. The function $u = \mathcal{T}_2 v$ is the unique solution of the limit equation which proves that the pointwise limit of $\{u^m\}$ is equal to u. Again the uniqueness of the limit implies that the whole sequence converges.

The equicontinuity of $\{u^m\}$ also implies that $\mathcal{T}$ is compact. It then follows by the Schauder fixed point theorem that there exists a $u \in X$ such that $\mathcal{T}u = u$, implying that the pair (u,v) with $v = \mathcal{T}_1 u$ solves Problem P_n.

We denote the solutions of P_n obtained in this way by (u_n, v_n) and we let n tend to infinity. By the comparison principle on bounded domains (Theorem 2.5) $\{u_n\}$ and $\{v_n\}$ form increasing sequences. We pass to the limit in equations (1.12) and (1.13).

The regularity assumptions of Definition 1.3 are satisfied for v by the uniform boundedness of $v_t = \mathcal{F}(u,v)$; the continuity of the function u follows from the equicontinuity of the sequence u^m due to Sacks, and the condition $u_x \in L^2_{loc}(Q)$ is a well-known consequence of the approximation procedure. •

4. Construction of a semigroup

By Theorems 1.4 and 1.5 we can define a semigroup operator that maps the initial data (u_0, v_0) to the corresponding solution $(u(t), v(t))$. For future convenience we consider pairs of the form $(\beta(u_0), v_0)$ instead of (u_0, v_0) and define $S(t)(\beta(u_0), v_0)$ as the solution $(\beta(u(t)), v(t))$ corresponding to the initial data $(\beta(u_0), v_0)$. The domain of definition of the semigroup $S(t)$ is the set

$$\mathcal{L} = \left\{ z_0 = (\beta(u_0), v_0) \in L^\infty(\mathbb{R})^2 : 0 \le u_0 \le \bar{u}, 0 \le v_0 \le \bar{v} \right\}.$$

We introduce the norm of $L^1(\mathbb{R})^2$ for elements $z = (\beta(u), v) \in \mathcal{L}$:

$$\|z\|_{L^1(\mathbb{R})^2} = \int_{\mathbb{R}} \Big(|\beta(u)| + |v| \Big),$$

which can be finite or infinite, and a partial ordering $\ge$ on $\mathcal{L}$:

$$z_1 \ge z_2 \quad \text{if} \quad u_1 \ge u_2 \text{ and } v_1 \ge v_2 \text{ on } \mathbb{R}.$$

Using this notation the semigroup $S(t)$ has the following properties:

1. $S(t)$ preserves L^1, i.e.

$$\|S(t)z_1 - S(t)z_2\|_{L^1(\mathbb{R})^2} \le \|z_1 - z_2\|_{L^1(\mathbb{R})^2} \tag{4.1}$$

   for each $t > 0$ and z_1, $z_2 \in \mathcal{L}$ such that $z_1 - z_2 \in L^1(\mathbb{R})^2$.
2. $S(t)$ preserves order, i.e. if $z_1, z_2 \in \mathcal{L}$ and $z_1 \ge z_2$, then $S(t)z_1 \ge S(t)z_2$ for all $t > 0$.

These properties are simply reformulations of Theorem 1.5. As a by-product of the proof of Theorem 2.1 we obtain the following extra information:

Proposition 4.1. *For $z_1, z_2 \in \mathcal{L}$, $z_1 - z_2 \in L^1(\mathbb{R})^2$, set $(\beta(u_i(t)), v_i(t)) = S(t)z_i$ for $t > 0$. If $t > 0$ is such that the set*

$$\{(x, \tau) \in \mathbb{R} \times (0, t) : (u_1 - u_2)(v_1 - v_2)(x, \tau) < 0\}$$

has non-zero measure, then

$$\|S(t)z_1 - S(t)z_2\|_{L^1(\mathbb{R})^2} < \|z_1 - z_2\|_{L^1(\mathbb{R})^2}.$$

Proof. The proof follows from an inspection of inequality (2.6). By Theorem 1.5 the integral $\int_{Q_t} \tilde{u}_+$ is finite, and therefore we can set $\lambda = 0$ in (2.6). This leads to

$$\int_{\mathbb{R}} (\tilde{\beta}_+(t) + \tilde{v}_+(t)) - \int_{\mathbb{R}} (\tilde{\beta}_+(0) + \tilde{v}_+(0)) \\ \le \int_{Q_t} (\mathcal{F}(u_1, v_1) - \mathcal{F}(u_2, v_2))(H(\tilde{v}) - H(\tilde{u})).$$

