Aggregation Methods for Markov Reward Chains with Fast and Silent Transitions

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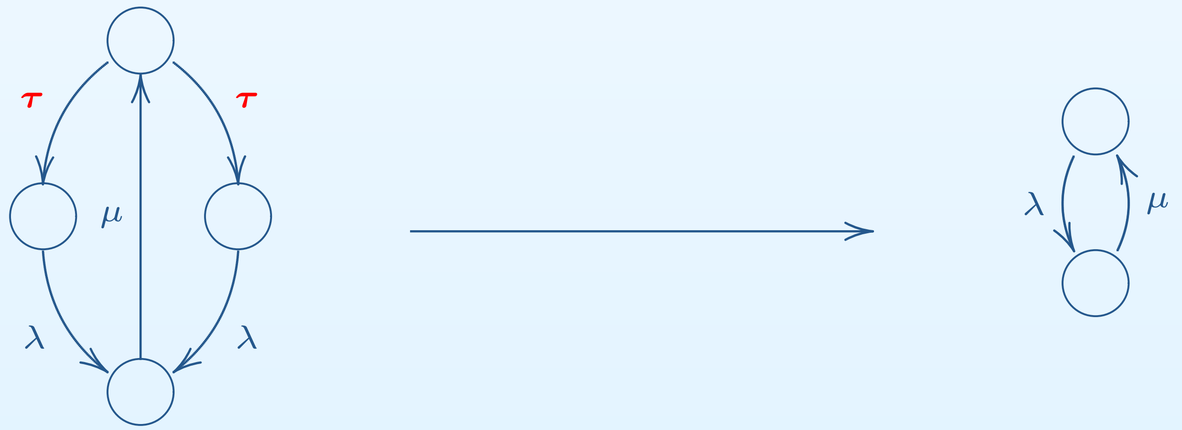

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Markov Chains

Exponential Distribution

- $X \in Exp(\lambda)$ iff
$$P(X \leq t) = 1 - e^{-\lambda t}.$$
- Expected value of $X \in Exp(\lambda)$ is $\frac{1}{\lambda}$.
- The only memoryless continuous distribution
$$P(X > t + \Delta t | X > t) = P(X > \Delta t)$$
- Closed under minimum:

  $X \in Exp(\lambda)$ and $Y \in Exp(\mu)$ implies $min(X, Y) \in Exp(\lambda + \mu)$

Continuous Time Markov Chains

- Stochastic process with the history-less property (future independent of the past).
- Widely popular for quantitative analysis.

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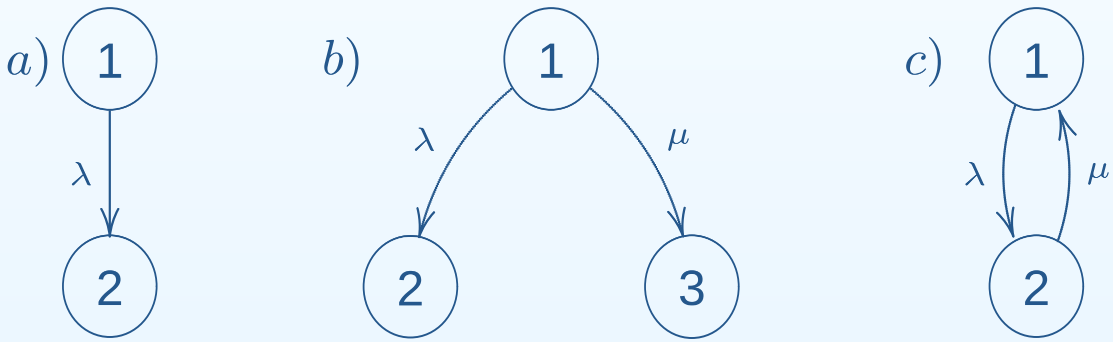

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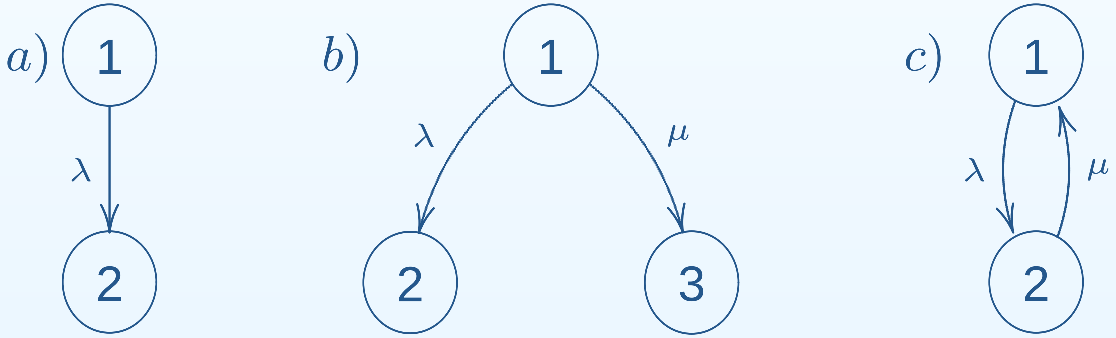


Performance measures:

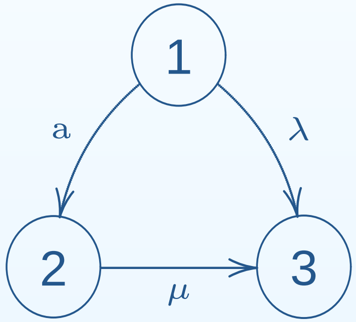
- probability to be in some state before time t,
- probability to enter some state for the first time before t,
- steady state probabilities.

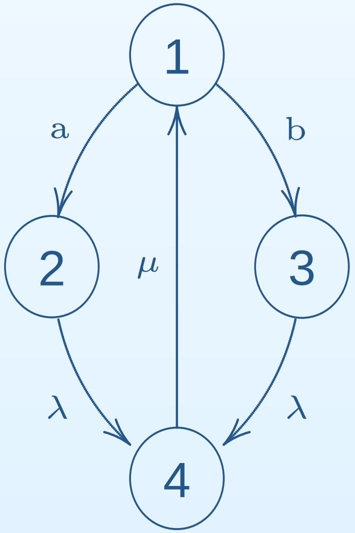
Motivation

- Compositional generation of Markov chains.
- **Idea:** Build big chains from small components.
- Components can interact.
- High level formalisms. Most popular:
  - stochastic (Markovian) process algebras (PEPA, EMPA, IMC, Markovian χ_σ, etc.)
  - stochastic Petri nets.
- Intermediate models obtained first.
- Might contain instantaneous transitions and internal nondeterminism.
- How do we know that the final Markov chain is correct?

Interactive Markov Chains - Hermanns, 2002

IMC
Open system





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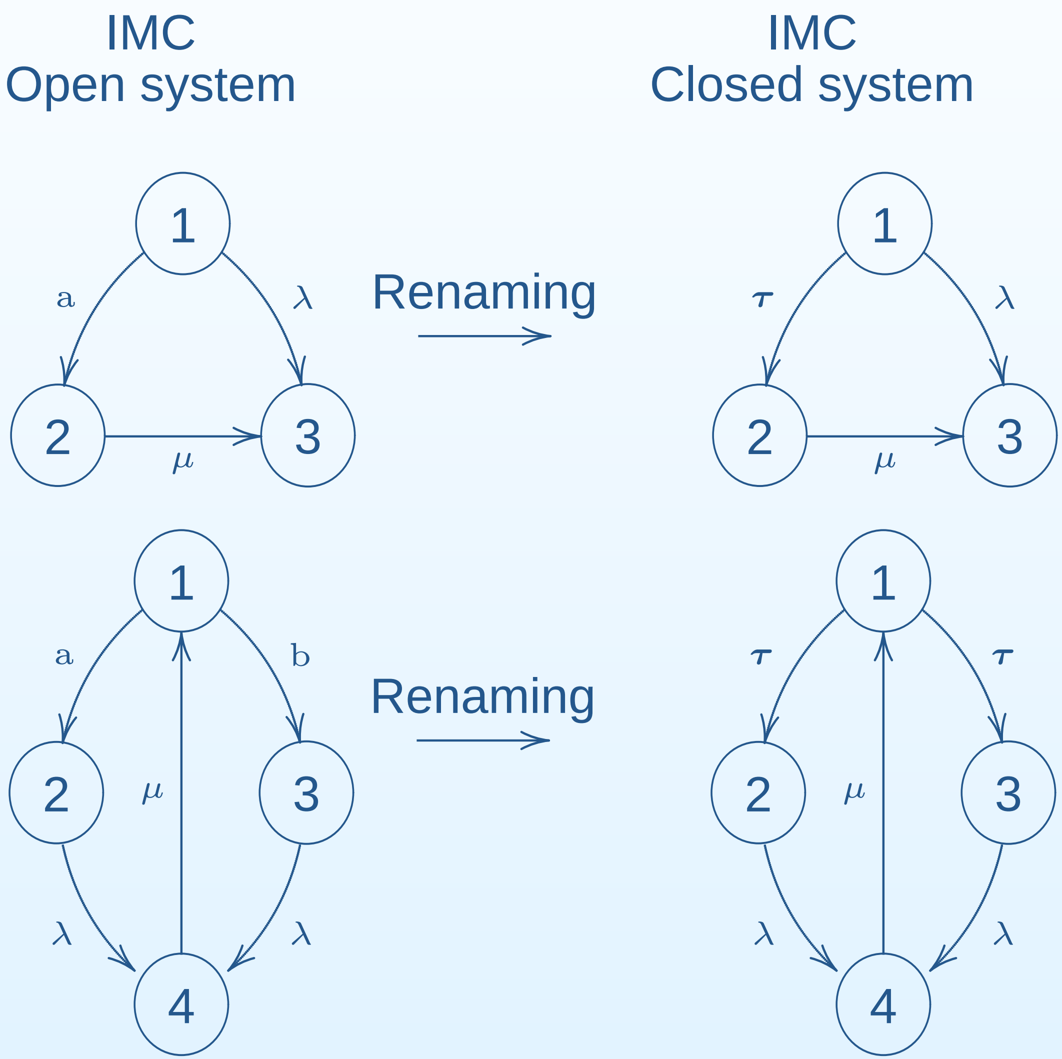

