Harmonic algorithm for 3-Dimensional Strip Packing Problem

Nikhil Bansal∗ Maxim Sviridenko†

Abstract

In the three dimensional strip packing problem, we are given a set of three-dimensional rectangular items \( I = \{ (x_i, y_i, z_i) : i = 1, \ldots, n \} \) and a three dimensional box \( B \). The goal is to pack all the items in the box \( B \) without any overlap, such that the height of the packing is minimized. We consider the most basic version of the problem, where the items must be packed with their edges parallel to the edges of \( B \) and cannot be rotated.

Building upon Caprara’s work [4] for the two dimensional bin packing problem we obtain an approximation algorithm with a similar performance guarantee of \( T_\infty \approx 1.69 \) where \( T_\infty \) is the well known Harmonic number that occurs naturally in the context of bin packing. The previously known approximation algorithms for this problem had worst case performance guarantees of 2 [7], 2.64 [13], 2.67 [14], 2.89 [10] and 3.25 [11].

1 Introduction

In the three dimensional (3D) strip packing problem, we are given a set three dimensional rectangular items specified by their depth, width and height. The goal is to pack these items into a single three dimensional rectangular box of unlimited height such that the height of the packing is minimized. We consider the most basic variant of the problem, the so called “orthogonal packing without rotations”, where the items are not allowed to be rotated and must be packed parallel to the edges of the box. In any feasible packing, items are not allowed to overlap. By scaling the dimensions appropriately, it can be assumed that the box has width and depth of 1, and that the items have size at most 1 in any dimension. The 2D strip packing problem is defined similarly, and the goal is to find a minimum height packing of two dimensional items into a strip of unit width and unlimited height.

Strip packing and the closely related bin packing problems are one of the most fundamental combinatorial optimization problems and have been studied extensively both theoretically and practically. In the \( d \)-dimensional bin packing problem, we are given a collection of \( d \)-dimensional rectangular items with size at most 1 in each dimension, and the goal is to pack these into the minimum number of unit sized \( d \)-dimensional bins. We will mainly be interested in the case of \( d = 1, 2 \) or 3.

The 3D strip packing is a common generalization of both the 2D bin packing problem and the 2D strip packing problem. When each item has height exactly 1, the 3D strip packing problem is identical to the 2D bin packing problem. Similarly, when the width (or depth) of each item is exactly 1, the 3D strip packing problem is identical to the 2D strip packing problem. The NP-Hardness of 3D Strip packing and 2D bin packing follows easily from the NP-Hardness of the 1D bin packing problem, and hence we will be interested in polynomial time approximation algorithms for these problems. For packing problems, the
worst case approximation ratio usually occurs only for specialized “small” instances, and hence the standard measure used is the asymptotic approximation ratio \( R^\infty \). Given a polynomial time algorithm \( A \), the ratio \( R^\infty_A \) is defined by

\[
R^n_A = \max \{ A(I)/\text{Opt}(I) | \text{Opt}(I) = n \}
\]

\[
R^\infty_A = \lim_{n \to \infty} \sup R^n_A
\]

where \( I \) ranges over the set of all problems instances and \( A(L) \) (resp. \( \text{Opt}(L) \)) is the value of \( A \) (resp. the optimum algorithm)on \( L \). A problem is said to admit an asymptotic approximation scheme (APTAS) if for every \( \epsilon > 0 \), there is an polynomial time algorithm \( A_\epsilon \) with an asymptotic approximation ratio of \( (1 + \epsilon) \).

In their celebrated work, Fernandez de la Vega and Lueker [6] gave the first APTAS for the 1D bin packing problem. Later, this was substantially improved by Karmarkar and Karp [8] who gave a guarantee of \( \text{Opt}(I) + O(\log^2 \text{Opt}(I)) \). For the 2D strip packing problem (which is identical to 1D bin packing if all items have height 1, and hence at least as hard), starting with an asymptotic approximation guarantee of 3 due to Baker, Coffman and Rivest [1] the ratio was improved in a sequence of papers until the breakthrough work of Kenyon and Rémiña [9], where they gave an APTAS for the problem. The algorithm of Kenyon and Rémiña is based on elegant extensions of the ideas of [6] and will play a key role in this paper.

On the negative side, Bansal et al [3] showed that the 2D bin packing problem does not admit an APTAS unless P=NP. This directly implies that the 3D strip packing problem does not admit an APTAS either. Currently, the best known algorithm for the 3D strip packing problem is a relatively recent result due to Jansen and Solis-Oba [7] and has an asymptotic approximation ratio of 2. This was a substantial improvement upon a long sequence of works that achieved approximation guarantees of 3.25 [11], 2.89 [10], 2.67 [14] and 2.67 [13] for the problem. However, better results are known for the 2D bin packing problem. Caprara [4] gave the first algorithm which broke the barrier of 2 and achieved an asymptotic approximation ratio of \( T_\infty \approx 1.69 \). Here \( T_\infty \) is the well known Harmonic number that appears ubiquitously in bin packing. Very recently, the asymptotic approximation ratio for the problem has been further improved to \( 1 + \ln(T_\infty) \approx 1.52 \) [2].