By hypothesis (1.5) the integral on the right-hand side is strictly negative. ●

On a region $\Omega \times (0,T)$ on which a solution (u,v) is bounded away from zero, assumptions (1.4) and (1.5) imply that equation (1.1) is uniformly parabolic, and therefore by classical regularity theory u_t and u_{xx} are functions and equation (1.1) is satisfied almost everywhere. We use this to prove a Strong Comparison Principle:

Proposition 4.2. *Let (u_1,v_1) and (u_2,v_2) both satisfy equations (1.1) and (1.2) on a domain $\Omega \times (0,T)$ and let u_1 and u_2 be bounded away from zero. Suppose that $|u_{2t}|$ is bounded on $\Omega \times (0,T)$. If $u_1 \geq u_2$ and $v_1 \geq v_2$, then*

$$\textit{either} \quad u_1 \equiv u_2 \quad \textit{or} \quad u_1 > u_2$$

on $\Omega \times (0,T)$.

Note that we do not explicitly impose any regularity on u_{1t} or $\beta(u_1)_t$, other than follows from the uniform parabolicity.

Proof. The strict positiveness of u_1 and u_2 and (1.4) imply that β'' is bounded on the values of u_1 and u_2. Then we have after setting $w = u_1 - u_2$,

$$\beta'(u_1)w_t - w_{xx} + w_x + \gamma w = -\mathcal{F}(u_1,v_1) + \mathcal{F}(u_2,v_2).$$

where

$$\gamma = \frac{\beta'(u_1) - \beta'(u_2)}{u_1 - u_2} u_{2t}$$

is a bounded function of x and t. Using the monotonicity of $\mathcal{F}$ in v we have

$$-\mathcal{F}(u_1,v_1) + \mathcal{F}(u_2,v_2) \geq -\frac{\mathcal{F}(u_1,v_1) - \mathcal{F}(u_2,v_1)}{u_1 - u_2} w,$$

and the result then follows from standard parabolic theory. •

5. Proof of Theorem 1.7 (convergence to travelling waves)

By the hypothesis

$$\int_{\mathbb{R}} \Big(|\beta(u_0) - \beta(f)| + |v_0 - g| \Big) < \infty, \tag{5.1}$$

we can assume without loss of generality that

$$\int_{\mathbb{R}} \Big(\beta(u_0) - \beta(f) + v_0 - g \Big) = 0. \tag{5.2}$$

This follows from the observation that if a locally integrable function $\varphi : \mathbb{R} \to \mathbb{R}$ has finite limits at plus and minus infinity, then

$$\begin{aligned} \int_{\mathbb{R}} (\varphi(x+h) - \varphi(x))\, dx &= \lim_{R\to\infty} \left\{ \int_R^{R+h} \varphi - \int_{-R}^{-R+h} \varphi \right\} \\ &= (\varphi(\infty) - \varphi(-\infty))h. \end{aligned}$$

Condition (5.2) pins down the travelling wave that (u, v) will converge to. We define the translation operator τ_h, for $h \in \mathbb{R}$, by

$$(\tau_h y)(x) = y(x - h) \quad \text{for all} \quad x \in \mathbb{R}$$

for any function y on $\mathbb{R}$.

In order to compare the general solution of Problem P with the travelling wave solution we introduce a change of variables: we set $\eta = x - ct$ where c is the wave speed of the travelling wave (U, V) and consider the solution as a function of (η, t) instead of (x, t). This amounts to considering instead of $S(t)$ the semigroup

$$\Sigma(t) = \tau_{-ct} \circ S(t).$$

The properties of $S(t)$ discussed above are passed unchanged to $\Sigma(t)$, and by construction $\zeta = (\beta(f), g)$ is a fixed point of $\Sigma(t)$.

First we prove stability in a special case. Recall that $\zeta = (\beta(f), g)$ is supposed throughout to satisfy condition (5.2).