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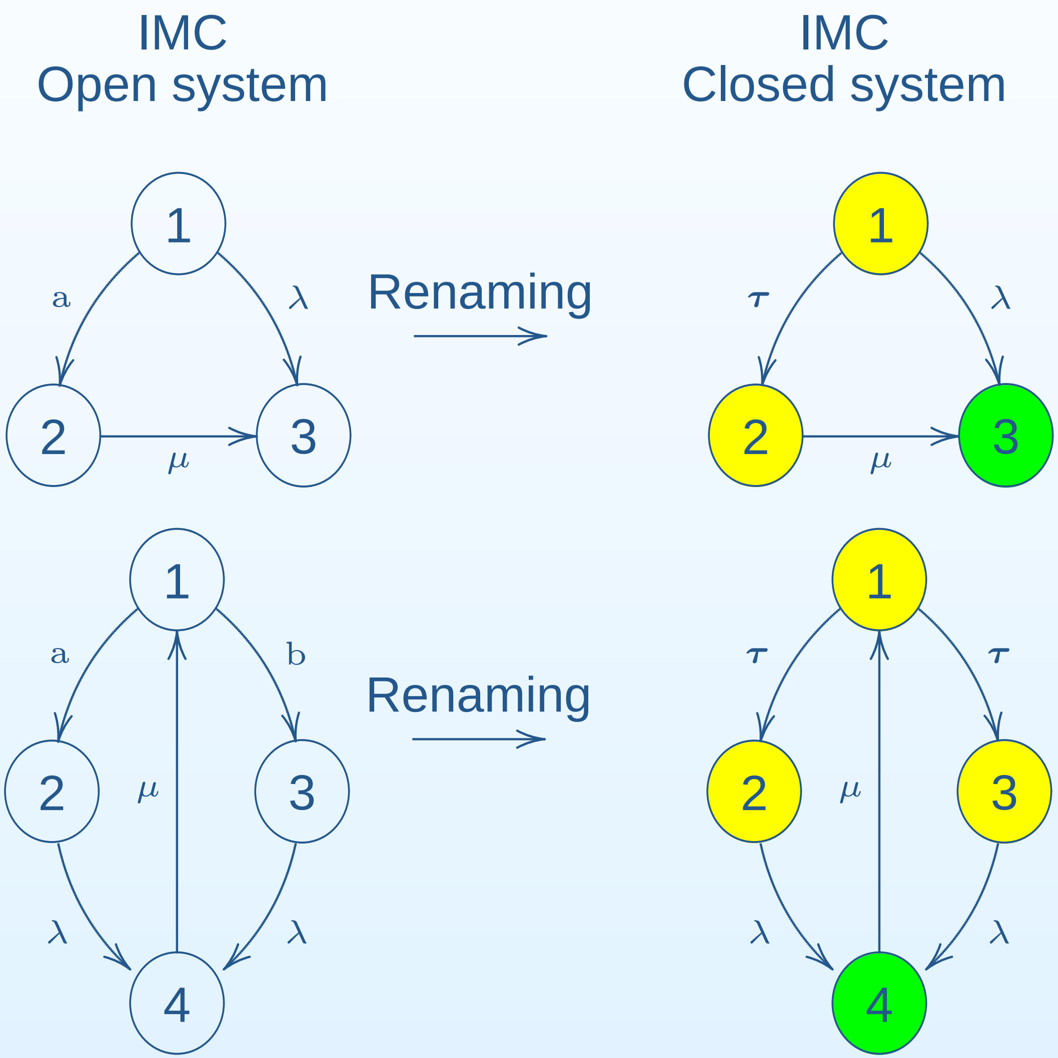

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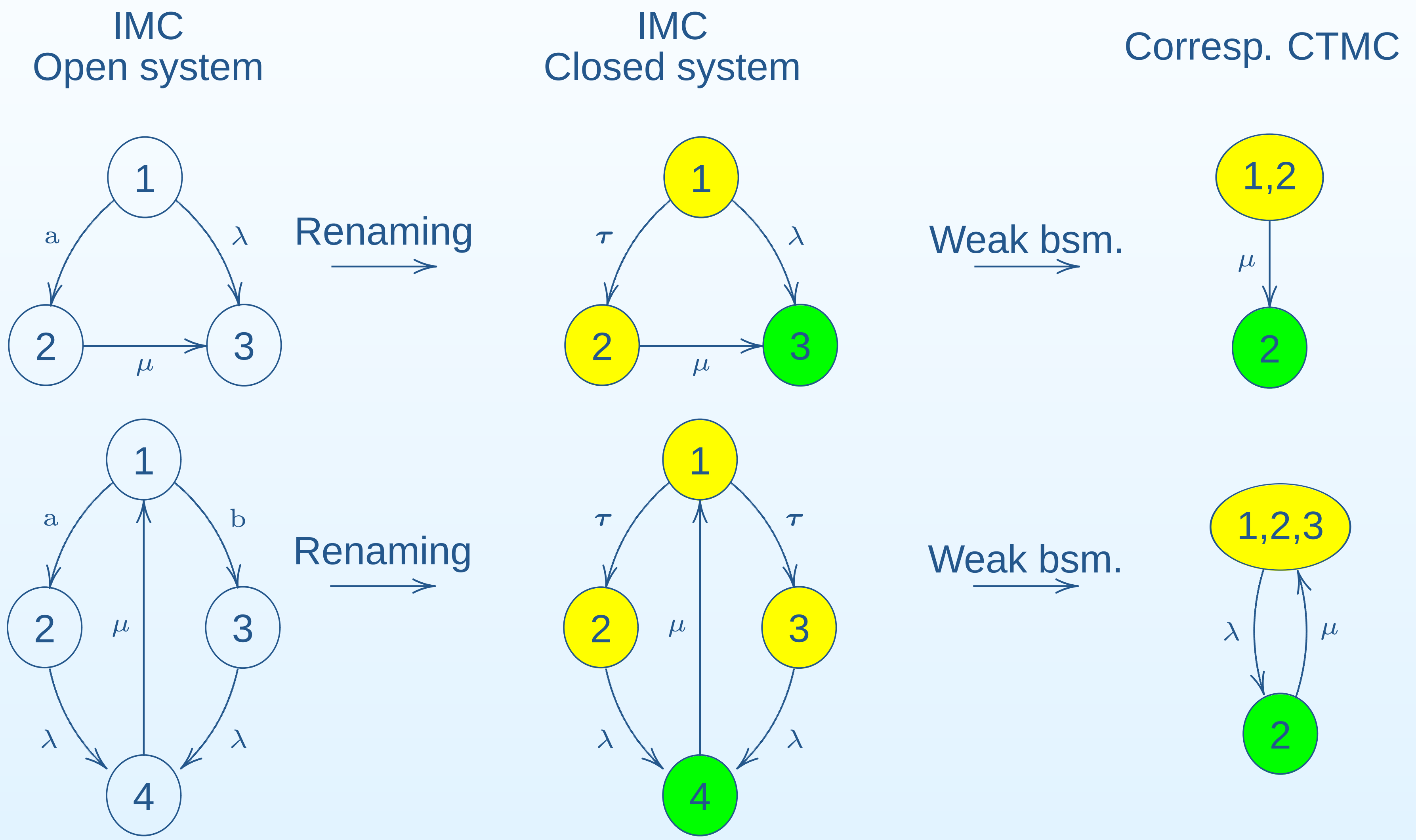

Extended Markovian Models

Discontinuous Markov Chains (Doeblin 1939.)

They:

- generalize Markov Chains,
- contain instantaneous states,
- can perform infinitely many transitions in finite time,
- have sample functions discontinuous everywhere on some intervals,
- are usually considered pathological.

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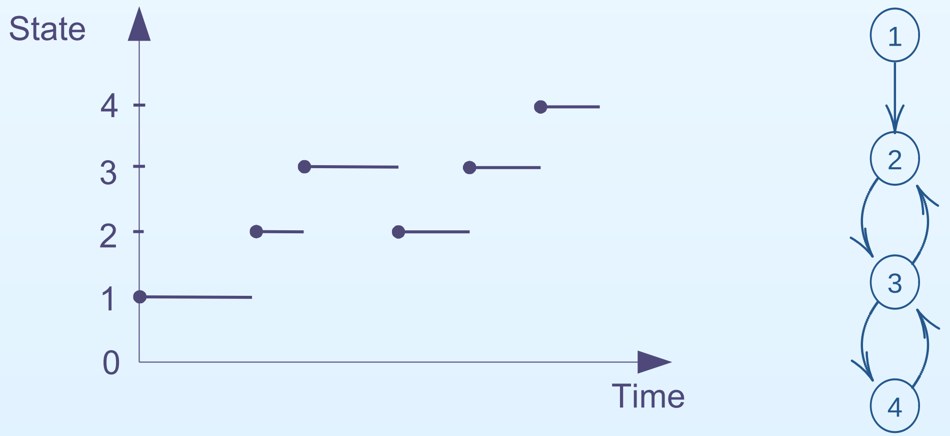

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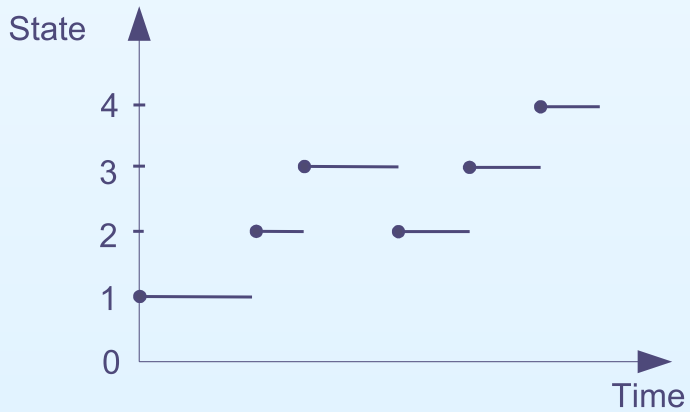
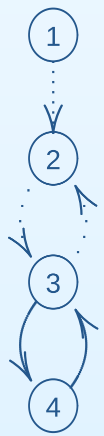
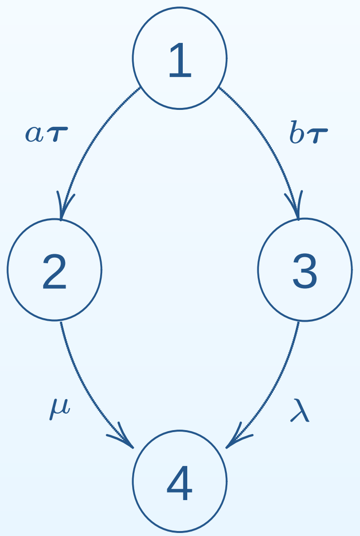




Discontinuous Markov Chain

Markov Chains with Fast Transitions

- Parameterized Markov Chains.
- Some transitions (linearly) depend on $\boldsymbol{\tau} > 0$.