In this paper, we give an algorithm for the 3D strip packing problem with an asymptotic approximation ratio of \( T_\infty \). Even though our algorithm improves the previously known guarantees it is very natural and simple and is based on relating strip packing and bin packing in easy (but non-obvious) ways. These relations between strip packing and bin packing also give more intuitive and simpler proofs (in our opinion) of previously known results. In fact, we begin by giving an alternate view of Caprara’s \( T_\infty \approx 1.69 \) approximation for 2D bin packing. Even though we are reproving a known result, this introduces the main ideas and techniques that will be used in our result for 3D strip packing. We begin by describing the preliminaries in Section 2. In Section 3, we give the alternate proof of Caprara’s result for 2D bin packing and finally we consider the 3D strip packing problem in Section 4.

## 2 Notation and Preliminaries

We will often use bins of sizes different than 1. A bin of size \((h_x, h_y)\) is a rectangular box with \( x \)-dimension \( h_x \) and \( y \)-dimension \( h_y \). We use \((h_x, \infty)\) to denote a strip with unlimited height in the \( y \) direction and \( x \)-dimension equal to \( h_x \). In two dimensions we refer to the \( x \)-dimension as width and \( y \)-dimension as height. For three dimensional boxes and strips, we define the bin size \((h_x, h_y, h_z)\) analogously. Similarly, we use \((h_x, h_y, \infty)\) to denote a strip which is unlimited in the \( z \)-direction. In three dimensions, we refer to the \( x \)-direction as depth, \( y \)-dimension as width and the \( z \)-dimension as height (see Figure 2 for a pictorial view for our labeling of the axes in 3D).
Let $\text{Opt}_{(h_x, h_y)}(I)$ denote the optimum number of bins with size $(h_x, h_y)$ required to pack $I$. Thus $\text{Opt}_{(1,1)}(I)$ is the optimum number of unit size bins required to pack $I$. We also use $\text{Opt}_{(h_x, \infty)}(I)$ to denote the optimum height (y-dimension) required to pack $I$ in a strip of width $h_x$. We use $\text{Apx}_{(h_x, h_y)}(I)$ to denote the number of bins used by algorithm Apx to pack an instance $I$, and use analogous notation for 2D strip packing and 3D strip and bin packing.

We now describe three well-known results which are key building blocks in most packing algorithms.

**Harmonic transformation:** This idea was first introduced by Lee and Lee [12]. Let $k$ be a positive integer and $x$ be a positive real in $(0, 1)$. The Harmonic transformation $f_k$ with parameter $k$ is defined as follows: $f_k(x) = 1/q$ if $x \in (1/(q+1), 1/q]$ for some integer $q = 1, \ldots, k-1$ and $f_k(x) = kx/(k-1)$ if $x \in (0, 1/k]$. The crucial property of this transformation is that

\begin{equation}
\sum_{i=1}^n f_k(x_i) \leq T_k \text{ where } T_k \leq T_\infty + 1/(k-1). \text{ Here } T_\infty \text{ is the Harmonic constant and approximately equal to } 1.691.
\end{equation}

For our purposes, we will use a slight variant of the transformation $f_k$ described above. We call this variant $h_k$. For $x \in (1/(q+1), 1/q]$ where $q = 1, \ldots, k-1$, we set $h_k(x) = 1/q$ as previously, however for $x \in (0, 1/k]$ we set $h_k(x) = x$. Since, $h_k(x) \leq f_k(x)$ for all $x$, it easily follows that Lemma 1 holds for our variant too. Moreover, $T_k \leq T_\infty$ for our variant.

**Next Fit Decreasing Height (NFDH) Algorithm:** This idea was first introduced by Coffman et al [5]. Suppose we are given a collection of rectangles that need to be packed in a 2D bin. The NFDH algorithm orders the rectangles in the decreasing order of their heights and greedily packs them in this order into shelves. A set of rectangles is said to be packed in a shelf if their bottom faces all lie on the same horizontal line. More specifically, starting from the bottom of the bin, the rectangles are packed left justified in a shelf until the next rectangle will not fit. This rectangle is then used to define a new shelf and packing continues until no new shelf can be opened. The previously packed shelves are never revisited.

If $w$ and $h$ are relatively small compared to the dimensions of the bin, then the packing produced by NFDH is very efficient. In particular,

**Lemma 2 ([5]):** Given a set of rectangles with width at most $w$ and height at most $h$, if NFDH is used to pack these items in a bin of width $a$ and height $b$, then the total used volume in that bin is at least $(a - w)(b - h)$ (provided there are enough rectangles that NFDH never runs out of them).

