1 Introduction

In this set of lecture notes we will go over the $\alpha$-strongly convex proof, show a lower bound for gradient descent, examine regularization, and discuss a new group of algorithms to tackle the online convex optimization setting. Such algorithms include follow the leader (FTL), be the leader (BTL), and follow the regularized leader (FTRL).

2 Gradient Descent in $\alpha$-strongly convex case

A function $f : X \to \mathbb{R}$ is $\alpha$-strongly convex if:

$$f(y) \geq f(x) + \nabla f(x)^T(y - x) + \frac{\alpha}{2}||y - x||_2^2$$

2.1 Proof

**Theorem 1** If we have a function $f$ that is $\alpha$-strongly convex and we set our learning rate for Gradient Descent to be $\eta_t = \frac{1}{\alpha t}$, then our regret is at most $O\left(\frac{G^2}{\alpha} \ln T\right)$, where $G$ is the bound on the gradient.

**Proof:** Let $y$ denote the fixed optimum solution in hindsight.

As $f$ is $\alpha$-strongly convex:

$$f_t(y) \geq f_t(x_t) + \nabla f_t(x_t)^T(y - x_t) + \frac{\alpha}{2}||y - x_t||_2^2$$

and hence

$$f_t(x_t) - f_t(y) \leq \nabla f_t(x_t)^T(x_t - y) - \frac{\alpha}{2}||x_t - y||_2^2 \tag{1}$$

Consider the following potential function:

$$\Phi(t) = \frac{1}{2\eta_{t-1}}||x_t - y||_2^2$$

The change in potential is defined as the following:

$$\Phi(t + 1) - \Phi(t) = \frac{1}{2\eta_t}||x_{t+1} - y||_2^2 - \frac{1}{2\eta_{t-1}}||x_t - y||_2^2$$

Recall from projecting onto a convex set $K$: $x_{t+1} = \Pi_K(x_t - \eta_t \nabla f_t(x_t))$. We use the notation $\nabla_t := \nabla f_t(x_t)$ and plugging the previous in we get
\[ \Phi(t+1) - \Phi(t) \leq \frac{1}{2\eta_t} \|x_t - y - \eta_t \nabla_i\|^2 - \frac{1}{2\eta_t} \|x_t - y\|^2 \]
\[ \leq \frac{1}{2\eta_t} (\|x_t - y\|^2 + \eta_t^2 \nabla_i^2 - 2\eta_t \nabla_i^T (x_t - y)) - \frac{1}{2\eta_{t-1}} \|x_t - y\|^2 \]
\[ \leq \frac{\alpha}{2} \|x_t - y\|^2 + \frac{\eta_t}{2} \|\nabla_i\|^2 - \nabla_i^T (x_t - y) \quad (2) \]

Note, the first line to the second line follows from expanding out the \(\ell_2\) norm. We also use \(1/\eta_t = \alpha t\).

Combining (1) and (2) we get:

\[ f_t(x_t) - f_t(y) + \Phi(t+1) - \Phi(t) \leq \frac{\eta_t}{2} \|\nabla_i\|^2 \]

Summing over time \(t\),

\[ \sum_{t=1}^{T} (f_t(x_t) - f_t(y) + \Phi(T+1) - \Phi(0)) \leq \sum_{t=1}^{T} \left( \frac{\eta_t}{2} \|\nabla_i\|^2 \right) \leq \sum_{t=1}^{T} \frac{1}{(2\alpha t)} G^2 \leq G^2 \ln T / 2 \alpha \]

Summing over \(t\), the first term gives regret. Then we note that \(\Phi(T+1) \geq 0\) and \(\Phi(0) = 0\).

**Theorem 2** The bound \(\Omega(DG\sqrt{T})\) is a tight bound for any algorithm for online convex optimization.

**Proof:** We define the following:

1. \(K := \text{hypercube where } x = \{\pm 1\}^n \text{ vertices} \)

2. \(f_v(x) = v^T x\) where \(v = [-1,1,-1,1,...(\pm 1)^n]\). In other words, we have \(2^n\) linear cost functions, one for each vertex in \(v\).

3. \(D \leq 2\sqrt{n}\) where \(D\) is the diameter. A sketch of this is as follows:

\[ D = \sup\{d(x,y) : x, y \in K\} \]
\[ D \leq \|x - y\|^2 \text{ where } x = \{1\}^n, y = \{-1\}^n \]
\[ D \leq \sqrt{\sum_{i=1}^{n} 2^2} = 2\sqrt{n} \]

4. \(G \leq \sqrt{n}\) where \(G\) is the norm of the cost function gradients.

\[ G \leq \sqrt{\sum_{i}^{n} (\pm 1)^2} = \sqrt{n} \]
At each time step $t$, the adversary gives the function $f_t = f_{v_t}$ where $v_t$ is picked uniformly at random. As the function is random at each step, and $E_v[f_v(x) = 0]$, the online algorithm has zero expected regret, no matter what it does.