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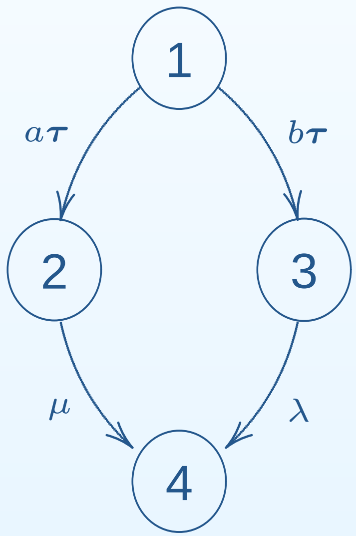

- **Theorem:** When $\boldsymbol{\tau} \to \infty$, Markov Chain with Fast Transitions behaves as a Discontinuous Markov Chain.
- States that can perform a fast transition become instantaneous.

Markov Chains with Silent Steps

- Abstraction from "speeds" in Markov Chains with Fast Transitions.

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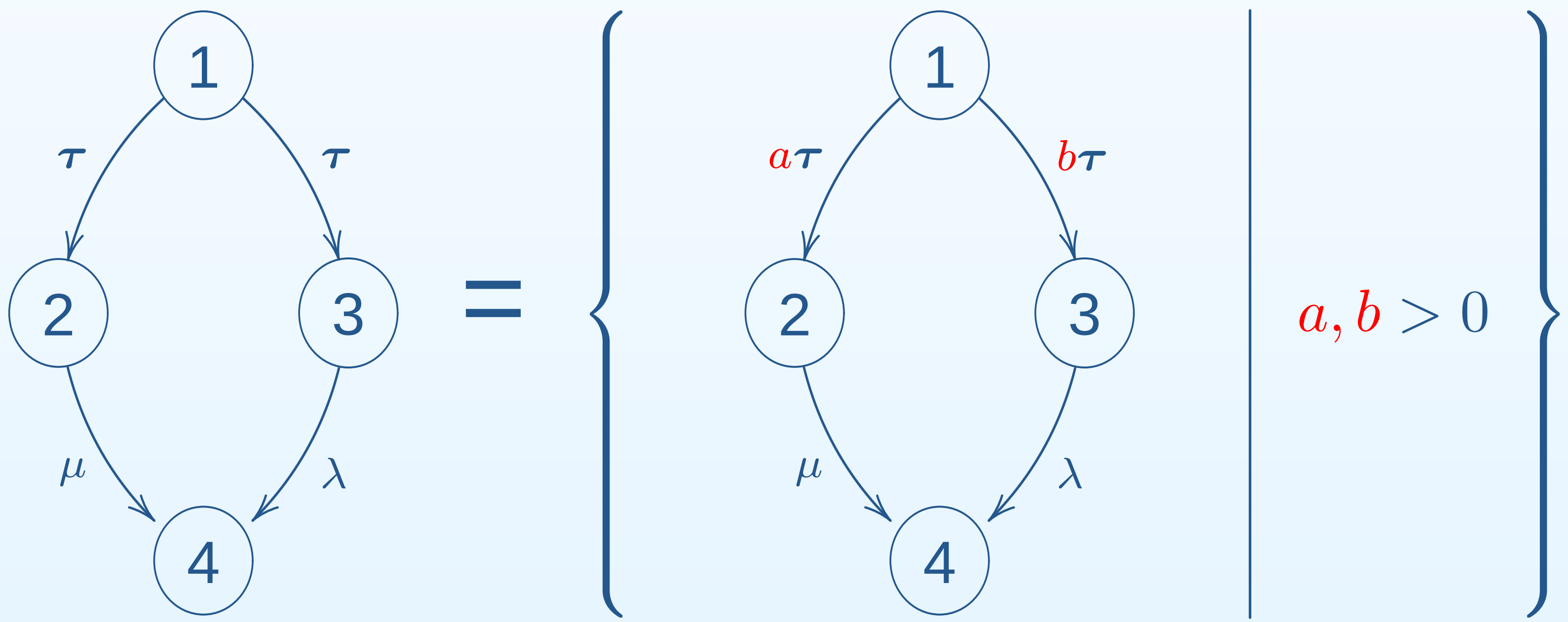
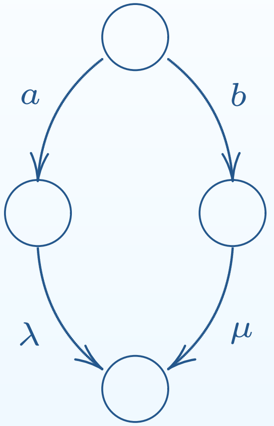
- **Definition:**

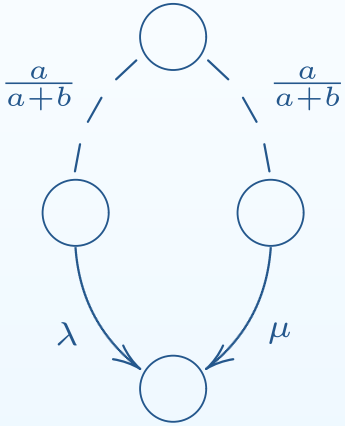


Summary

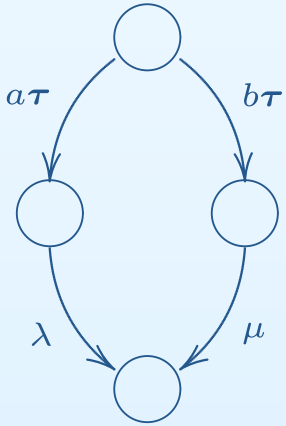
Markov chain



Discontinuous
Markov chain



Markov chain
with fast transitions



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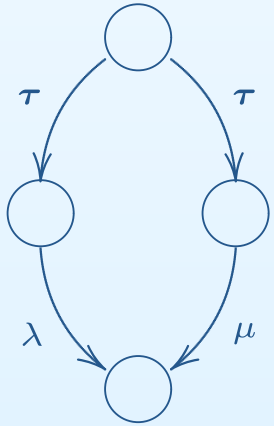

Aggregation by Lumping

Ordinary Lumping for Markov Chains

- The most common way of aggregation.
- States joined into equivalence classes.
- All relevant properties preserved.

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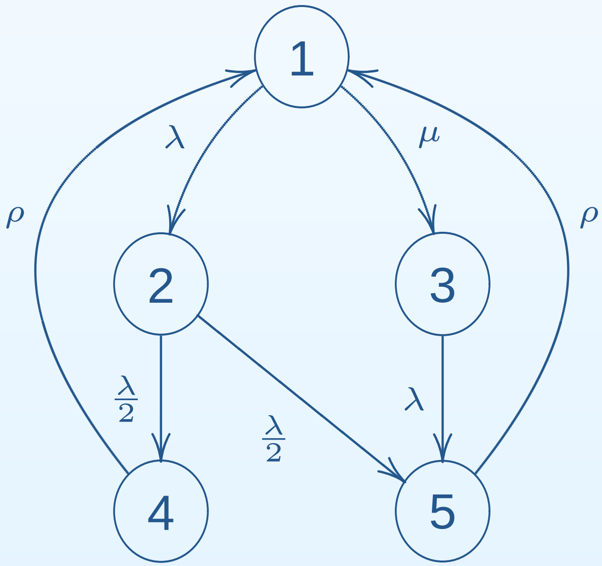

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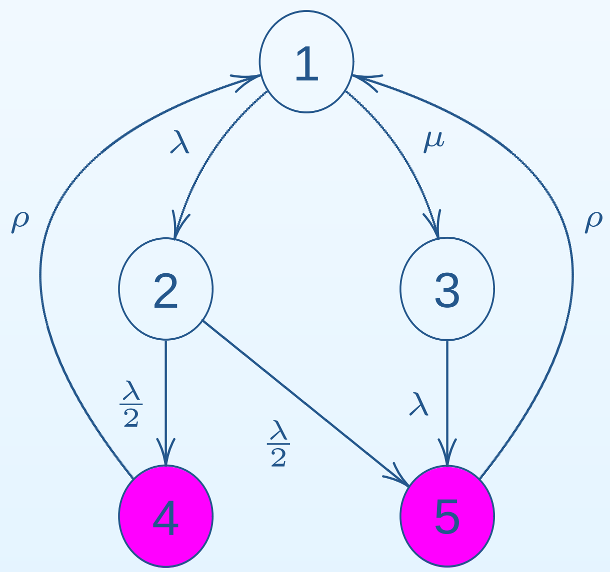

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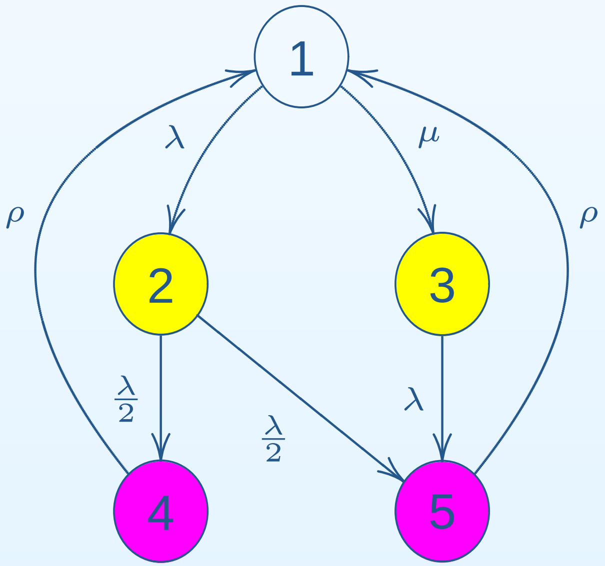