**APTAS of Kenyon and Rémiña for 2D Strip Packing [9]:** Given a collection of rectangles $I$, let $\text{Opt}(I)$ denote the height of some optimum strip packing of $I$. The algorithm of Kenyon and Rémiña (KR) takes as input a parameter $\epsilon > 0$ and produces a packing with height $(1 + \epsilon)\text{Opt}(I) + O(1/\epsilon^2)$. We now sketch the algorithm and describe the main ideas. Given a parameter $\epsilon > 0$, the algorithm begins by classifying items with width greater than $\epsilon/(1 + \epsilon)$ as wide and thin otherwise. It then rounds up the widths of wide items into $l = O(1/\epsilon^2)$ different values $w_1, \ldots, w_l$, in such a way that the optimum solution of this rounded instance is at most $(1 + \epsilon)$ times that of the original instance. Let a configuration denote some positive integer linear combination of these rounded widths that sums up to at most 1. Thus, a configuration is some valid way of placing the wide items along the width of the bin. Let $H_i$ denote the cumulative height of items of width $w_i$ in the instance. The KR algorithm considers a natural linear programming relaxation of the problem. This linear program returns a collection of $l = O(1/\epsilon^2)$ configurations with minimum cumulative height,
such that for each width \( w_i \) the cumulative amount of height of width \( w_i \) in these configurations is at least \( H_i \). In other words, this linear program produces an optimum (fractional) packing of the instance, where the packing is fractional in the sense that a wide item is allowed to be split horizontally into many pieces of the same width. To round this solution, we consider each configuration sequentially and start placing the items of appropriate width as determined by this configuration. The only place some item cannot be placed completely is when a configuration finishes and hence at most one unit of height might be wasted. Since there are at most \( l = O(1/\epsilon^2) \) configurations, this adds at most \( O(1/\epsilon^2) \) to the height of the packing. Finally, the configurations are pushed to left side of the strip, leaving \( O(1/\epsilon^2) \) empty rectangular regions to the right which are used to pack thin items using the NFDH algorithm. Figure 1 shows an example of these regions. If all thin items do not fit in these regions, the remainder are packed on the top using NFDH.

3 Two Dimensional Bin Packing

The following is a well known technique to obtain a \((2+\epsilon)\) approximation for 2D bin packing using the AP-TAS for 2D strip packing. Given an input instance \( I \), find an almost optimum strip packing \( \text{AP}_{(1,\infty)}(I) \approx \text{Opt}_{(1,\infty)}(I) \). Cut this strip into slices of height 1 using the horizontal lines \( y = i \) for \( i = 1, \ldots, \lfloor \text{AP}_{(1,\infty)}(I) \rfloor \). For each such \( i \), remove the rectangles that intersect the line \( y = i \) and pack them into a separate bin. Clearly this gives a feasible bin packing using at most \( 2\text{AP}_{(1,\infty)}(I) + 1 \approx (2+\epsilon)\text{Opt}_{(1,\infty)}(I) \) bins.

Our approach for 2D bin packing is based on a natural refinement of this idea: Suppose we can produce a strip packing of \( I \) with the property that any item that is cut by the horizontal line \( y = i \) has height at most \( \epsilon \). We call this the \textit{tall not sliced} property. In this case we could take up to \( 1/\epsilon \) such slices and pack them in a single bin. This would yield a bin packing where the number of bins used is at most \((1 + \epsilon)\) times the height of the strip. In general one cannot guarantee an almost optimum strip packing which has the \textit{tall not sliced} property mentioned (indeed, if possible it would imply an APTAS for the 2-D bin packing problem which is impossible unless \( P = NP \) [3]). However, we can show that if the heights of the items have harmonic sizes (i.e. all items with height more than \( \epsilon \) have an height equal to \( 1/q \) for some integer \( q \)), then a simple modification of the KR algorithm can be used to find an almost optimum strip packing that additionally satisfies the \textit{tall not sliced} property. Thus our overall algorithm is to convert an arbitrary instance into one which has Harmonic heights and then find an almost optimum strip packing of this instance that also satisfies the \textit{tall not sliced} property. The approximation factor \( T_\infty \approx 1.69 \) is incurred only in the process of converting an arbitrary instance into one where the heights are Harmonic.

We now describe the details. Given an input instance \( I \), we derive an input instance \( \tilde{I} \) by applying the Harmonic transformation with some parameter \( k > 1/\epsilon \) to the height of each item in \( I \). The following straightforward lemma relates \( \tilde{I} \) to \( I \).

**Lemma 3** The number of bins of size \((1, T_\infty)\) required to pack \( \tilde{I} \) is no more than the number of unit size bins required to pack \( I \). That is, \( \text{Opt}_{(1,T_\infty)}(\tilde{I}) \leq \text{Opt}_{(1,1)}(I) \).