We claim that the offline cost is less than $-cn\sqrt{T}$ for some constant $c$ in expectation. This follows as if we consider the overall cost function $F = \sum_t f_t$, then in each coordinate $i$, it is just a sum of $T$ random $\pm$ variables, there is constant probability that it is more than $c\sqrt{T}$ or less than $-c\sqrt{T}$ for some $c$. So the adversary can pick signs appropriately for each coordinate (and hence a vertex on the hypercube), so that the cost is at most $-c'n\sqrt{T}$, and hence the regret is $\Omega(DG\sqrt{T})$.

\section{Follow the Leader, Be the Leader, and Regularization}

\subsection{Follow the Leader}

Follow the Leader (FTL) is an algorithm that at each steps mimics the best offline solution. If the game were to end at time $t-1$, the offline would be at $\arg\min_x \sum_{s=1}^{t-1} f_s(x)$. So this is what online sets $x_t$ to be.

\subsubsection{Explanation}

However, this procedure can be arbitrarily bad. Consider the following example. If we have a set $K: \{-1, ..., 1\}$ and at time step $t = 1$, a function $f_1 \leftarrow (x/2)$, the logical choice would be to choose $-1$. However at time step $t = 2$, we have $f_2 \leftarrow -x$ it makes sense to choose 1. Now we have an online cost of at least $T$ and offline cost of 0. This can be seen as ”over optimizing”. As a solution we introduce the concept of regularization were we add some $R(x)$ term to ”regularize” and prevent too much changing to our function.

As a thought experiment, we pose the following algorithm called Be The Leader (BTL). It is a hypothetical algorithm assuming that the algorithm could see one time step in the future. But it has an interesting guarantee.

\subsection{Be The Leader}

As previously, let $x_{t+1}$ be what FTL would play at time $t+1$.

$$x_{t+1} = \arg\min_{x \in K} \sum_{i=1}^{t} f_i(x)$$

BTL plays $x_{t+1}$ at time $t$.

\textbf{Theorem 3}

$$\sum_{t=1}^{T} f_t(x_{t+1}) \leq \sum_{t=1}^{T} f_t(u) \quad \forall u \in K$$

What this theorem is intuitively saying is that we have a lower bound on the cost of any fixed static optimum.
**Proof:** We do this through a proof by induction. Assume the following expression is true for $T - 1$.

\[
\sum_{t=1}^{T-1} f_t(x_{t+1}) \leq \sum_{t=1}^{T-1} f_t(u) \quad \forall u \in K
\]

Now we set $u$ to be $x_{T+1}$. Now if add $f_T(x_{T+1})$ to both sides we get the following:

\[
\sum_{t=1}^{T-1} f_t(x_{t+1}) + f_T(x_{T+1}) \leq \sum_{t=1}^{T-1} f_t(x_{T+1}) + f_T(x_{T+1})
\]

Now, the lhs is $\sum_{t=1}^{T} f_t(x_{t+1})$. The rhs becomes $\sum_{t=1}^{T} f_t(x_{T+1})$, but as $x_{T+1}$ is the minimizer of $\sum_{t=1}^{T} f_t$, the rhs is at most $\sum_{t=1}^{T} f_t(u)$ for any $u \in K$. So,

\[
\sum_{t=1}^{T} f_t(x_{t+1}) \leq \sum_{t=1}^{T} f_t(u) \quad \forall u \in K
\]

Now we state the significance of this theorem by providing a lower bound to our regret. As

\[
\text{Regret} = \sum_{t} (f_t(x_t) - \arg\min_u \sum_{t} f_t(u))
\]

Because we have just showed:

\[
\sum_{t=1}^{T} f_t(u) \geq \sum_{t=1}^{T} f_t(x_{t+1})
\]

Which implies:

\[
\text{Regret} \leq \sum_{t=1}^{T} f_t(x_t) - f_t(x_{t+1})
\]

Now we move onto Follow The Regularized Leader (FTRL)

### 3.3 Follow the Regularized Leader

#### 3.3.1 Introduction

The idea of adding a strongly convex regularizing term is to prevent excessive oscillating when we optimize.

