Lumping Markov Chains with Silent Steps

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Formal Methods Group

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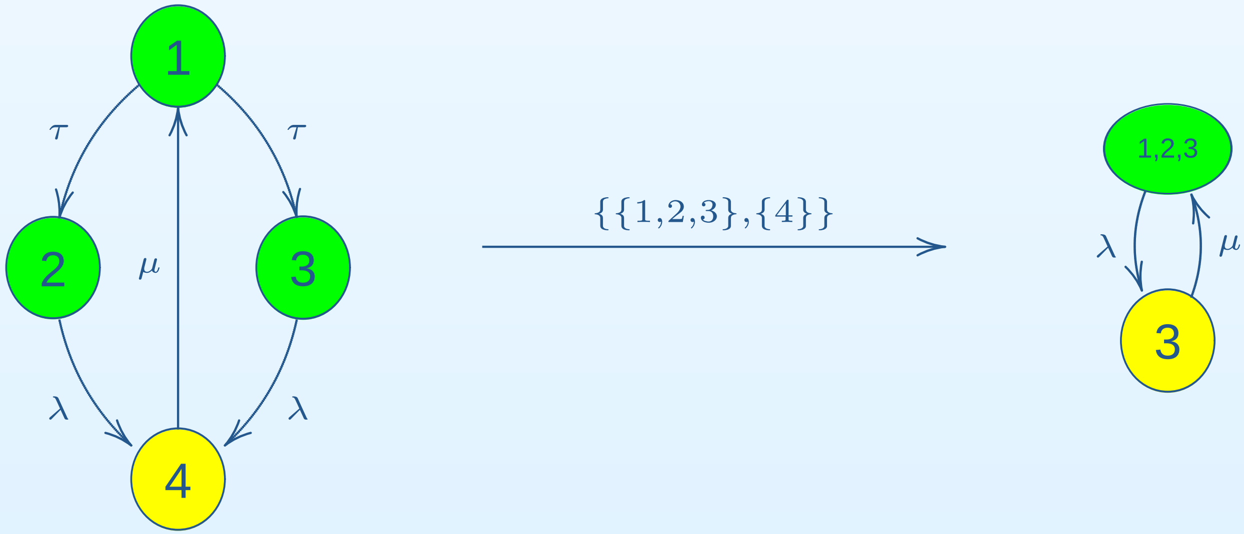

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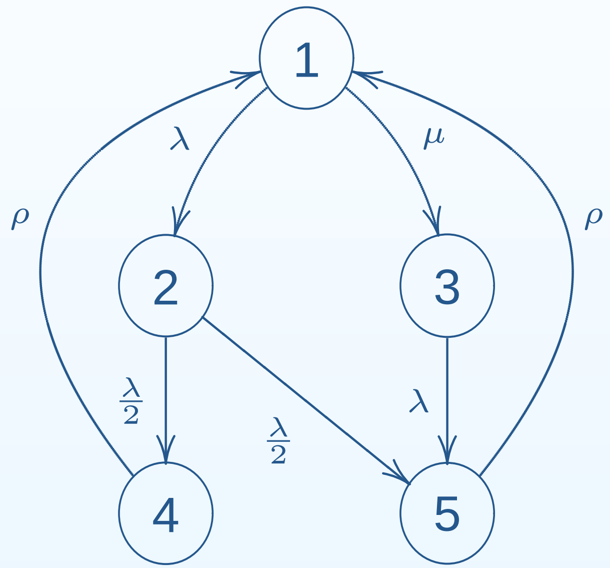
6. Future work.

Lumping for Markov Chains



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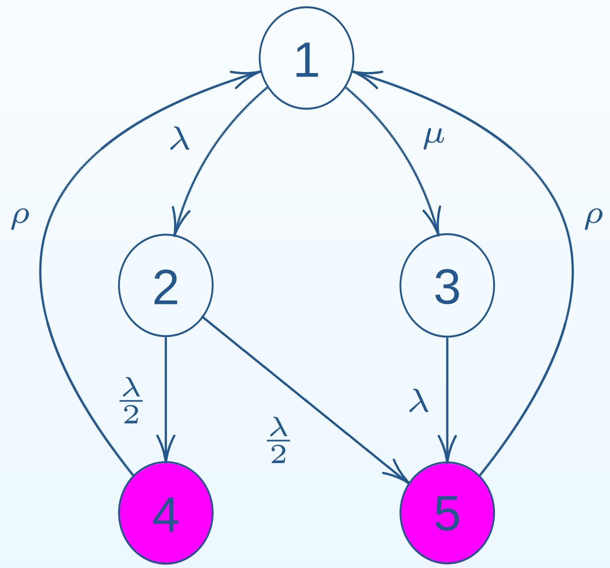