**Proof:** Fix some optimum solution \( \text{Opt}_{(1,1)}(I) \) and use same packing for corresponding items in \( \tilde{I} \). By Lemma 1, if a subset of items in \( I \) fit in a \((1,1)\) bin, then the corresponding items in \( \tilde{I} \) also fit in a bin of size \((1,T_k)\). This gives a feasible packing of \( \tilde{I} \) using \( \text{Opt}_{(1,1)}(I) \) bins of size \((1,T_k)\). As \( T_k \leq T_\infty \) for our variant of the Harmonic rounding, the result follows.

We will be interested in the strip packing of \( \tilde{I} \).
Lemma 4 A slight modification of the Kenyon Rénila algorithm applied on \( \tilde{I} \) produces an almost optimum strip packing that in addition satisfies the tall not sliced property. In particular, the height of this packing produced is at most \((1 + \epsilon)\text{Opt}_{(1,\infty)}(\tilde{I}) + O(1/\epsilon^3)\).

Proof: Consider the KR algorithm described in Section 2. We will form the configuration LP exactly as in the KR algorithm. The only difference will be in rounding the LP solution. In particular, we classify the items in \( \tilde{I} \) as wide and thin and round the widths of wide items into \( l = O(1/\epsilon^2) \) different types such that we lose at most a factor of \((1 + \epsilon)\) in the guarantee. We define the configuration LP on wide items and consider the (at most \( l \)) configurations obtained by solving this LP.

To round the solution consider the configurations in an arbitrary order. In each configuration start placing the items of appropriate width (as dictated by this configuration) in decreasing height order. While placing “tall” items of height \(1/q\), we ensure the property that the starting \(y\)-position of each such item is an integral multiple of \(1/q\). In fact, for simplicity we will assume that whenever we start placing items of a different (new) height in a configuration, we start these items from an integral \(y\)-position. For short items (with height smaller than \(\epsilon\)), we do not care where they start. Figure 1 shows an example of this transformation applied to the KR solution.

Figure 1: The left figure shows the fractional packing and the right side shows the rounded solution. The fractional solution consists of two configurations, the first has height 2.5 and the second has height 1.5. The items denoted with \(U\) have height 1, those with \(H\) have height 1/2, those with \(T\) have height 1/3 and those with \(F\) have height 1/4. Note that on the right, we continue placing items for each width type in a configuration in the decreasing height order. If a particular height for some width type in a configuration is over, we move to the next integer starting position. For example, in the second width type in the first configuration we leave empty space after placing \(H_3\) and start \(T_1\) from height \(y = 2\).

In this rounding, we lose some space only when we change configurations or when we shift from one Harmonic height class to another for some width type. Since there are \(O(1/\epsilon)^2\) different widths for wide items, and \(O(1/\epsilon)\) different possible heights for each of these widths, this can happen at most \(O(1/\epsilon^3)\) times and hence add at most \(O(1/\epsilon^3)\) to the total height used.

It now remains to pack thin items. As in the KR algorithm, the wide items are pushed to left, leaving \(l = O(1/\epsilon^2)\) empty rectangular regions on the right for thin items which are packed using the NFDH algorithm. As before we ensure that tall items of height \(1/q\) start at \(y\) position equal to an integral multiple of \(1/q\). We do it by starting a new shelf each time when the current item has height in a different class from the previous item and ensuring that this shelf starts at some integral multiple of the item size. We do not care where the short items start. Again this adds at most \(O(1/\epsilon) + l = O(1/\epsilon^2)\) to the height, since the height changes \(O(1/\epsilon)\) times and configurations change at most \(l\) times.

By construction, the packing produced satisfies the tall not sliced property. Moreover, since the height of the packing produced above is at most \(O(1/\epsilon^3)\) more than that produced by the KR algorithm, the desired guarantee follows from the guarantee for the KR algorithm.

Final Algorithm: The final algorithm works as follows. Given an instance \(I\), we round up the heights of items and obtain the instance \(\tilde{I}\). After that we apply the modification of KR algorithm in Lemma 4 to \(\tilde{I}\) to obtain a packing \(P\). Since \(P\) satisfies the tall not sliced property, we convert it into a feasible 2D bin packing solution while losing a factor of at most \((1 + \epsilon)\). Since \(\tilde{I}\) was obtained from \(I\) by rounding each item up, this also produces a feasible packing of \(I\).
**Theorem 5** Let $\text{Apx}_{(1,1)}(I)$ denote the number of bins used by our algorithm. Then $\text{Apx}_{(1,1)}(I) \leq (1 + \epsilon)^2 T_\infty \text{Opt}_{(1,1)}(I) + O(1/\epsilon^3)$.