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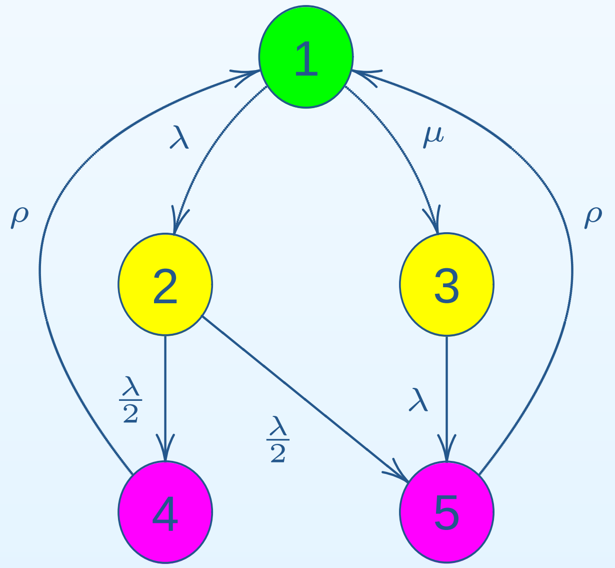

Partitioning $\{\{1\},\{2,3\},\{4,5\}\}$

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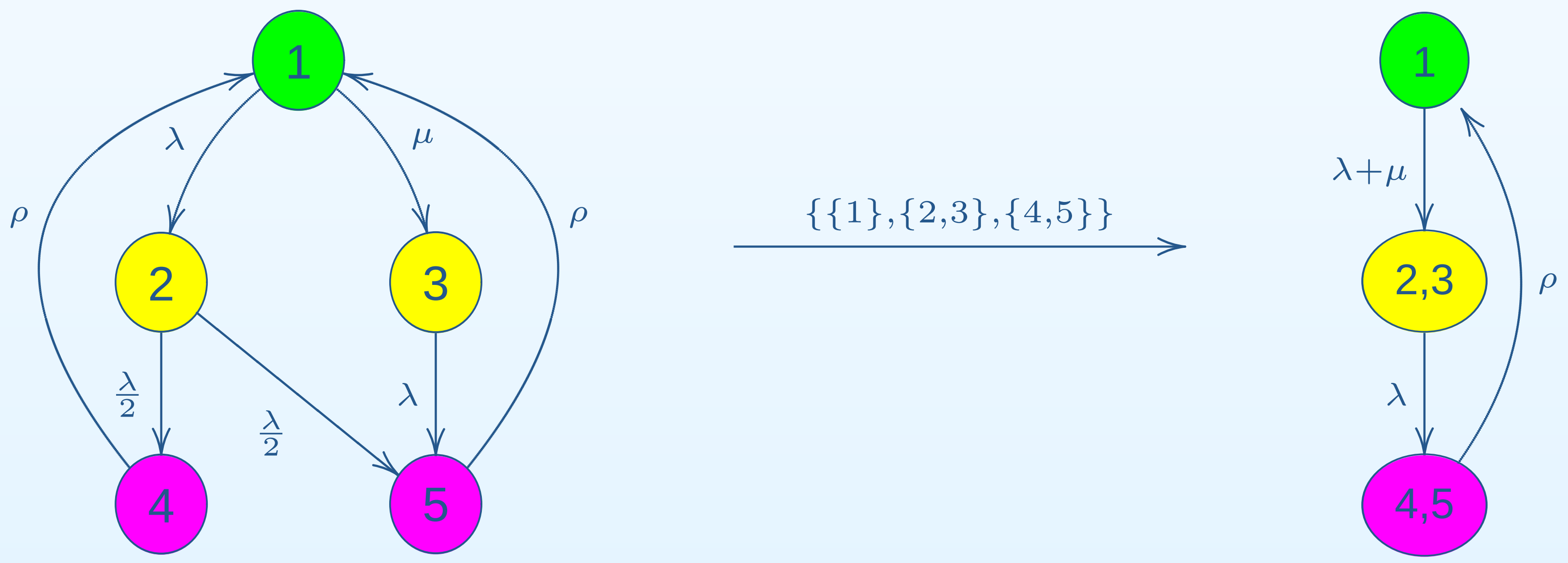

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Lumped process

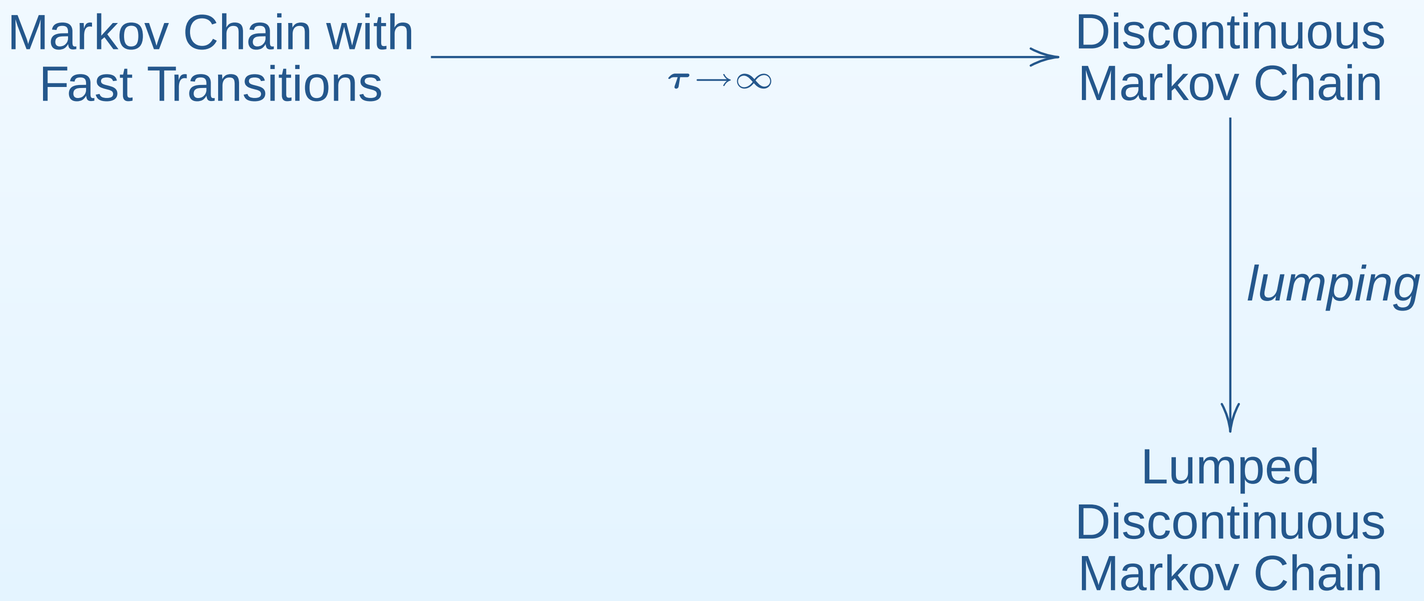
Lumping for Discontinuous Markov Chains

$$\text{Discontinuous Markov Chain} \xrightarrow{\textit{Lumping by } \mathcal{P}} \text{Discontinuous Markov Chain.}$$

- Lumping condition for regular states is the same as in the lumping for Markov Chains.
- For instantaneous states no real intuition.
- Generalization of the lumping condition is based on the generalization of the corresponding matrix conditions.

τ-lumping for Markov Chains with Fast Transitions

- **Definition:** A partitioning is a τ-lumping of a Markov Chain with Fast Transitions if it is a lumping of the limit Discontinuous Markov Chain.



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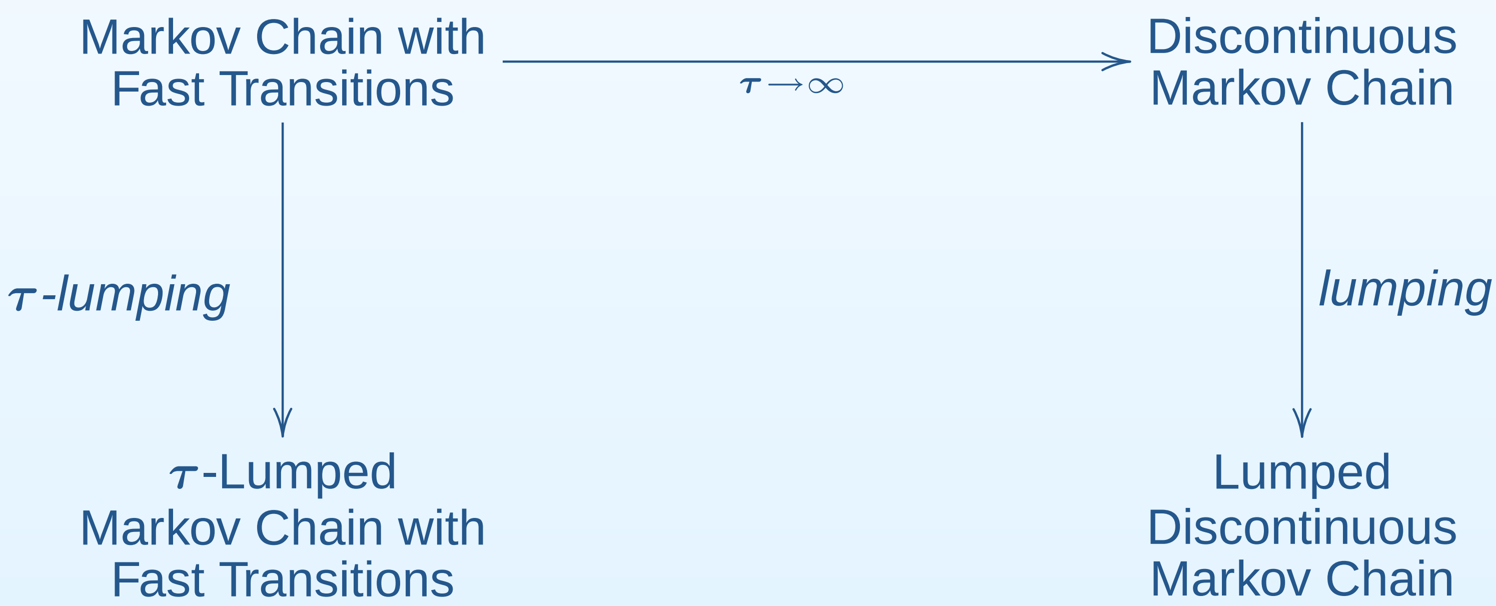