Proposition 5.1. *Suppose that z_0 lies between two travelling waves, i.e. there exist $h_1, h_2 \in \mathbb{R}$ such that*

$$\tau_{h_1}\zeta \leq z_0 \leq \tau_{h_2}\zeta.$$

Then $\|z(t) - \zeta\|_{L^1(\mathbb{R})^2} \to 0$ as $t \to \infty$.

Proof. The proof consists of six steps. For the length of this proof set $z(t) = \Sigma(t)z_0$.

Step 1 *The set $\{z(t) - \zeta\}_{t>0}$ is compact in $L^1(\mathbb{R})^2$.*

By (4.1),

$$\|\tau_h z(t) - z(t)\|_{L^1(\mathbb{R})^2} \leq \|\tau_h z_0 - z_0\|_{L^1(\mathbb{R})^2}, \tag{5.3}$$

for all $h \in \mathbb{R}$. This implies that $\|\tau_h z(t) - z(t)\|_{L^1(\mathbb{R})^2} \to 0$ uniformly in t as $h \to 0$. By the comparison principle the fact that initially the solution z lies between travelling waves implies that the same holds for all positive time t. Therefore for instance

$$\int_{\{|\eta|>R\}} |\beta(u(\eta, t) - \beta(f(\eta))|\, d\eta \leq \int_{\{|\eta|>R\}} [\beta(\tau_{h_2} f(\eta)) - \beta(\tau_{h_1} f(\eta))]\, d\eta$$

and this integral tends to zero as R tends to infinity. Consequently $\beta(u) - \beta(f)$ has tails at infinity that are uniformly integrable in t, and the same holds for $v-g$. We can then apply for instance Corollary IV.26 of [7] or Theorem IV.8.20 of [11] to conclude that $\{\beta(u(t)) - \beta(f)\}_{t>0}$ and $\{v(t) - g\}_{t>0}$ both are compact in $L^1(\mathbb{R})$.

Let the ω-limit set ω be defined as

$$\omega = \{y \in \zeta + L^1(\mathbb{R})^2 : \exists t_n \to \infty, z(t_n) - y \to 0 \text{ in } L^1(\mathbb{R})^2\}.$$

By Step 1, $\omega \neq \emptyset$.

Remark 5.2 By the results of Sacks [26] solutions of (1.1) have a modulus of continuity in space and time that does not depend on t for large t. Consequently if $y = (\beta(y_u), y_v) \in \omega$, then y_u is necessarily uniformly continuous in space, and the first component of $\Sigma(t)y$ is uniformly continuous in space and time. The second component of $\Sigma(t)y$ is Lipschitz continuous in time by equation (1.2). Finally, it follows from Corollary 2.3 that

$$\int_{\mathbb{R}} (\beta(y_u) - \beta(f) + y_v - g) = 0. \tag{5.4}$$

•

We introduce a class of Lyapunov functionals V_h, for $h \in \mathbb{R}$,

$$V_h(y) = \|y - \tau_h\zeta\|_{L^1(\mathbb{R})^2} \quad \text{for all} \quad y \in \zeta + L^1(\mathbb{R})^2.$$

Step 2 *The functional V_h is constant on ω.*

This is a classical result in the theory of dynamical systems. By (4.1) the functional V_h decreases along the trajectory $z(t)$. Since it is bounded from below it has a limit V_h^* as $t \to \infty$. If $y_1, y_2 \in \omega$, then we can find a sequence $t_n \to \infty$ such that $z(t_{2n}) - y_1 \to 0$ and $z(t_{2n+1}) - y_2 \to 0$ in $L^1(\mathbb{R})^2$. It follows that $V_h(y_1) = V_h(y_2) = V_h^*$.

Step 3 *If $y \in \omega$, then $\Sigma(t)y \in \omega$ for all $t > 0$.*

Again this is classical: if the sequence $t_n \to \infty$ is such that $z(t_n) - y \to 0$, then

$$\begin{aligned}\|z(t_n + t) - \Sigma(t)y\|_{L^1(\mathbb{R})^2} &= \|\Sigma(t)z(t_n) - \Sigma(t)y\|_{L^1(\mathbb{R})^2} \\ &\leq \|z(t_n) - y\|_{L^1(\mathbb{R})^2}\end{aligned}$$

and this last term tends to zero by hypothesis.