**Proof:** As each item is $I$ is no larger than the corresponding item in $\tilde{I}$, it follows that $\text{Apx}_{(1,1)}(I) \leq \text{Apx}_{(1,1)}(\tilde{I})$. Since the packing produced by Lemma 4 has the tall not sliced property $\text{Apx}_{(1,1)}(\tilde{I})$ is at most $(1 + \epsilon)$ times the height of this strip, and hence it follows that

$$\text{Apx}_{(1,1)}(I) \leq \text{Apx}_{(1,1)}(\tilde{I}) \leq (1 + \epsilon)^2 \text{Opt}_{(1,\infty)}(\tilde{I}) + O(1/\epsilon^3)$$  \hspace{1cm} (1)

Consider the optimum bin packing solution $\text{Opt}_{(1,T_\infty)}(\tilde{I})$ of $\tilde{I}$ into bins of width 1 and height $T_\infty$. Since placing these bins on top of each other yields a feasible strip packing solution for $\tilde{I}$, it trivially follows that

$$\text{Opt}_{(1,\infty)}(\tilde{I}) \leq T_\infty \cdot \text{Opt}_{(1,T_\infty)}(\tilde{I})$$  \hspace{1cm} (2)

By Lemma 3 that relates $I$ and $\tilde{I}$, we have that $\text{Opt}_{(1,T_\infty)}(\tilde{I}) \leq \text{Opt}_{(1,1)}(I)$. This, together with equations 1 and 2 implies desired result. $lacksquare$

## 4 The Algorithm for the 3D Strip Packing

Let $I$ denote the instance of three dimensional items $(x_i, y_i, z_i)$ for $i \in I$. We will assume that the strip is infinite in the $z$ direction. Thus, in our notation we wish to find an approximation to $\text{Opt}_{(1,1,\infty)}(I)$. We will use an approach similar to that in the previous section of applying the Harmonic transformation to one dimension and then relating bin packing and strip packing on this rounded instance. We start with an overview below.

Let $c$ be a large integer, say greater than $1/\epsilon$ for some sufficiently small $\epsilon > 0$. By slicing a strip packing solution in the $z$-dimension at integral multiples of $c$ and repacking the intersecting items (see Lemma 6 for a formal proof), it can be seen that strip packing is related to packing in bins of size $(1,1,c)$ to within a factor of $(1 + 1/c) \leq (1 + \epsilon)$. Thus we will seek to approximate $\text{Opt}_{(1,1,c)}(I)$. To do this, we will consider the strip $(1,\infty,c)$ which is infinite in the $y$-direction. Our idea will be to find a good packing of $(1,\infty,c)$ and then approximate $\text{Opt}_{(1,1,c)}(I)$ by slicing the packing along the $y$-direction. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property. We call this the wide not sliced property.

We show that when all the items have harmonic widths, then we can find an almost optimum packing of the instance into the strip $(1,\infty,c)$ that in addition satisfies the wide not sliced property. As previously in the case for 2D packing, we will incur a loss of a factor of $T_\infty \approx 1.69$ in the overall approximation guarantee in transforming an arbitrary instance into that with Harmonic widths. We now describe the details.

**Lemma 6** For any instance $I$, we have that $\text{Opt}_{(1,1,\infty)}(I)/c \leq \text{Opt}_{(1,1,c)}(I) \leq (1 + 1/c)\text{Opt}_{(1,1,\infty)}(I)/c$

**Proof:** The first inequality is clear since given any packing in bins of size $(1,1,c)$, we can place the bins on top of each other to produce a valid 3D strip packing solution. For the second inequality, consider some optimal solution for the 3D strip packing problem. Make cuts in the $z$-dimension at the integral multiples of $c$. These cuts define $\text{Opt}_{(1,1,\infty)}(I)/c$ bins. Pack items that are cut in the separate bins. Since each item has width at most 1, the number of additional bins of size $(1,1,c)$ required is upper bounded by the number of cuts divided by $c$ which is $\text{Opt}_{(1,1,\infty)}(I)/c^2$. $lacksquare$

Given an instance $I$ of three dimensional items, let $\tilde{I}_1$ denote the instance obtained by applying the Harmonic transformation with some big parameter $k$ to the item widths (i.e. the $y$-dimension). For technical reasons later, we will also require that $\sqrt{k}$ be an integer. The following simple lemma relates $I$ and $\tilde{I}_1$.
Lemma 7 If a set of items in $I$ can be packed in a bin of size $(1, 1, c)$, then the corresponding set of items in $	ilde{I}_1$ can be packed in a bin of size $(1, T_{\infty}, c)$. In particular, this implies that $\text{Opt}_{(1, T_{\infty}, c)}(\tilde{I}_1) \leq \text{Opt}_{(1, 1, c)}(I)$.

Proof: By the argument identical to the one in Lemma 3 each solution to the original instance and bins of size $(1, 1, c)$ is a feasible solution for the rounded instance and bins of size $(1, T_{\infty}, c)$.