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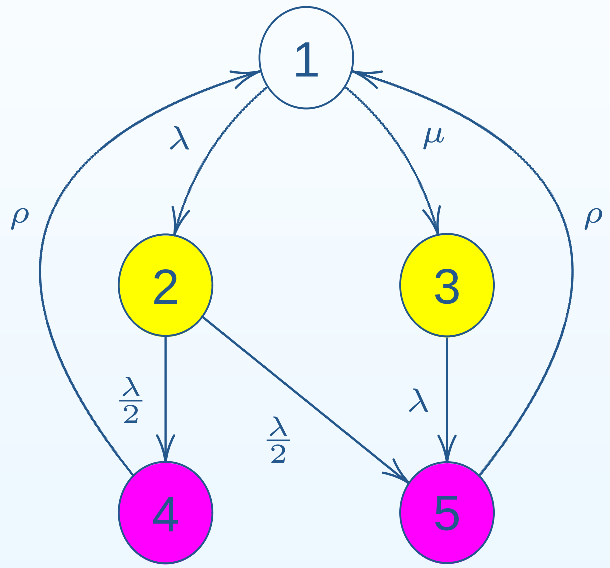

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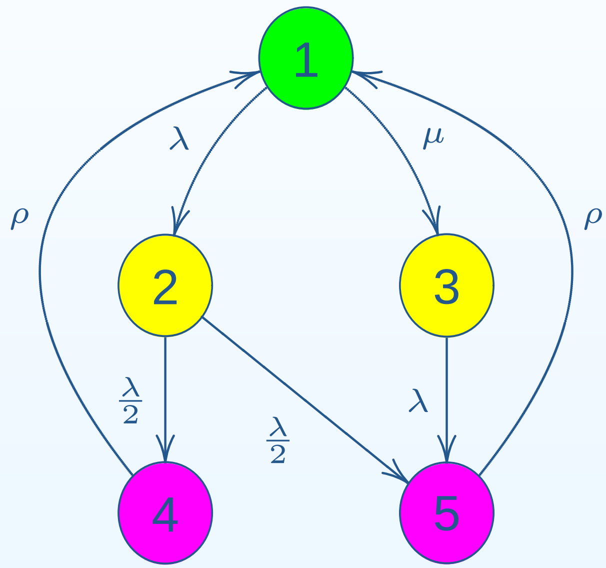

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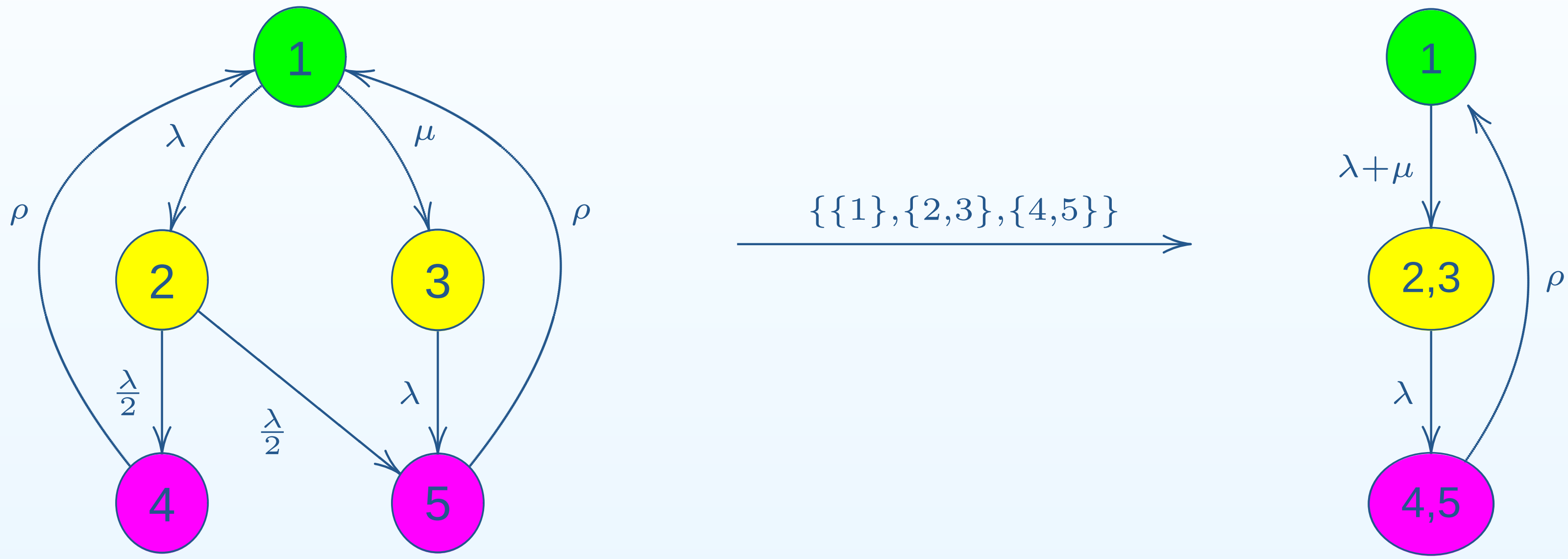

