1 Introduction

In this lecture, we will consider online problems, in which the input arrives in a random order. That is, we have again an adversary who defines the input we have to optimize over but it cannot determine the order in which we see it. A variant of the secretary problem (as seen in last lecture) would be that we have an adversary who defines weights $w_1, \ldots, w_n$. We see the weights in random order and have to stop the sequence at some point. Our reward is the weight at which we stop the sequence.

We will always do a competitive analysis, in which we compare the expected value that our algorithm achieves, $E[\text{ALG}]$, with the value of the optimal offline solution, $\text{OPT}$. The expectation is over the random arrival order. For the secretary problem, the best competitive ratio $E[\text{ALG}]/\text{OPT}$ that can be achieved is $\frac{1}{e}$. We showed essentially this result in the last lecture for algorithms that can only do pairwise comparisons. But it also holds generally, even if the algorithm can observe the numbers.

2 Edge-Weighted Matching

We first consider a variant of edge-weighted matching, introduced by Korula and Pál [5].

1. Adversary chooses edge-weighted bipartite graph $G = (L, R, w)$

2. Nature draws permutation of $L$ uniformly

3. Algorithm sees vertices $L$ one after the other with edges and weights; has to select at most one edge; all selected edges must be a matching at all times

We are interested in the competitive ratio $E[\text{ALG}]/\text{OPT}$, where $\text{OPT}$ denotes the value of the max-weight matching in $G$. Note that the algorithm is never allowed to drop previously selected edges. (Otherwise, the problem become significantly different.) We consider the algorithm from [2], which is $\frac{1}{e}$-competitive. This is optimal because this problem generalizes the secretary problem, for which $\frac{1}{e}$ is already optimal.
Algorithm.

**Input**: vertex set $R$, $n = |L|$

**Output**: matching Accept

Discard the first $T - 1$ vertices in $L$;

Accept := $\emptyset$;

for each subsequent vertex $\ell \in L$ do

\[ t := \text{number of revealed vertices in } L; \]

\[ M^{(t)} := \text{max-weight matching in revealed graph}; \]

\[ e^{(t)} := \text{edge assigned to } \ell \text{ in } M^{(t)}; \]

if $\text{Accept} \cup e^{(t)}$ is a matching then

Accept := Accept $\cup e^{(t)}$;

end

end

Notation.

- $e^{(t)} = (\ell_t, r_t)$: tentative edge in round $t$

- $F_t = \begin{cases} 1 & \text{if tentative edge in round } t \text{ is feasible} \\ 0 & \text{otherwise} \end{cases}$

We want to bound $\mathbb{E}[w(M)] = \mathbb{E}\left[ \sum_{t=T}^{n} w(e^{(t)}) F_t \right]$

**Lemma 1** $\mathbb{E}[w(e^{(t)})] \geq \frac{1}{n} w(\text{OPT})$

**Proof**: We have $\mathbb{E}[w(M^{(t)})] \geq \frac{1}{n} w(\text{OPT})$ because we can restrict OPT to the edges that have already arrived by round $t$. The matching $M^{(t)}$ is at least as good at this matching.

Furthermore, we have $\mathbb{E}[w(e^{(t)}) \mid M^{(t)}] \geq \frac{1}{T} w(M^{(t)})$ for the following reason. No matter in which order the online vertices arrive in rounds $1, \ldots, t$, we always get the same matching $M^{(t)}$. Therefore, we can consider the vertex $t$ being drawn uniformly among them.

**Lemma 2** For every choice of $e^{(t)}$, we have $\Pr[F_t = 1 \mid e^{(t)}] \geq \frac{T-1}{t-1}$.

To prove this lemma, observe that it is enough to show the following lemma.