The items in the rounded instance $	ilde{I}_1$ are naturally partitioned into $k$ classes according to their rounded $y$-dimension. Let $I_q$ be the set of items with $\tilde{y}_i = 1/q$ for $q = 1, \ldots, k - 1$ and $I_k$ be the set of items with $\tilde{y}_i = y_k \in (0, 1/k]$. We order items in each group according to their $x$-dimension (large to small). Let $i_1^q, i_2^q, \ldots, i_{|I_q|}^q$ be the ordering of items in the set $I_q$.

We now form slabs of items in $I_q$ as follows: For each group $I_q, q = 1, \ldots, k - 1$, we greedily aggregate items in $I_q$ into slabs of height $c$ and width $1/q$. Specifically, let $x_{i_1}^q$ be the maximal $x$-coordinate for items in $I_q$. We greedily pack items in a slab of size $(x_{i_1}^q, 1/q, c)$ by placing them on top of each other in the $z$-coordinate until the last item does not fit, i.e. the total height (or $z$-coordinate) of packed items is at least $c - 1$. After that we repeat the process, we choose the item $i_1^q$ with largest $x$-coordinate out of remaining items in $I_q$, define a slab of size $(x_{i_1}^q, 1/q, c)$ and so on. The following lemma states the simple properties of this packing.

Lemma 8 If the above algorithm packs items $i_1^q, \ldots, i_{|I_q|}^q$ for $t \leq \tau$ from the instance $	ilde{I}_1$ into the slab of size $(x_{i_1}^q, 1/q, c)$, then the sum of $z$-coordinates of packed items is at least $c - 1$ and at most $c$. All items packed in the box have $x$-coordinate at least $x_{i_1}^q$ and at most $x_{i_1}^q$.

Next we define a similar process and prove a similar lemma for the last set of (thin) items $I_k$. In this case we need to be more careful since the items within the set $I_k$ can have different $y$-coordinate. To handle this, we will pack these items in slabs of width $1/\sqrt{k}$ and height $c$. Since, the items in $I_k$ have width less than $1/k$, we will be able to argue that the packing in slabs is “efficient”. To form the slabs, we take the first $l$ items $i_1^k, \ldots, i_l^k$ in the set $I_k$ and try to pack them into a box of size $(x_{i_1}^k, 1/\sqrt{k}, c)$ using a variation of the Next Fit Decreasing Height (NFDH) algorithm that we describe below. Let $l_{\max}^{i_1}$ be the maximal number $l$ for which the algorithm can pack all items $i_1^k, i_2^k, \ldots, i_l^k$ into one bin of size $(x_{i_1}^k, 1/\sqrt{k}, c)$.

We use the following variation of NFDH to form slabs. Consider the items $i_1^k, \ldots, i_{l_{\max}}^k$ and reorder them by the $y$-coordinate (large to small). Let $i_1', i_2', \ldots, i_{l_{\max}}'$ be this order. Starting with the first, place items on top of each other until the last item cannot be placed since the total height of items would exceed $c$. This forms a shelf of size $(x_{i_1'}, y_{i_1'}, c)$, This last item $i_{s}'$ that could not be packed in the previous shelf, starts a new shelf of size $(x_{i_1'}, y_{i_1'}, c)$ and the algorithm places next items on top of each other until the total height of the shelf exceeds $c$. Then we start a new shelf and so on. Note that all shelves have depth $x_{i_1}$ and hence the $x$-dimension is irrelevant and this algorithm is exactly NFDH applied to widths instead of heights. The following lemma states simple properties of this packing which essentially imply that the packing is area efficient.

Lemma 9 If the above algorithm packs items $i_1^k, \ldots, i_{l_{\max}}^k$ for $t \leq \tau$ in a slab of size $(x_{i_1}^k, 1/\sqrt{k}, c)$ then the sum of $z$-coordinates of packed items on any shelf in the slab is at least $c - 1$ and at most $c$. All items packed in the slab have $x$-coordinate at least $x_{i_1}^k$ and at most $x_{i_1}^k$. The set of items packed in the slab satisfies the following inequality $\sum_{i=1}^{l_{\max}} y_i z_i \geq (c - 1)(\frac{1}{\sqrt{k}} - \frac{1}{k})$.