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Lumped process

Discontinuous Markov Chains - Definition

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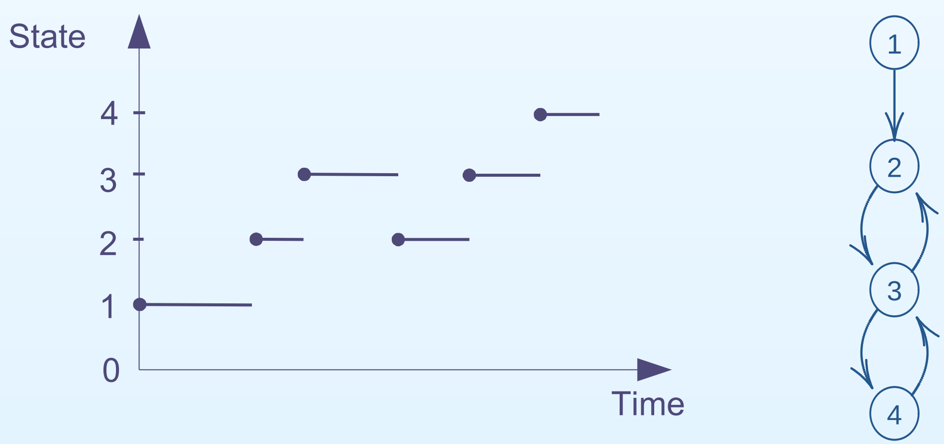

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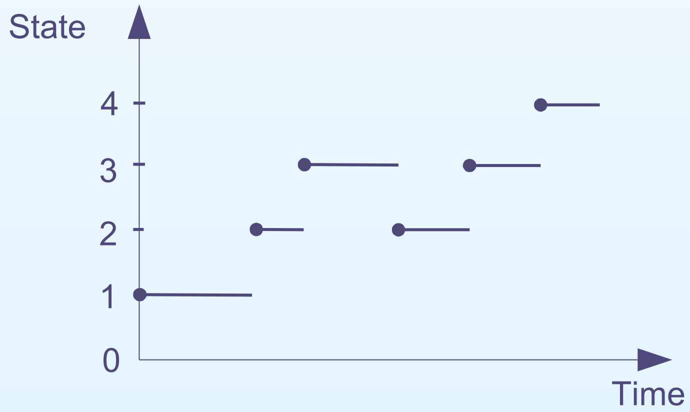