**Claim 3** For all $t' \geq T$, all $r \in R$ and all choices of $\{u_{t'+1}, \ldots, u_n\} \subseteq L$, we have

$$\Pr\left[ \bigwedge_{k=T}^{t'} r \notin e^{(k)} \mid \ell_{t'+1} = u_{t'+1}, \ldots, \ell_n = u_n \right] \geq \frac{T-1}{t'}$$

**Proof**: We prove this claim by induction on $t'$. The claim holds for $t' = T$ because, in this case, we only have to consider the matching $M^{(T)}$. In this matching, there are at most $T$ edges and the vertex $u_T$ is chosen uniformly at random from $L^T$. So the probability that $r$ is matched in step $T$ is $\Pr[r \in e^{(T)}] \leq \frac{1}{T}$.

From now on, we assume that the claim holds for $t' - 1$ and we derive that it also holds for $t'$. Let $E_{t'}$ denote the event that $\ell_{t'+1} = u_{t'+1} \wedge E_{t'+1}$ for some fixed $u_{t'+1}, \ldots, u_n$. 

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Note that by fixing all arrivals in rounds \( t' + 1, \ldots, n \), we also fix the matching \( M^{(t')} \). Therefore, after conditioning on \( E_{t'} \), \( M^{(t')} \) is not random anymore. Let \( S \subseteq V \setminus \{u_{t'+1}, \ldots, u_n\} \) denote the set of online vertices that are not matched to \( r \) in \( M^{(t')} \). We now write the probability as the sum of probabilities of disjoint events, depending on which online vertex arrives in round \( t' \), as follows

\[
\Pr \left[ \bigwedge_{k=T}^{t'} r \notin e^{(k)} \bigg| E_{t'} \right] = \sum_{u \in S} \Pr [\ell_{t'} = u \big| E_{t'}] \Pr \left[ \bigwedge_{k=T}^{t'-1} r \notin e^{(k)} \bigg| \ell_{t'} = u, E_{t'} \right].
\]

The size of the set \( S \) is at least \(|S| \geq t' - 1\) because at most one online vertex is matched to \( r \) in \( M^{(t')} \). Furthermore, for all \( u \in V \setminus \{u_{t'+1}, \ldots, u_n\} \), we have \( \Pr [\ell_{t'} = u \big| E_{t'}] = \frac{1}{t'} \) because conditioning on the event \( E \) effectively restricts which set of online vertices arrives in round \( 1, \ldots, t' \) but not their respective order. Finally, by induction hypothesis, we have

\[
\Pr \left[ \bigwedge_{k=T}^{t'-1} r \notin e^{(k)} \bigg| \ell_{t'} = u, E_{t'} \right] \geq \frac{T - 1}{t' - 1}.
\]

This way, we get

\[
\Pr \left[ \bigwedge_{k=T}^{t'} r \notin e^{(k)} \bigg| E_{t'} \right] \geq (t' - 1) \frac{1}{t'} \frac{T - 1}{t' - 1} = \frac{T - 1}{t'}.
\]

\[\square\]

**Theorem 4** For \( T = \lceil \frac{n}{e} \rceil \), we have \( \mathbf{E}[w(M)] \geq \frac{1}{e} w(\text{OPT}) - o(1). \)

**Proof:**

\[
\mathbf{E}[w(M)] = \sum_{t=T}^{n} \mathbf{E}[w(e^{(t)})F_t] \\
\geq \sum_{t=\lceil \frac{n}{e} \rceil}^{n} \left( \frac{w(\text{OPT})}{n} \cdot \left\lfloor \frac{n}{e} \right\rfloor - 1 \right) \\
= \frac{w(\text{OPT})}{n} \cdot \left\lfloor \frac{n}{e} \right\rfloor \cdot \sum_{t=\lceil \frac{n}{e} \rceil}^{n} \frac{1}{t - 1} \\
\geq \left( \frac{1}{e} - \frac{1}{n} \right) \cdot w(\text{OPT}).
\]

\[\square\]

### 3 Multiple-Choice Secretary Problem

Next, we turn to the multiple-choice secretary problem \[1, 4]. It is the natural extension of the secretary problems, in which one has to make up to \( k \) choices.