Proof: All items have $x$-coordinate of at least $x_{i_1}$ since the items are considered in the decreasing order of the $x$-coordinate. The height and width of the slab are exactly $c$ and $1/\sqrt{k}$ respectively. As each item in $I_k$ has height at most 1 and width at most $1/k$, the final bound follows from Lemma 2.
Consider the problem of packing the slabs defined above into the strip \((1, \infty, c)\). Since each slab has height \(c\), we can ignore the \(z\)-dimension and hence this is equivalent to packing items in the 2D strip \((1, \infty)\). In particular, for each slab of size \((x, y, c)\) we define one item of size \((x, y)\). Let \(\tilde{I}_3\) denote this 2D strip packing problem. Observe that the width of each item in the instance \(\tilde{I}_3\) is Harmonic (it is either \(1/q\) for \(q = 1, \ldots, k\) corresponding to slabs of items in \(I_q\) or \(1/\sqrt{k}\) corresponding to slabs of items in \(I_k\)). Thus by Lemma 4 applying the (slightly modified) KR algorithm will produce almost optimum 2D strip packing solution of \(\tilde{I}_3\) that also has the wide not sliced property. Thus our final algorithm is the following:

**Final Algorithm:** Given the instance \(I\), apply the Harmonic transformation to the widths of items in \(I\) to obtain \(\tilde{I}_1\). Form slabs of items in \(\tilde{I}_1\) as described above to form the 2D strip packing instance \(\tilde{I}_3\). Apply the algorithm in Lemma 4 to \(\tilde{I}_3\) and convert this solution to a 2D bin packing solution by cutting the strip by lines \(y=1, y=2, \ldots\). This gives a packing of items in \(\tilde{I}_1\) into bins of size \((1, 1, c)\). Place these bins on top of each other in \(z\)-dimension to produce a feasible 3D strip packing.

**Analysis:** Let \(\text{Apx}_{(1,\infty)}(I)\) denote the height of the 3D strip packing produced by our algorithm. Let \(\text{KR}_{(1,\infty)}(\tilde{I}_3)\) denote the value produced by the (variant) of KR algorithm applied to the intermediate 2D strip packing instance \(\tilde{I}_3\). As this solution satisfies the wide not sliced property, by Lemma 4 we obtain

\[
\text{Apx}_{(1,\infty)}(I) \leq c(1 + \epsilon)\text{KR}_{(1,\infty)}(\tilde{I}_3) \leq c(1 + \epsilon)((1 + \epsilon)\text{LP}_{(1,\infty)}(\tilde{I}_3) + O(1/\epsilon^3)) \tag{3}
\]

Here \(\text{LP}_{(1,\infty)}\) refers to the configuration LP lower bound for 2D strip packing. In the configuration LP, each item is allowed to be sliced arbitrarily into many pieces of the same width which can be packed separately. It is easily seen (we defer the details to the full version) that the quantity \(\text{Opt}_{(1,\infty)}(\tilde{I})\) in Lemma 4 can be replaced by \(\text{LP}_{(1,\infty)}(\tilde{I})\).

On the other hand, we can lower bound the optimum 3D strip packing solution as follows. By Lemma 6 that relates 3D strip packing to packing in bins of size \((1, 1, c)\) we get \(\text{Opt}_{(1,1,\infty)}(I) \geq c^2/(c + 1) \cdot \text{Opt}_{(1,1,c)}(I)\). By Lemma 7 that relates the packing of \(I\) to that of \(\tilde{I}_1\), we get \(\text{Opt}_{(1,1,c)}(I) \geq \text{Opt}_{(1,T_{\infty},c)}(\tilde{I}_1)\). By placing the \((1, T_{\infty}, c)\) sized bins side by side along the \(y\)-dimension, we get that \(T_{\infty} \cdot \text{Opt}_{(1,T_{\infty},c)}(\tilde{I}_1) \geq \text{Opt}_{(1,\infty,c)}(\tilde{I}_1)\). As \(c \geq 1/\epsilon\), together these inequalities imply that

\[
\text{Opt}_{(1,\infty,c)}(\tilde{I}_1) \leq (1 + \epsilon)T_{\infty}\text{Opt}_{(1,1,\infty)}(\tilde{I}_1)/c \tag{4}
\]

By equations 3 and 4 it suffices to relate the optimum packing of \(\tilde{I}_1\) in the strip \((1, \infty, c)\) to the LP lower bound on the 2D strip packing solution of the derived instance \(\tilde{I}_3\). This is accomplished in the next lemma. The proof is somewhat technical but the main idea is to exploit the fact that the slabs of items in \(\tilde{I}_1\) (which correspond to items in \(\tilde{I}_3\)) have a very efficient packing.

**Lemma 10** \(\text{LP}_{(1,\infty)}(\tilde{I}_3) \leq \frac{c\sqrt{k}}{(c-1)(\sqrt{k}-1)} \text{Opt}_{(1,\infty,c)}(\tilde{I}_1) + k\). As \(c\) and \(k\) are both at least \(1/\epsilon\), it follows that

\[
\text{LP}_{(1,\infty)}(\tilde{I}_3) = (1 + O(\sqrt{\epsilon}))\text{Opt}_{(1,\infty,c)}(\tilde{I}_1) + O(1/\epsilon).
\]