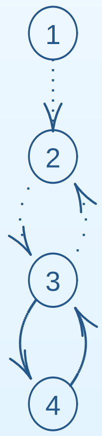

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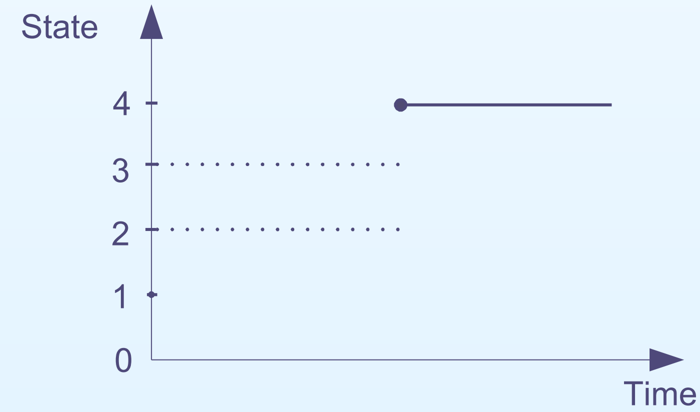
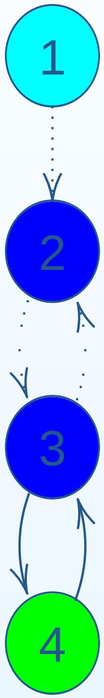


Discontinuous Markov Chain - Classification of States



- States 1, 2 and 3 are instantaneous.
  Process spends no time here (with probability 1).
- State 1 is also evanescent.
  Probability to be here at any time is 0.
- State 4 is regular.
  Sojourn time exponentially distributed.

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- Generalization of the lumping condition is based on the generalization of the corresponding matrix conditions.

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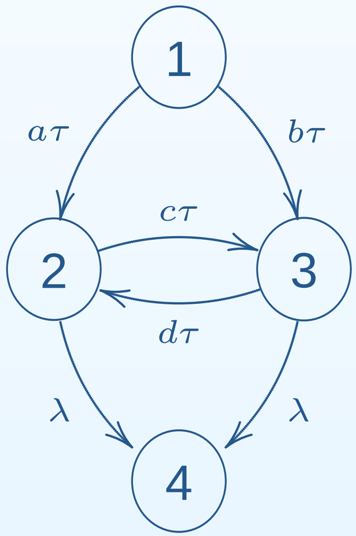

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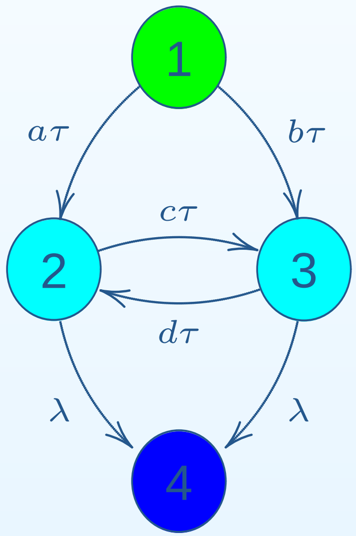

Classification of states:

- Ergodic states - those that form closed τ-communicating classes.
- Transient states - all other states.

Markov Chains with Fast Transitions - Limiting Behavior

Theorem: When $\tau \to \infty$, Markov chain with fast transitions behaves as a discontinuous Markov chain.

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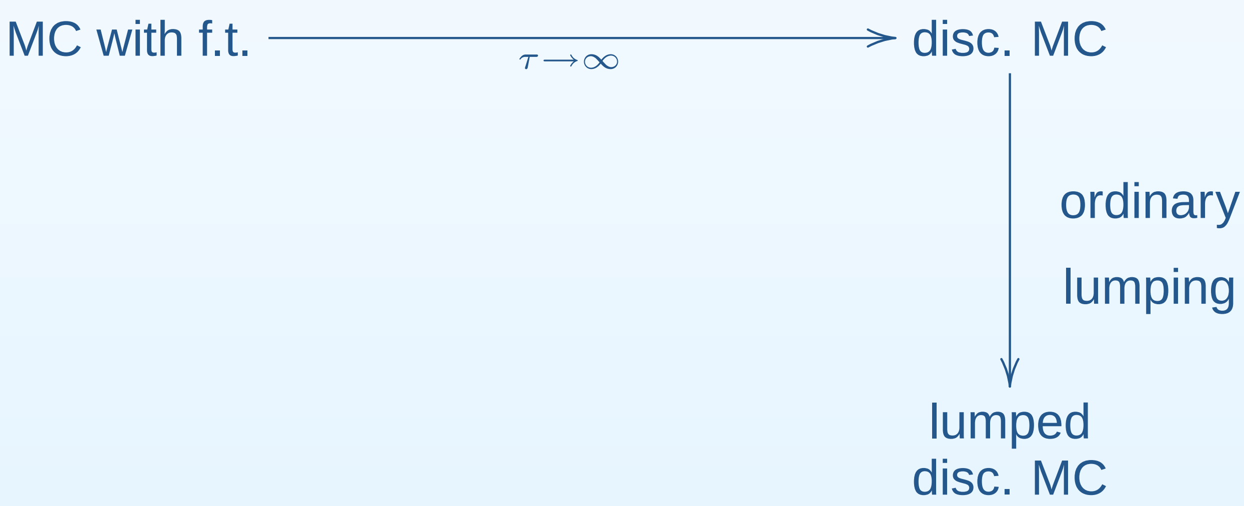
- States that perform a fast transition become instantaneous.
- Transient states become evanescent.
- Regular states are considered as ergodic classes with one element.