Step 4 *If $y \in \omega$, $V_0(y) > 0$, then there exists $h \in \mathbb{R}$ such that*

$$V_h(\Sigma(t)y) < V_h(y) \quad \textit{for all time} \quad t > 0.$$

Set $y = (\beta(u), v)$. When we compare u with $\tau_h f$ for different values of h, there are two possibilities:

1. There exists $h \in \mathbb{R}$ such that $u \equiv \tau_h f$ on $\mathbb{R}$;
2. There exists $h \in \mathbb{R}$ and $I = [\eta_1, \eta_2] \subset \mathbb{R}$ such that $u - \tau_h f$ is of two signs on I and $u, f > 0$ on I.

In case 1 the possibility $v \equiv \tau_h g$ implies $h = 0$ by the mass restriction (5.4). Therefore $V_0(y) = 0$ which is ruled out by hypothesis. From $v \not\equiv \tau_h g$ it follows that by choosing h close to 0 we can obtain that the set

$$\{\eta \in \mathbb{R} : (u - \tau_h f)(v - \tau_h g) < 0\}$$

has non-zero measure. We then conclude by the continuity in time (Remark 5.2) and Proposition 4.1.

In case 2 we obtain the result by the strong maximum principle. Denote by $\beta(u(t))$ and $v(t)$ the first and second components of $\Sigma(t)y$, so that $(\beta(u(0)), v(0)) = y$. We define the initial conditions (on $\mathbb{R}$)

$$\check{u}_0 = \max\{u(0), \tau_h f\} \quad \text{and} \quad \check{v}_0 = \max\{v(0), \tau_h g\},$$

and we set

$$\check{u}(t) = \max\{u(t), \tau_h f\} \quad \text{and} \quad \check{v}(t) = \max\{v(t), \tau_h g\}.$$

Let $(\beta(\bar{u}(t)), \bar{v}(t)) = \Sigma(t)(\beta(\check{u}_0), \check{v}_0)$. By the comparison principle we have that $u, \tau_h f \le \bar{u}$ and $v, \tau_h g \le \bar{v}$. Since by construction $\tau_h f \not\equiv \check{u}_0$ on I, we deduce from the strong comparison principle (Proposition 4.2) that $u, \tau_h f < \bar{u}$ on $I \times (0, 1]$. Here we use the fact that f is continuously differentiable so that the travelling wave has a bounded time derivative. Then

$$\check{u}(t) < \bar{u}(t) \quad \text{on} \quad I \times (0, 1].$$

If we construct analogous functions $\hat{u}$, $\underline{u}$, $\hat{v}$, and $\underline{v}$, where max is replaced by min, then we have for $0 < t < 1$,

$$\begin{aligned}
\int_{\mathbb{R}} \Big(|\beta(u(t)) - \beta(\tau_h f)| + |v(t) - \tau_h g| \Big) \\
&= \int_{\mathbb{R}} \Big(\beta(\check{u}(t)) - \beta(\hat{u}(t)) + \check{v}(t) - \hat{v}(t) \Big) \\
&< \int_{\mathbb{R}} \Big(\beta(\bar{u}(t)) - \beta(\underline{u}(t)) + \bar{v}(t) - \underline{v}(t) \Big) \\
&\le \int_{\mathbb{R}} \Big(\beta(\check{u}_0) - \beta(\hat{u}_0) + \check{v}_0 - \hat{v}_0 \Big) \\
&= \int_{\mathbb{R}} \Big(|\beta(u(0)) - \beta(\tau_h f)| + |v(0) - \tau_h g| \Big).
\end{aligned}$$

This implies that $V_h(\Sigma(t)y) < V_h(y)$ for all $t > 0$.

Step 5 *Conclusion.*

We combine these building blocks in the following manner: by Step 1, the ω-limit set ω is not empty. By Step 3, it consists of trajectories, and by Step 2 every functional V_h is constant along these trajectories. By Step 4, this implies that $V_0(\omega) = 0$. Therefore $\omega = \{\zeta\}$. •

Theorem 1.7 is a simple consequence of Proposition 5.1 and the property of contraction in L^1:

Proof of Theorem 1.7. By (5.1) we can approximate z_0 by functions z_{0n} such that z_{0n} lies between two travelling waves, $\|z_{0n} - z_0\|_{L^1(\mathbb{R})^2} \le 1/n$, and

$$\int_{\mathbb{R}} \Big(\beta(u_{0n}) - \beta(u_0) + v_{0n} - v_0 \Big) = 0.$$

By applying Proposition 5.1 to the sequence of functions $z_n(t) = \Sigma(t)z_{0n}$ it follows that $\|z_n(t)-\zeta\|_{L^1(\mathbb{R})^2} \to 0$ as $t\to\infty$ for all n. Then

$$\begin{aligned}\|z(t)-\zeta\|_{L^1(\mathbb{R})^2} &\le \|z(t)-z_n(t)\|_{L^1(\mathbb{R})^2} + \|z_n(t)-\zeta\|_{L^1(\mathbb{R})^2}\\ &\le \|z_0-z_{0n}\|_{L^1(\mathbb{R})^2} + \|z_n(t)-\zeta\|_{L^1(\mathbb{R})^2},\end{aligned}$$

and by choosing n and t large enough these two norms can be made as small as necessary. •

A. Changes to the main theorem in [24]

In Section 2 we used the main result from [24], inequality (15), under conditions that are not explicitly allowed for in [24]. Here we explain the changes that are necessary to extend the theorem. We will not try to give an impression of the proof of the theorem of Otto, which is lengthy and technical; on the contrary, we will even assume that the reader disposes of a copy of [24], allowing us to be concise. Because of this assumption we shall also use Otto's notation, which corresponds to ours by the following translation table:

$$b\to\beta,\quad \mathrm{sign}^+\to H,\quad a(\nabla u, b(u))\to u_x,\quad f(b(u))\to\mathcal{F}(u,v).$$

Note that $b_i^0=\beta(u_{i0})$ is the initial datum.

The altered statement that we use reads

$$\begin{aligned}\int_Q \Big\{ &\left[(b_1^0-b_2^0)^+ - (b(u_1)-b(u_2))^+\right]\partial_t\gamma\\ &+\mathrm{sign}^+(u_1-u_2)\left[a(\nabla u_1,b(u_1))-a(\nabla u_2,b(u_2))\right]\cdot\nabla\gamma\\ &\qquad+\mathrm{sign}^+(b(u_1)-b(u_2))\left[f(b(u_1))-f(b(u_2))\right]\gamma\Big\}\le 0\end{aligned}\tag{A.1}$$

for all nonnegative $\gamma\in C_0^\infty((-\infty,T)\times\Omega)$. The only difference between this and inequality (15) in [24] lies in the restriction of γ to functions with spatial support inside Ω. This restriction allows us to relax the condition, present in [24] but absent in the application in this paper, that the Dirichlet data on the boundary $(0,T)\times\partial\Omega$ be constant.

The only parts of the proof that are affected by these changes are Lemmas 1 and 2. Essentially, the vanishing of the test function near $\partial\Omega$ implies that $\mathrm{sign}^+(u_1-u_2)\gamma$ is a valid test function even if u_1 and u_2 do not coincide on the boundary. With this remark it follows from an inspection of the proof (notably inequality (27)) that Lemma 1 of [24] is still valid without the hypothesis (19) if γ is restricted to the aforementioned set.

Because of the restriction in test functions the proof of Lemma 2 of [24] requires an intermediary step. Using the new version of Lemma 1 we first prove that

$$\operatorname*{ess\,lim}_{t\downarrow 0}\int_\Omega (b(u(t))-b^0)^+\chi = 0\tag{A.2}$$

for all $\chi\in C_0^\infty(\Omega)$, and a similar result for part b. Similarly to the proof of Lemma 1, we drop the boundary restriction $v^0=u^D$ from (31), but introduce an extra space-dependent test function $\chi \in C_0(\Omega)$ in the ensuing inequality

$$-\int_{(0,T)} \alpha'(t) \int_\Omega \left[\eta^*_\delta(b(u(t)), v^0) - \eta^*_\delta(b^0, v^0)\right] \chi \, dt \le \int_{(0,T)} \alpha(t)\theta(t)\, dt$$
$$\text{for all nonnegative } \alpha \in C_0^\infty((-\infty, T))$$

for some $\theta = \theta(\chi) \in L^1((0,T))$. From this inequality, (A.2) follows by the same argument as in [24]. The original statement of Lemma 2 then follows from (A.2) by a simple approximation argument.

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