**Proof:** Consider the optimal solution to pack the instance \(\tilde{I}_1\) into the strip \((1, \infty, c)\). We will show how to convert this solution into a feasible LP solution to the instance \(\tilde{I}_3\) while losing a factor of at most \((c\sqrt{k}/((c-1)(\sqrt{k}-1)))\). Each item defines two hyperplanes in \(z\)-dimension containing its faces. Let \(H_1, H_2, \ldots, H_m\) be the set of distinct hyperplanes ordered by their \(z\) coordinate. Let \(\delta_s\) for \(s = 1, \ldots, m - 1\) be the distance
between hyperplanes \( H_s \) and \( H_{s+1} \). Each set of items that cross the interior between two hyperplanes \( H_s \) and \( H_{s+1} \) defines an instance of the two dimensional bin packing problem and its packing into the strip \((1, \infty)\). Consider the set of \( m-1 \) such instances and their corresponding packings. We multiply the \( y \)-dimension (or height) of every rectangle in the 2-dimensional instance corresponding to the interior of hyperplanes \( H_s \) and \( H_{s+1} \) by \( \delta_s/(c-1) \). Let \( \tilde{I}_3 \) be the two dimensional instance consisting of the union of the \( m-1 \) instances defined by the interior of hyperplanes \( H_s \) and \( H_{s+1} \) for \( s = 1, \ldots, m-1 \). Each such instance has a feasible packing in the strip \((1, \infty)\) of height \( \delta_s \text{Opt}_{(1,\infty,c)}(\tilde{I}_1)/(c-1) \). Therefore, the instance \( \tilde{I}_3 \) has a feasible packing of height at most \( c\text{Opt}_{(1,\infty,c)}(\tilde{I}_1)/(c-1) \) since \( \sum_{s=1}^{m-1} \delta_s \leq c \).

For every three dimensional item \((x_i, y_i, z_i)\) in \( \tilde{I}_1 \) there is a set of corresponding rectangles in the instance \( \tilde{I}_3 \). The sum of their \( y \)-coordinates is equal to \( y_i \cdot z_i/(c-1) \) while their \( x \)-coordinates are equal to \( x_i \). For each slab \( b \) formed from the items in \( \tilde{I}_1 \), let \( S_b \) be the set of rectangles in \( \tilde{I}_3 \) corresponding to the items packed into \( b \).

We now transform the feasible packing for the instance \( \tilde{I}_3 \) into a feasible fractional packing for almost all rectangles in the instance \( \tilde{I}_3 \) the remaining items will be packed separately. Consider rectangles in \( \tilde{I}_3 \) corresponding to some \( I_p \) for \( q = 1, \ldots, k-1 \). There is a natural order on this set of rectangles according to their \( x \)-coordinate (large to small), \( r_1, r_2, \ldots \). The first rectangle \( r_1 \) in this order is packed separately. The second rectangle \( r_2 \) is packed in the fractional packing corresponding to the instance \( \tilde{I}_3 \) into the places occupied by \( S_1 \), i.e. by rectangles corresponding to the items packed in the first box. By Lemma 8 the \( x \)-coordinate of all rectangles in \( S_1 \) is lower bounded by the \( x \)-coordinate of \( r_2 \) and the sum of their \( y \)-coordinates is \( \sum_{i \in S_1} y_i z_i/(c-1) \geq y_i = 1/q \). Therefore, there is enough space in the places occupied by \( S_1 \) to fractionally pack \( r_2 \). We repeat the process, i.e. we fractionally pack \( r_3 \) into the places occupied by \( S_2 \) and so on.

We now show how to pack rectangles in \( I_k \subseteq I_3 \) which are ordered by their \( x \)-coordinate. First we scale all rectangles in the current fractional packing of the instance \( \tilde{I}_3 \) by the factor \( \sqrt[k]{k} \) in the \( y \)-dimension. After that we pack each rectangle in \( I_k \subseteq I_3 \) into the places occupied by the rectangles corresponding to the items packed in the previous box. By Lemma 9 we have enough space in the \( x \)-coordinate and the total sum of \( y \)-coordinates of all rectangles corresponding to the previous box is at least \( 1/\sqrt{k} \).

By the equations (3), (4) and Lemma 10 it follows that

**Theorem 11** The algorithm for 3D strip packing described above has an asymptotic approximation ratio arbitrarily close to \( T_\infty \).

## 5 Concluding Remarks

In this paper we designed a Harmonic approximation algorithm with performance guarantee \( T_\infty \approx 1.69 \) for the 3D Strip Packing Problem analogous to the algorithm by Caprara [4] for the 2D Bin Packing Problem. Recently, an approximation algorithm with better performance guarantee of \( 1 + \ln(T_\infty) \approx 1.52 \) has been designed for the 2D Bin Packing Problem [2] based on a combination of randomized rounding of the natural configuration LP and the Harmonic algorithm. While a similar configuration LP provides a lower bound for the 3D Strip Packing, the techniques from [2] do not seem to generalize to the strip packing problems.
References