Markov Chains with Fast Transitions - τ-lumping

- **Definition:** A partitioning is a τ-lumping of a Markov chain with fast transitions if it is a lumping of the limit discontinuous Markov chain.



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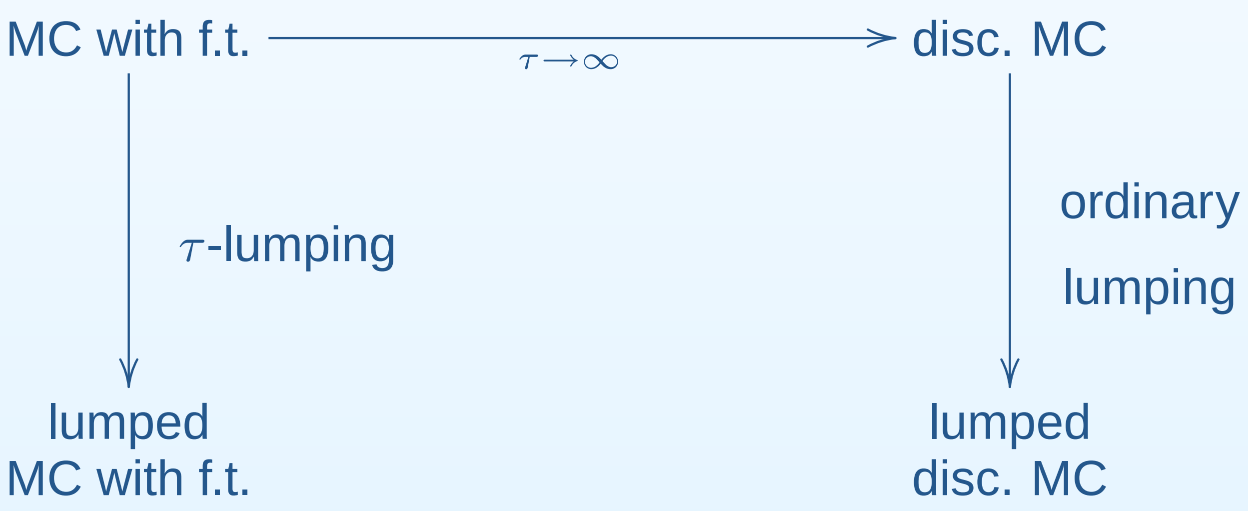

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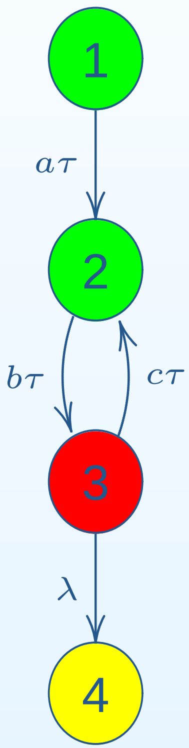
- Note the analogy with weak bisimulation for transition systems.