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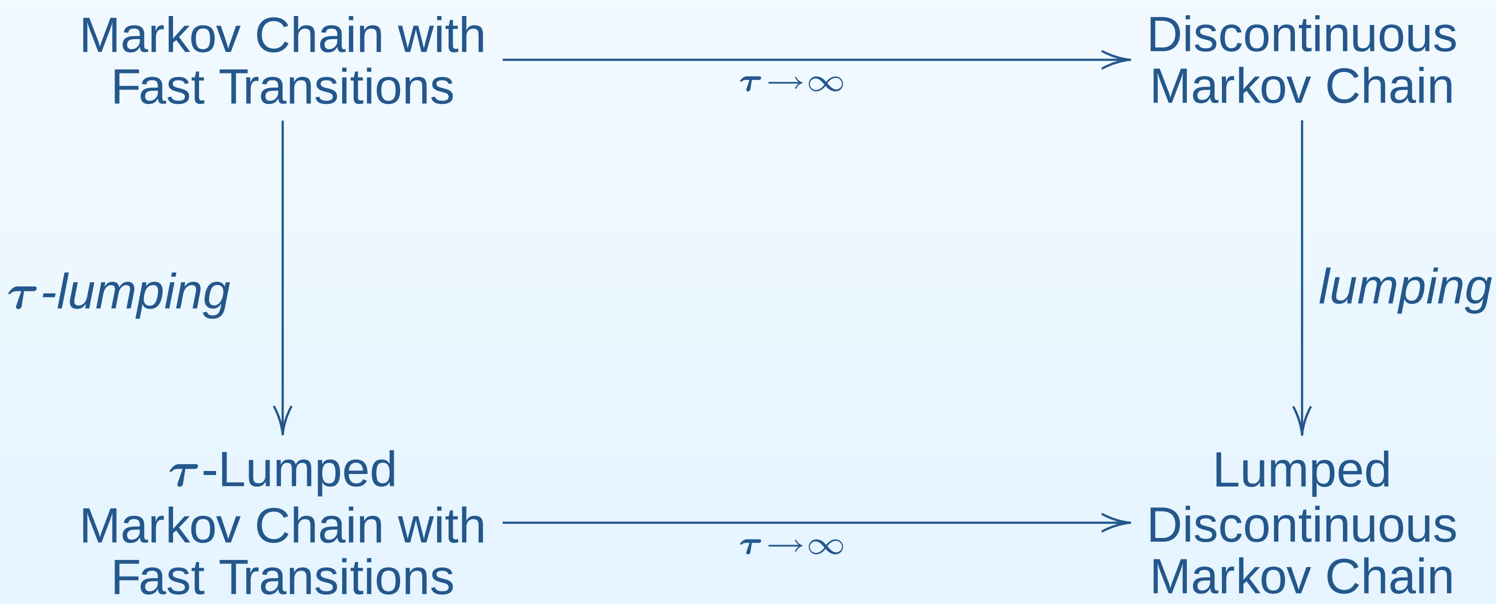
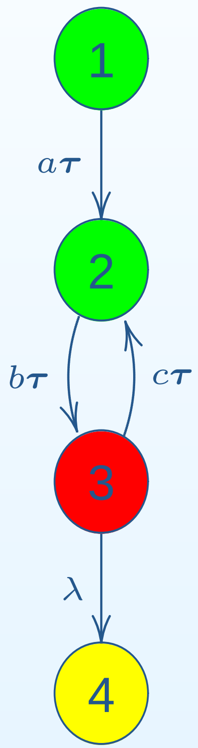
- **Theorem:** τ-lumping is sound.



$\boldsymbol{\tau}$-lumping - Example



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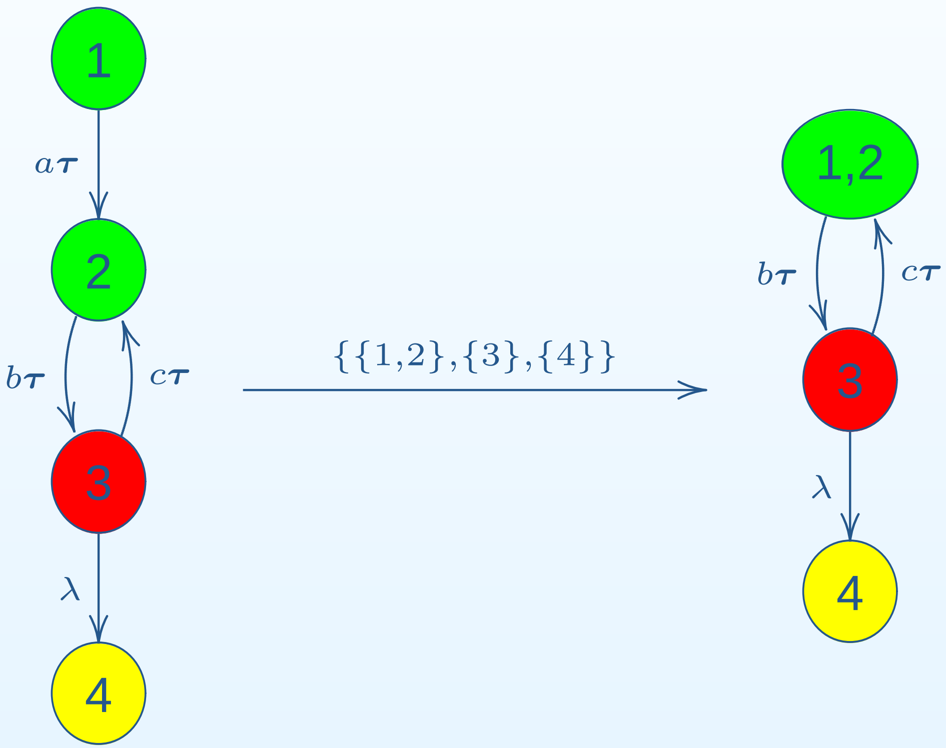

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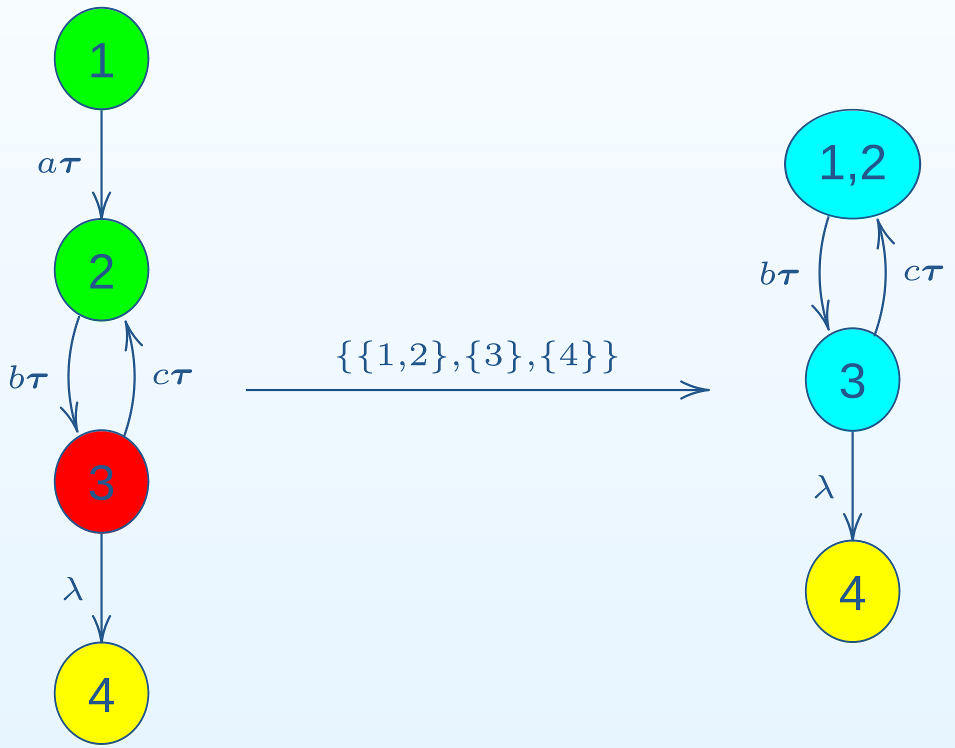

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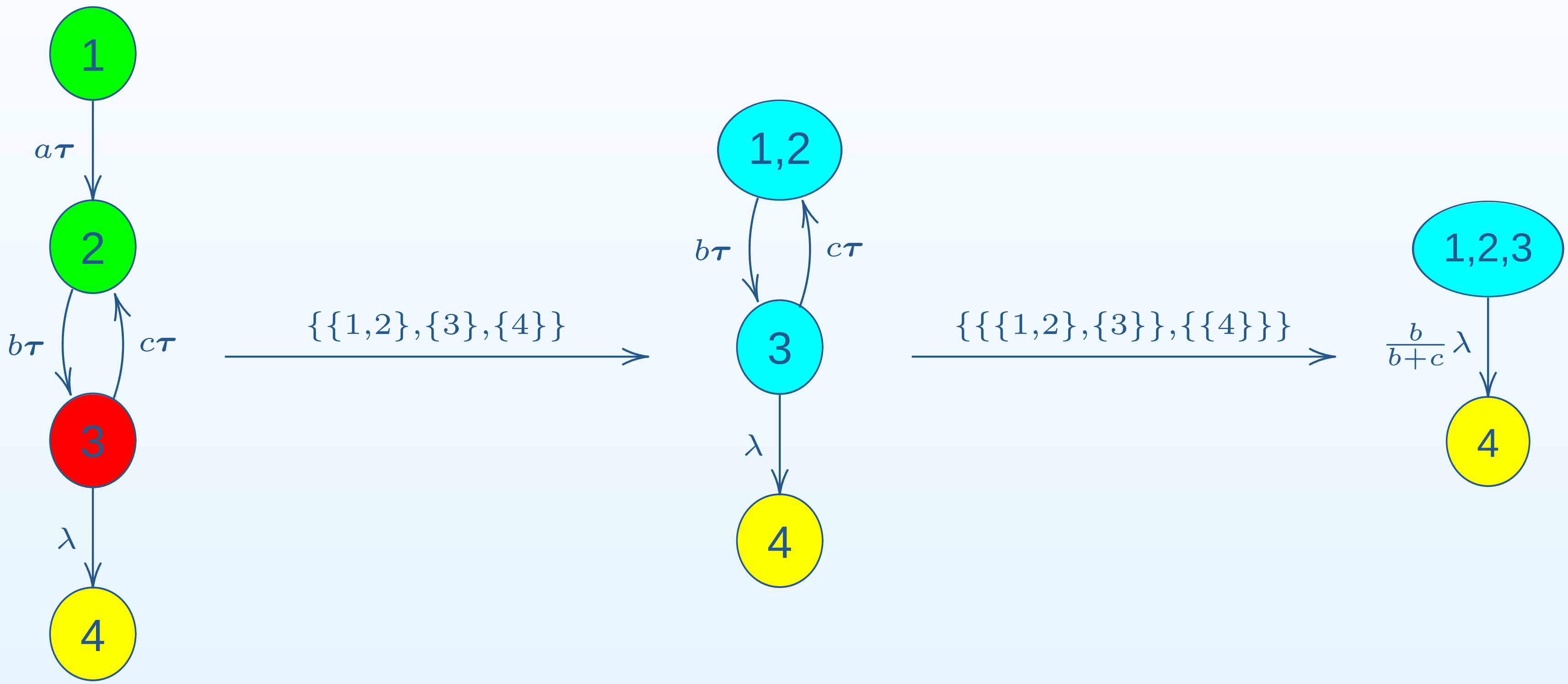

$\tau_{\sim}$-lumping for Markov Chains with Silent Steps

- Lifting of τ-lumping to sets of Markov Chains with Fast Transitions.
- Must be correct:
  1. it should be a $\boldsymbol{\tau}$-lumping for any representative Markov Chain with Fast Transitions;
  2. regular states in the lumped process should not depend on the representative.