1. Adversary chooses weights \( w_1, \ldots, w_n > 0 \)
2. Nature draws permutation of \( [n] \) uniformly
Algorithm sees item weights one after the other; can select at most \( k \) items

Our measure of performance is again the competitive ratio, \( \mathbb{E}[\text{ALG}] / \text{OPT} \), where ALG now denotes the sum of the weights that we accept and OPT is the sum of the \( k \) highest weights. Note that one way to solve this problem is via the matching algorithm from the first part of this lecture. This means, we already know how to get a \( 1/e \)-competitive algorithm. However, for large \( k \) we can do a lot better. Bobby Kleinberg [4] showed that there is a \( 1 - O(1/\sqrt{k}) \)-competitive algorithm and \( 1 - o(1/\sqrt{k}) \) is not possible. We will show the positive result here but with a different algorithm. The advantage of our algorithm is that it can be extended to general packing LPs [3].

**Algorithm.** In round \( t \), upon arrival of item \( j \)

- Let \( S(t) \) be \( \lfloor \frac{t}{n} k \rfloor \) items of highest value that arrived so far
- If \( j \in S(t) \) and \( |\text{Accepted}| < k \)
  Set \( \text{Accepted} := \text{Accepted} \cup \{ j \} \)

**Theorem 5** The algorithm is \( 1 - O(1/\sqrt{k}) \)-competitive.

**Notation.**
- \( V_t \): Weight of item coming in \( t \)th step
- \( C_t = 1 \) if item in \( t \)th step is in \( S(t) \), 0 otherwise
- \( F_t = 1 \) if at most \( k - 1 \) items are accepted in steps 1, \ldots, \( t - 1 \), 0 otherwise

In the first step, we relate the expected value of \( V_t C_t \) to the weight of the set \( w(S(t)) \).

**Lemma 6**

\[
\mathbb{E} \left[ V_t C_t \middle| S(t) \right] = \frac{1}{t} w(S(t))
\]

**Proof:** With probability \( \frac{|S(t)|}{t} \), we see an item from \( S(t) \), which is then uniformly drawn from \( S(t) \).

To bound the expectation of \( w(S(t)) \), we have to understand the effect of letting only \( \lfloor \frac{t}{n} k \rfloor \) items into this set. Observe that of OPT we have seen \( \frac{t}{n} k \) items in expectation at this point. Therefore, the following lemma about capping random variables at their expectation will be helpful.

**Lemma 7** For every random variable \( X \) with expectation \( \mu \) and variance \( \sigma^2 \), we have \( \mathbb{E} \left[ \min \{ X, \mu \} \right] \geq \mu - \left( 1 + \frac{\sigma^2}{\sigma} \right) \sigma \).

**Proof:** We can write \( \min \{ X, \mu \} = X - \max \{ X - \mu, 0 \} \).

Furthermore \( \mathbb{E} \left[ \max \{ X - \mu, 0 \} \right] \leq \sigma + \sum_{j=1}^{\infty} \sigma \Pr \left[ \max \{ X - \mu, 0 \} \geq j \sigma \right] \leq \sigma \left( 1 + \sum_{j=1}^{\infty} \frac{1}{j^2} \right) = \sigma \left( 1 + \frac{\sigma^2}{\sigma} \right) \).

Given this lemma, we can bound \( \mathbb{E} \left[ w(S(t)) \right] \).
Lemma 8 For all \( t \geq \frac{n}{k} \), we have
\[
E \left[ w(S^{(t)}) \right] \geq \frac{t}{n} w(OPT) \left( 1 - 4 \sqrt{\frac{1}{nk}} \right).
\]