#### 3.3.2 Algorithm

We assume linear functions and adopt the convention $\nabla_i = \nabla f_i(x_i)$. FTRL is defined as the following procedure:

\[
x_{t+1} = \arg\min_{x \in K} \eta(\nabla_1 + \ldots + \nabla_i)x + R(x)
\]
Theorem 4  FTRL’s regret is bounded by the following expression where $y$ is the optimal solution in hindsight and $\|\cdot\|_*$ is the dual norm and $R(x)$ is $\alpha$-strongly convex w.r.t. a norm $\|\cdot\|$.

$$\text{Regret} \leq \sum_t \frac{2\eta}{\alpha} \|\nabla_t\|^2 + \frac{R(y) - R(x_0)}{\eta}$$

Proof: Consider the following fake game as a thought experiment. At $t = 0$ we have the following function

$$g_0(x) = \frac{R(x)}{\eta}$$ and $g_t(x) = \nabla_T^T x \ \forall t : t \geq 1$

Thus by the previous discussion on FTL and BTL, we have the following bound on regret for FTL wrt costs $g$:

$$\text{Regret (FTL)}: \leq \sum_{t=0}^{T} g_t(x_t) - g_t(x_{t+1})$$

$$\sum_{t=0}^{T} g_t(x_t) - g_t(u) \leq \sum_{t=0}^{T} g_t(x_t) - g_t(x_{t+1})$$

Now in the real game, FTRL does the same moves as above, and regret of FTRL that the RHS is bounded by the following

$$\text{Regret (FTRL)} \leq \sum_{t=1}^{T} f_t(x_t) - f_t(u) = \sum_{t=1}^{T} g_t(x_t) - g_t(u)$$

Note that the summation is from $t = 1$, instead of $t = 0$ previously. But we have

$$\sum_{t=1}^{T} g_t(x_t) - g_t(u) \leq \sum_{t=0}^{T} g_t(x_t) - g_t(x_{t+1}) + \frac{1}{\eta}(R(x_0) - R(u))$$

So, we focus on bounding $\sum_{t=1}^{T} g_t(x_t) - g_t(x_{t+1})$. By definition of $g_t$ we have for $t \geq 1$,

$$g_t(x_t) - g_t(x_{t+1}) = \nabla_T^T f_t(x_t)T(x_t - x_{t+1})$$

$$= \frac{(\nabla_T^T(x_t - x_{t+1}))^2}{\nabla_T^T(x_t - x_{t+1})}$$

(3)

Let $\Phi_t$ denote the function that FTRL is minimizing (the symbol $\Phi$ should not be confused with any potential function here).

$$\Phi_t(x) := \eta(\nabla_1 + \ldots + \nabla_t)x + R(x)$$

As $\Phi_t = \Phi_{t-1} + \eta \nabla_t$,

$$\Phi_t(x_t) - \Phi_t(x_{t+1}) = \Phi_{t-1}(x_t) - \eta \nabla_T^T x_t - \Phi_{t-1}(x_{t+1}) - \eta \nabla_T^T x_{t+1}.$$ 

As $x_t$ is the minimizer of $\Phi_{t-1}$, we have $\Phi_{t-1}(x_t) \leq \Phi_{t-1}(x_{t+1})$ and thus,

$$\Phi_t(x_t) - \Phi_t(x_{t+1}) \leq \eta \nabla_T^T(x_t - x_{t+1})$$
By strong convexity we also have the following:

$$\Phi_t(x_t) - \Phi_t(x_{t+1}) \geq \nabla^T_t \Phi_t(x_{t+1})^T(x_t - x_{t+1}) + \frac{\alpha}{2} \|x_t - x_{t+1}\|^2$$

Now, as $x_{t+1}$ is the minimizer of $\Phi_t$, because standard optimality conditions, $\nabla^T_t \Phi_t(x_{t+1})^T(y - x_{t+1}) \geq 0$ for any $y \in K$ otherwise, one could decrease $\Phi_t(x_{t+1})$ by moving slightly in the direction of $y - x_{t+1}$.

So, $\nabla^T_t \Phi_t(x_{t+1})^T(x_t - x_{t+1}) \geq 0$ and putting everything together gives,

$$\eta \nabla^T_t (x_t - x_{t+1}) \geq \Phi_t(x_t) - \Phi_t(x_{t+1}) \geq \frac{\alpha}{2} \|x_t - x_{t+1}\|^2$$

Returning back to $(3)$

$$g_t(x_t) - g_t(x_{t+1}) = \frac{(\nabla^T_t (x_t - x_{t+1}))^2}{\nabla^T_t (x_t - x_{t+1})} \leq \frac{(\|\nabla_t\| [x_t - x_{t+1}] )^2}{\eta} \frac{\alpha}{2} \|x_t - x_{t+1}\|^2$$

Here we are upper bounding the numerator using the definition of dual norms,

$$\nabla^T_t (x_t - x_{t+1}) \leq \|\nabla_t\| [x_t - x_{t+1}]$$

and lower bounding the denominator using $(4)$. This gives,

$$g_t(x_t) - g_t(x_{t+1}) \leq \frac{2\eta}{\alpha} \|\nabla_t\|^2$$

which finishes the proof. \end{proof}

**References**