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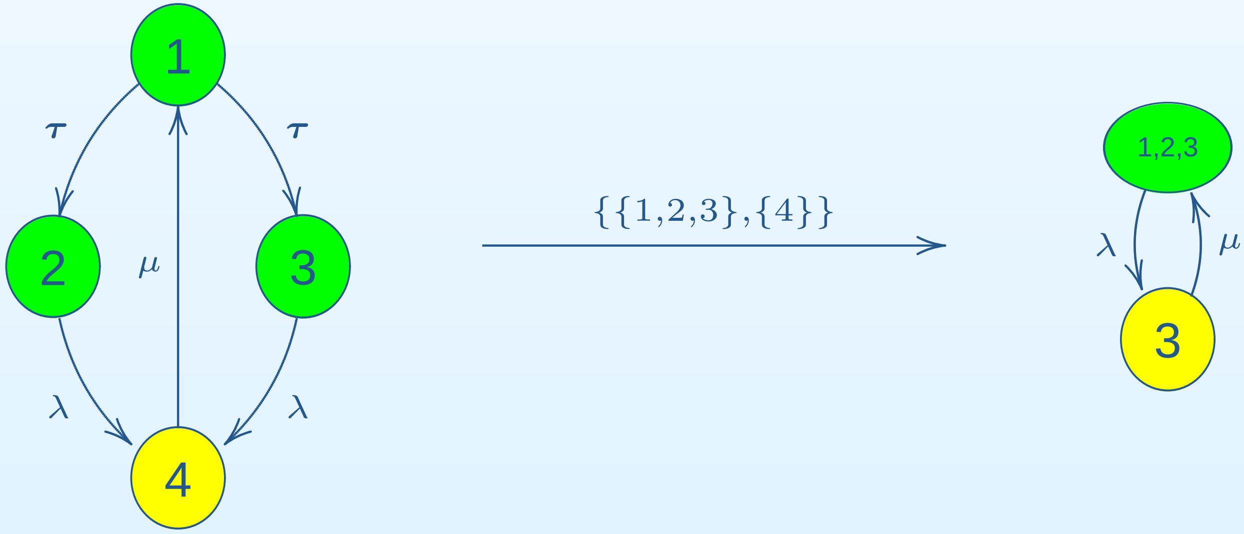

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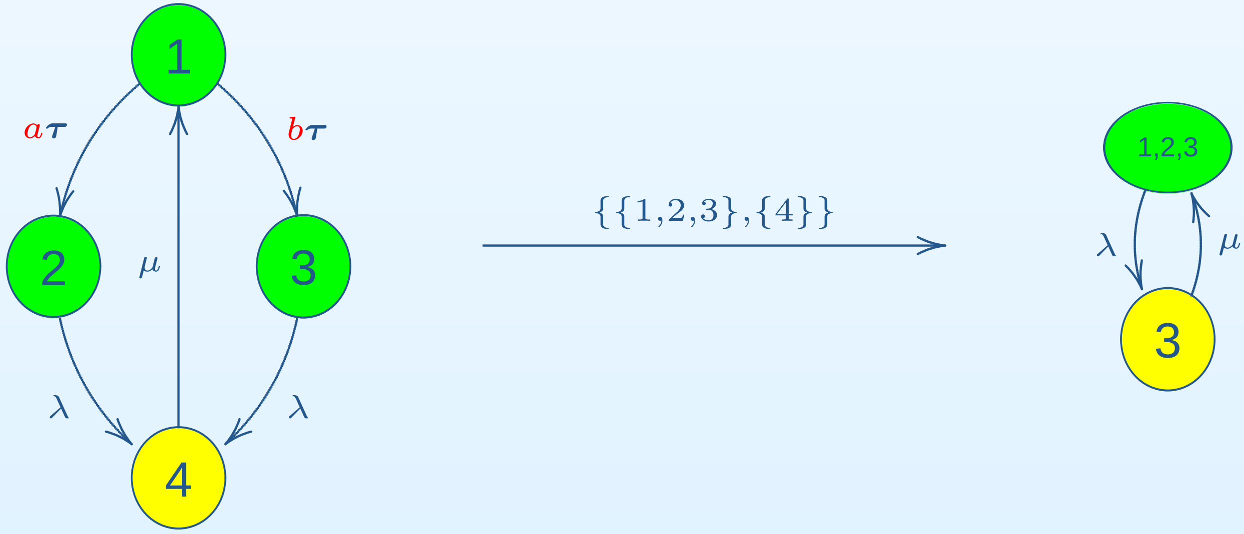

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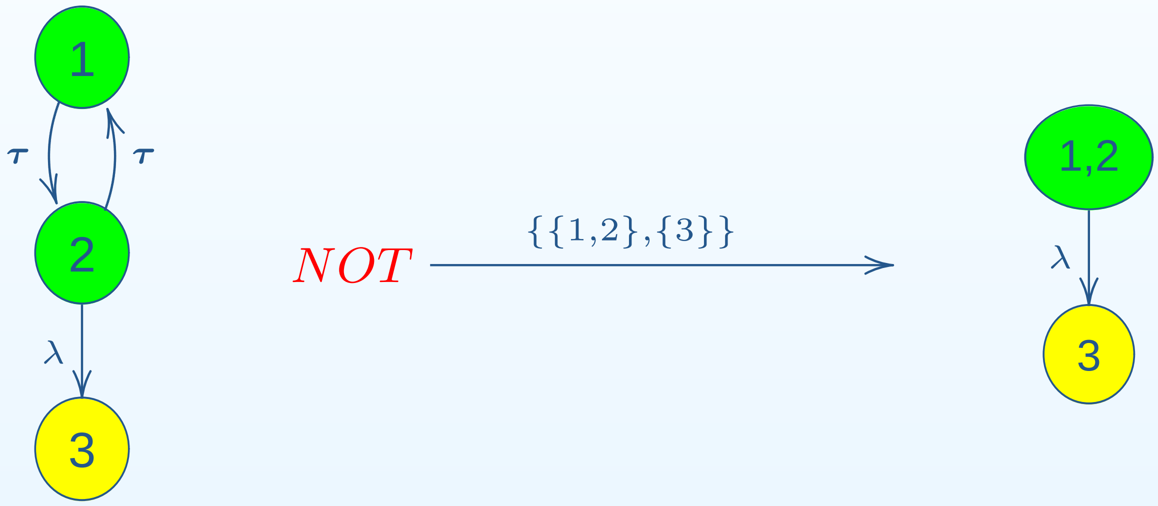

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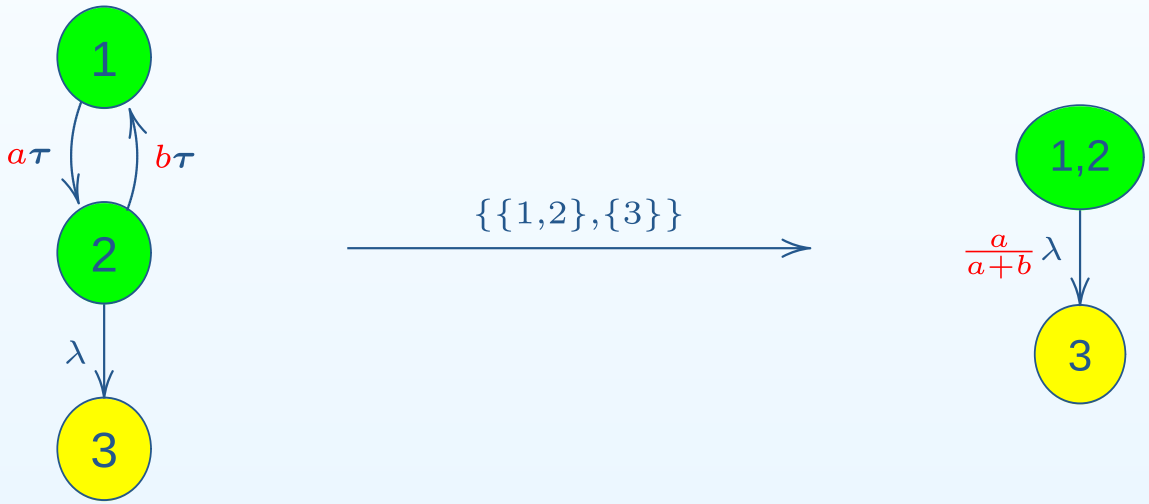
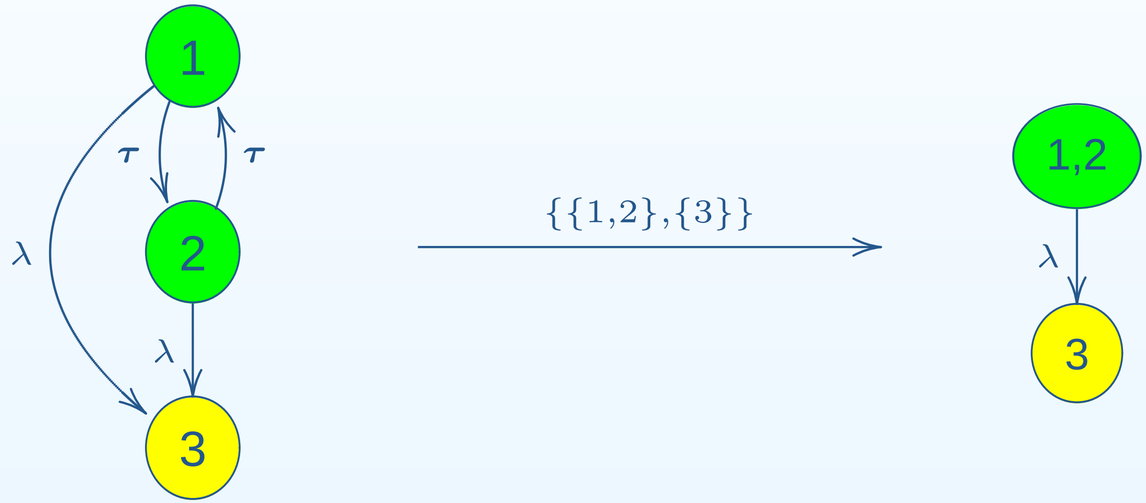


This is because:

- τ-lumping works for any a, b, but the result depends on a, b.