Proof: A simple lower bound on the value of \( S^{(t)} \) is given by only considering the OPT items that are included.
\[
E \left[ w(S^{(t)}) \right] \geq \sum_{i=1}^{k} w_i \Pr \left[ i \in S^{(t)} \right]
\]
Note that \( \Pr \left[ i \in S^{(t)} \right] \) is non-increasing in \( i \), therefore by Chebyshev’s sum inequality, we have
\[
\sum_{i=1}^{k} w_i \Pr \left[ i \in S^{(t)} \right] \geq \frac{1}{k} \sum_{i=1}^{k} w_i \sum_{i=1}^{k} \Pr \left[ i \in S^{(t)} \right] = \frac{w(OPT)}{k} E \left[ |S^{(t)} \cap OPT| \right].
\]

Letting now \( A^{(t)} \) denote the set of items that have arrived by time \( t \), we get
\[
|S^{(t)} \cap OPT| = \min \left\{ |A^{(t)} \cap OPT|, \frac{t}{n} k \right\} \geq \min \left\{ |A^{(t)} \cap OPT|, \frac{t}{n} \right\} - 1
\]
As \( |A^{(t)} \cap OPT| \) is binomially distributed, the variance is at most \( \frac{t}{n} k \).

If \( t \geq \frac{n}{k} \), then we get overall \( E \left[ |S^{(t)} \cap OPT| \right] \geq \frac{t}{n} k - 4 \sqrt{\frac{t}{n} k} \).

Overall, this already means that the expectation of \( V_t C_t \) is approximately \( \frac{1}{n} OPT \) in most steps (which mirrors Lemma 1). Now, it remains to bound the probability that a tentative selection is feasible.

Lemma 9 For all \( t \) such that \( \frac{6}{\sqrt{k}} n \leq t \leq 1 - \frac{6}{\sqrt{k}} n \), we have
\[
\Pr \left[ F_t = 1 \right] = \Pr \left[ \sum_{t'=1}^{t-1} C_{t'} \geq k \right] \leq \exp \left( - \frac{n - t}{\sqrt{k}} \right).
\]

Proof: Let us first observe that the \( C_{t'} \) are independent. Furthermore, we have
\[
E \left[ \sum_{t'=1}^{t-1} C_{t'} \right] \leq \frac{t}{n} k
\]
Now set \( \delta = 1 - \frac{t}{n} \), \( \mu = (1 - \delta) k \)
Using that \( \frac{6}{\sqrt{k}} \leq \frac{t}{n} \leq 1 - \frac{6}{\sqrt{k}} \) we have \( \delta(1 - \delta) \geq \frac{3}{\sqrt{k}} \). So, \( \delta^2 \mu = (1 - \delta)^2 k \geq \delta \frac{3}{\sqrt{k}} k = 3 \delta \sqrt{k} \).
As \( \delta \leq 1 \), \( E \left[ \sum_{t'=1}^{t-1} C_{t'} \right] \leq \mu \), Chernoff gives us
\[
\Pr \left[ \sum_{t'=1}^{t-1} C_{t'} \geq k \right] \leq \Pr \left[ \sum_{t'=1}^{t-1} C_{t'} \geq (1 + \delta)(1 - \delta) k \right] \leq \exp \left( - \frac{\delta^2 \mu}{3} \right) \leq \exp \left( - \frac{n - t}{\sqrt{k}} \right).
\]
Proof of Theorem 5. We now bound $E \left[ \sum_t V_t C_t F_t \right]$. Let $p = \frac{6}{\sqrt{k}}$.

\[
E \left[ \sum_t V_t C_t F_t \right] \geq E \left[ \sum_{t=pm} (1-p)n \right] \geq \sum_{t=pm} \frac{w(OPT)}{n} \left( 1 - 4 \sqrt{\frac{1}{2nk}} \right) \left( 1 - \exp \left( -\frac{n - t}{n} \sqrt{k} \right) \right) \\
\geq w(OPT) \left( 1 - 2p - \sum_{t=pm} \frac{4}{n} \sqrt{\frac{1}{2nk}} - \sum_{t=pm} \frac{1}{n} \exp \left( -\frac{n - t}{n} \sqrt{k} \right) \right) \\
= w(OPT) \left( 1 - O\left(1/\sqrt{k}\right) \right)
\]

References