$\tau_{\sim}$-lumping for Markov Chains with Silent Steps

Note, however, that:



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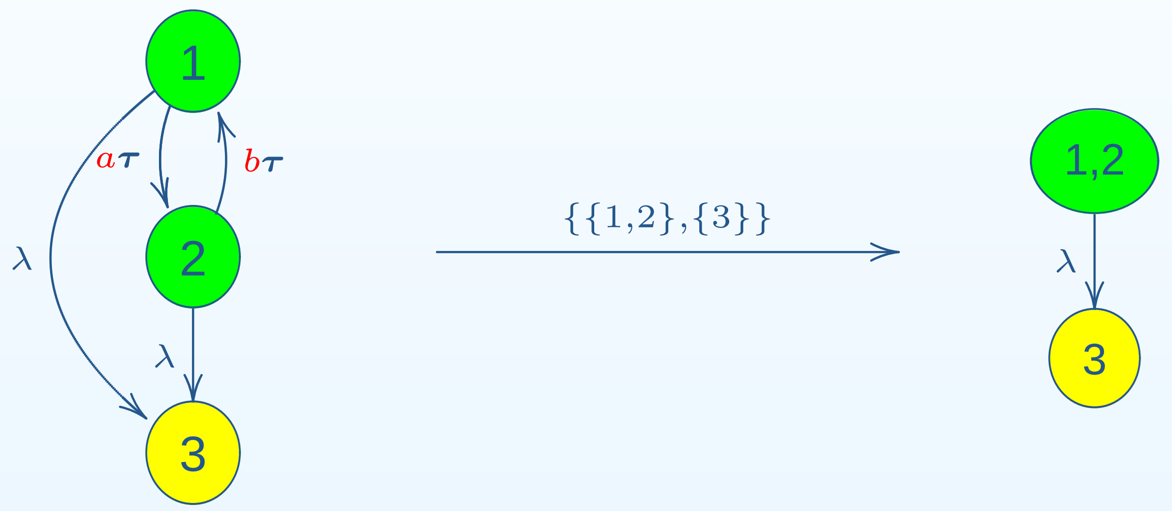
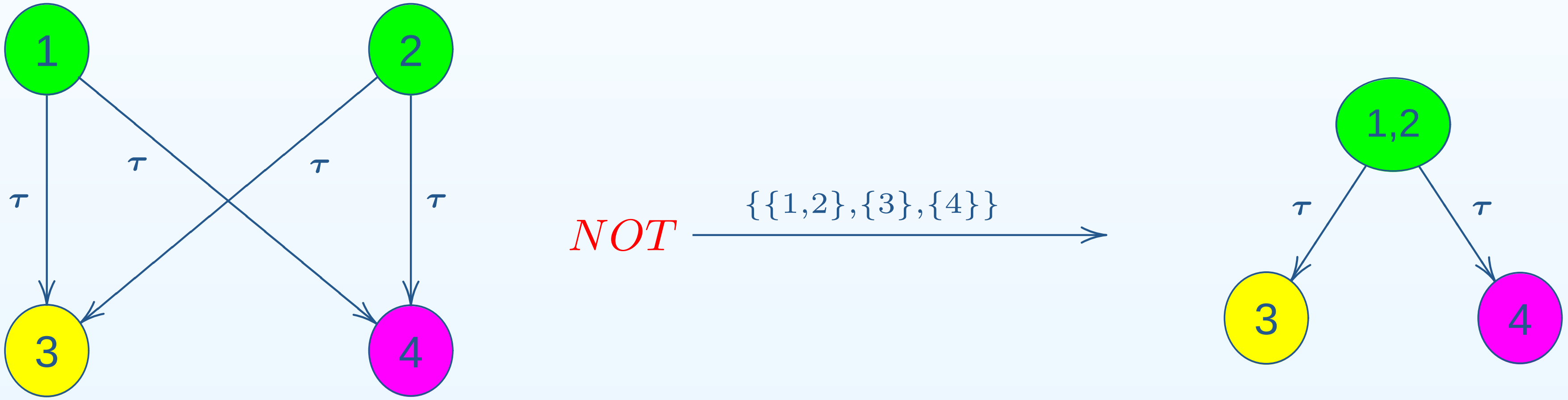


This is because:

- $\frac{a}{a+b}\lambda + \frac{b}{a+b}\lambda = \lambda$.

$\tau_\sim$-lumping for Markov Chains with Silent Steps

Counterexample 2:



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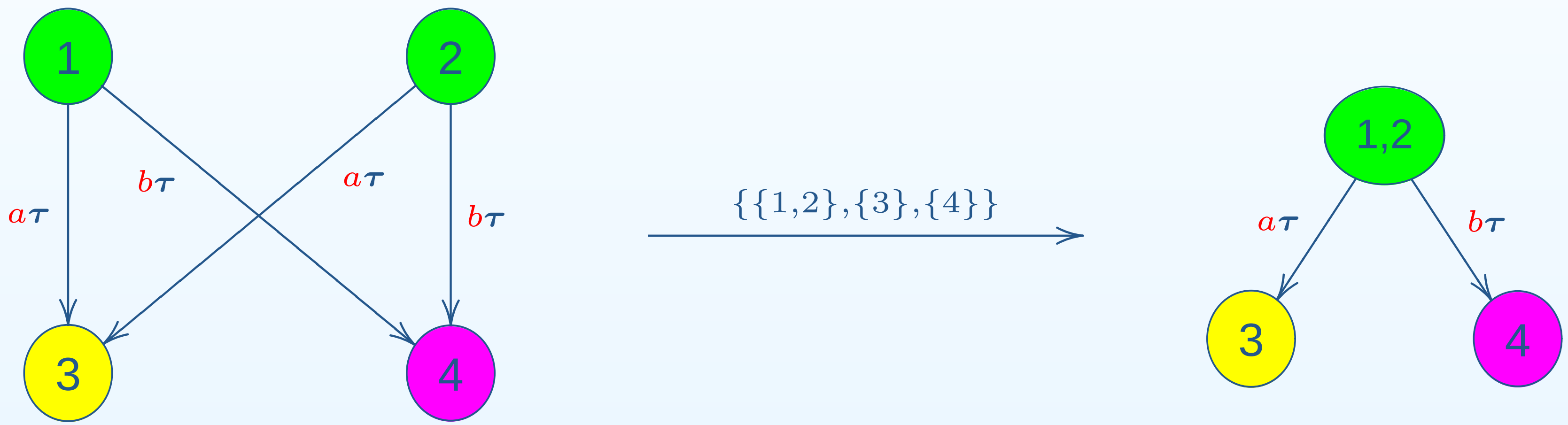

This is because:

- even though it works for arbitrary sequence a, b, a, b,

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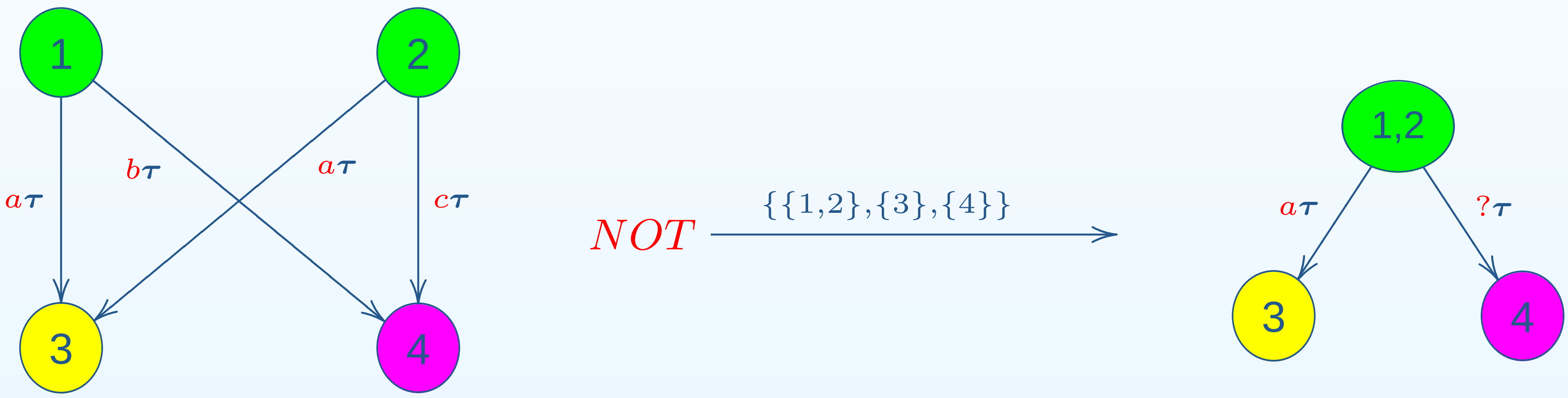


This is because:

- even though it works for arbitrary sequence a, b, a, b,
- it does not work for a, b, a, c.

$\tau_{\sim}$-lumping - Direct Definition

1. For every class from the partitioning at least one of the following holds:
   (a) Every ergodic class that can be reached by doing silent steps belongs to the same partitioning class.
   (b) All ergodic states that can be reached by doing silent steps belong to the same ergodic class.
   (c) The class contains only transient states and only one state can reach another class by doing a silent step.

   and
2. All ergodic states from one class reach other classes with the same accumulative rate.

Aggregation by Reduction

Reduction = Elimination of Stochastic Discontinuity

Discontinuous Markov Chain $\xrightarrow{\textit{Reduction}}$ Markov Chain.

- Result from perturbation theory (Delebecque-Quadrat 1981.)
- Closed loops of instantaneous states become regular states.
- Other instantaneous state merge with the closed loops that they can reach.
- Possible state splitting.

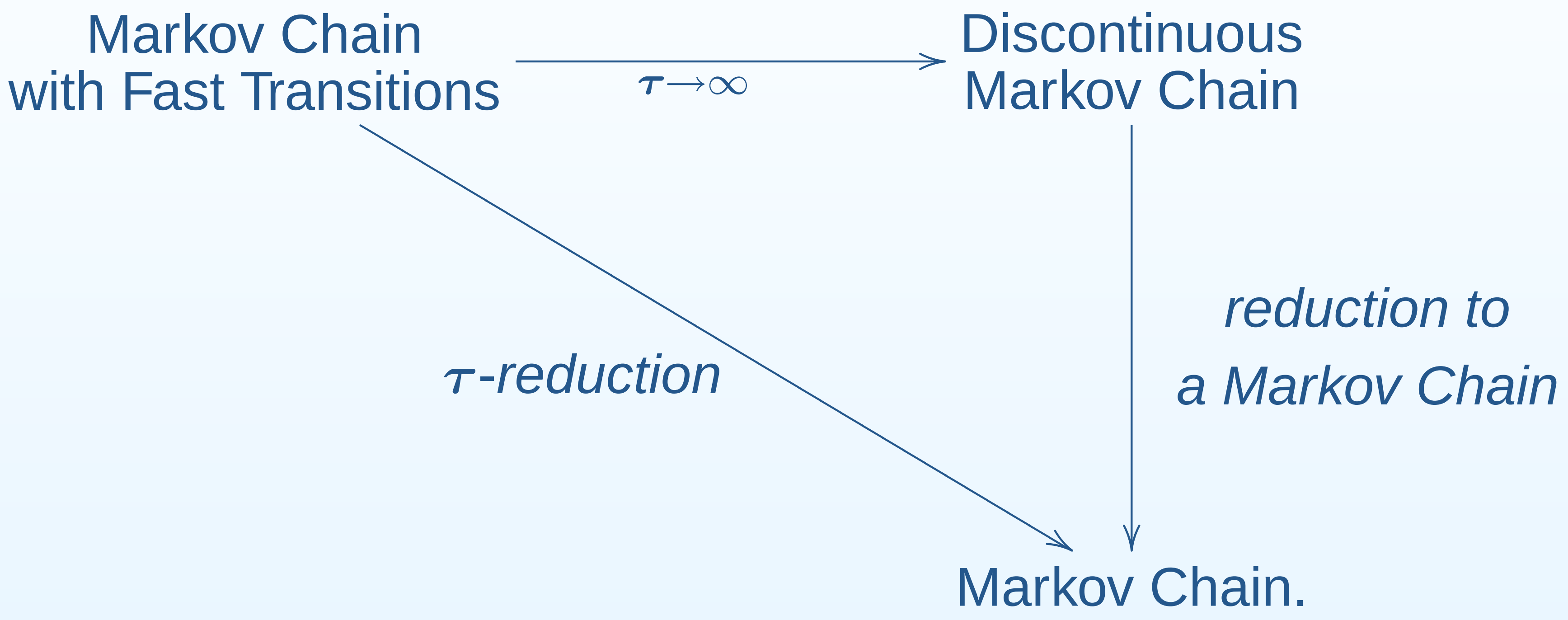
Markov Chains with Fast Transitions - τ-reduction

Markov Chain
with Fast Transitions $\xrightarrow{\tau \to \infty}$ Discontinuous
Markov Chain

reduction to
a Markov Chain

Markov Chain.

Markov Chains with Fast Transitions - τ-reduction



$\boldsymbol{\tau}$-reduction - Example

$\tau_\sim$-reduction for Markov chains with Silent Steps

- Lifting of τ-reduction to sets of Markov Chains with Fast Transitions.
- Combination of τ-reduction and Ordinary Lumping.
- Must give the same result for any representative Markov Chain with Fast Transitions.

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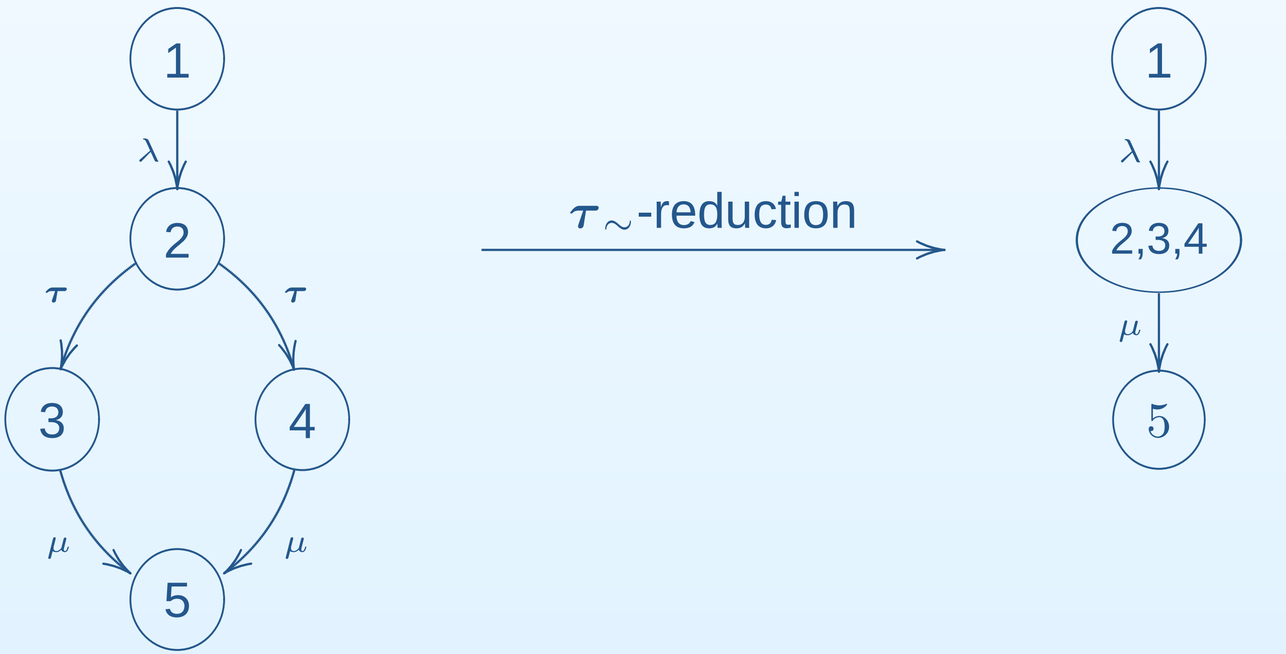

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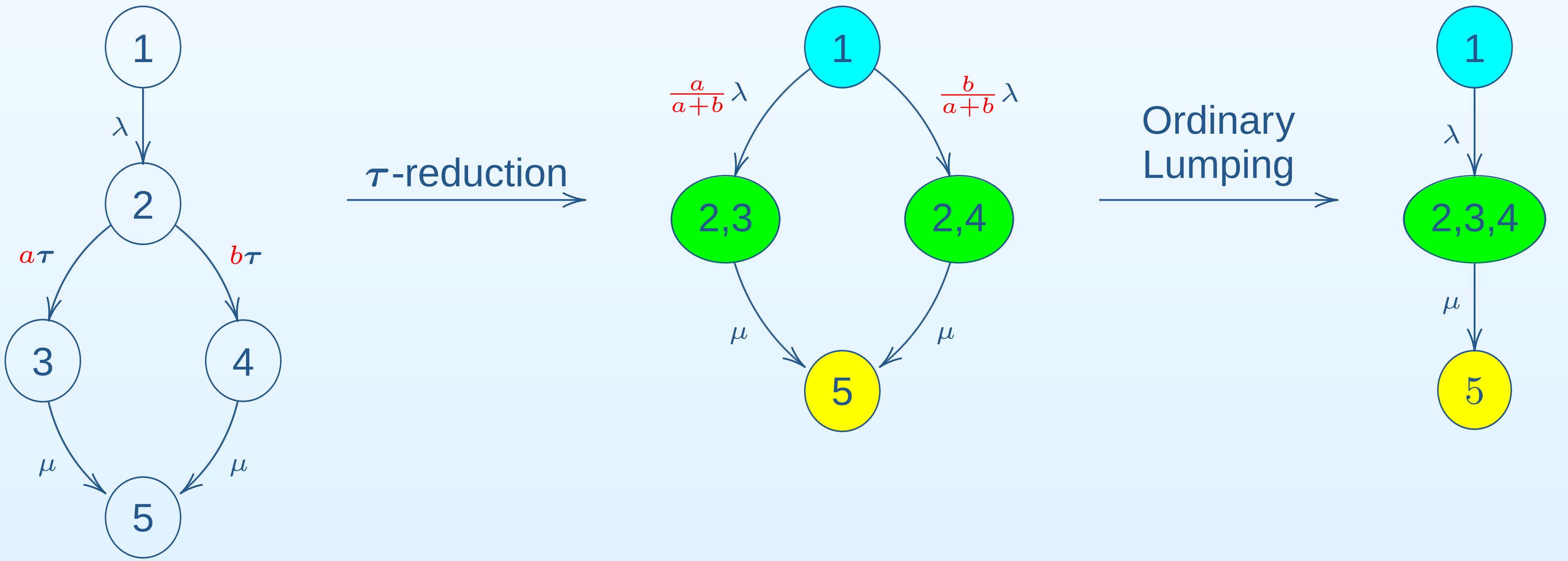

Comparison

- Lumping considered only when it gives a pure Markov Chain.

Markov Chains with Fast Transitions:

- τ-lumping vs. τ-reduction.
  - Incomparable.
  - Reduction can split states.
  - Lumping can combine regular states too.
  - τ-reduction + Ordinary Lumping better than τ-lumping.

Markov Chains with Silent Steps:

- $\tau_{\sim}$-lumping vs. $\tau_{\sim}$-reduction.
  - Equivalent.

Conclusions and Future Work

Conclusions:

- We treated silent steps in Markov chains as instantaneous transitions with unknown probability.
- We developed two methods for their elimination.
- This was done by extending the lumping method and the method for masking stochastic discontinuity.
- Both methods are shown to give equivalent Markov chains.

Future Work:

- Combine fast and silent steps.
- Algorithms.
- Introduce (passive) actions and formalize them in stochastic terms.

THANK YOU!