τ-lumping - Example



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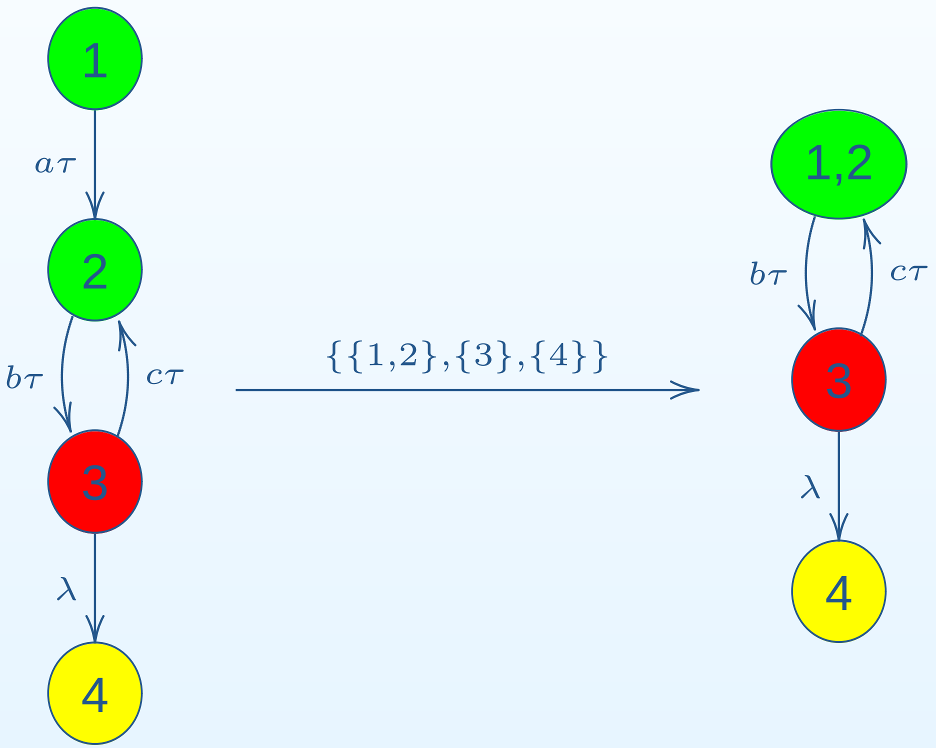

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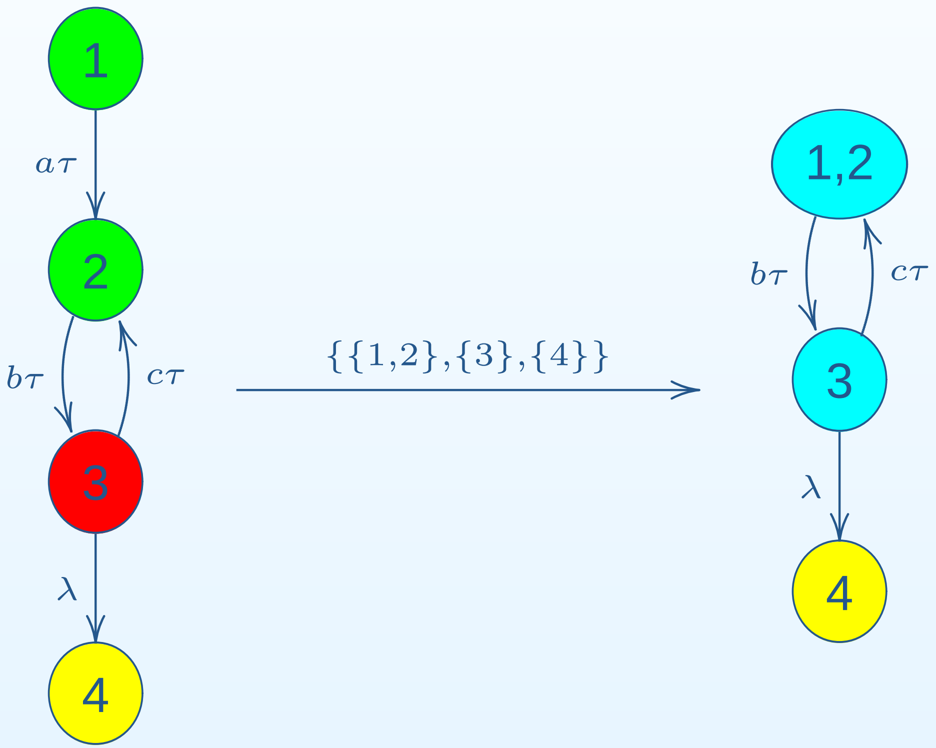

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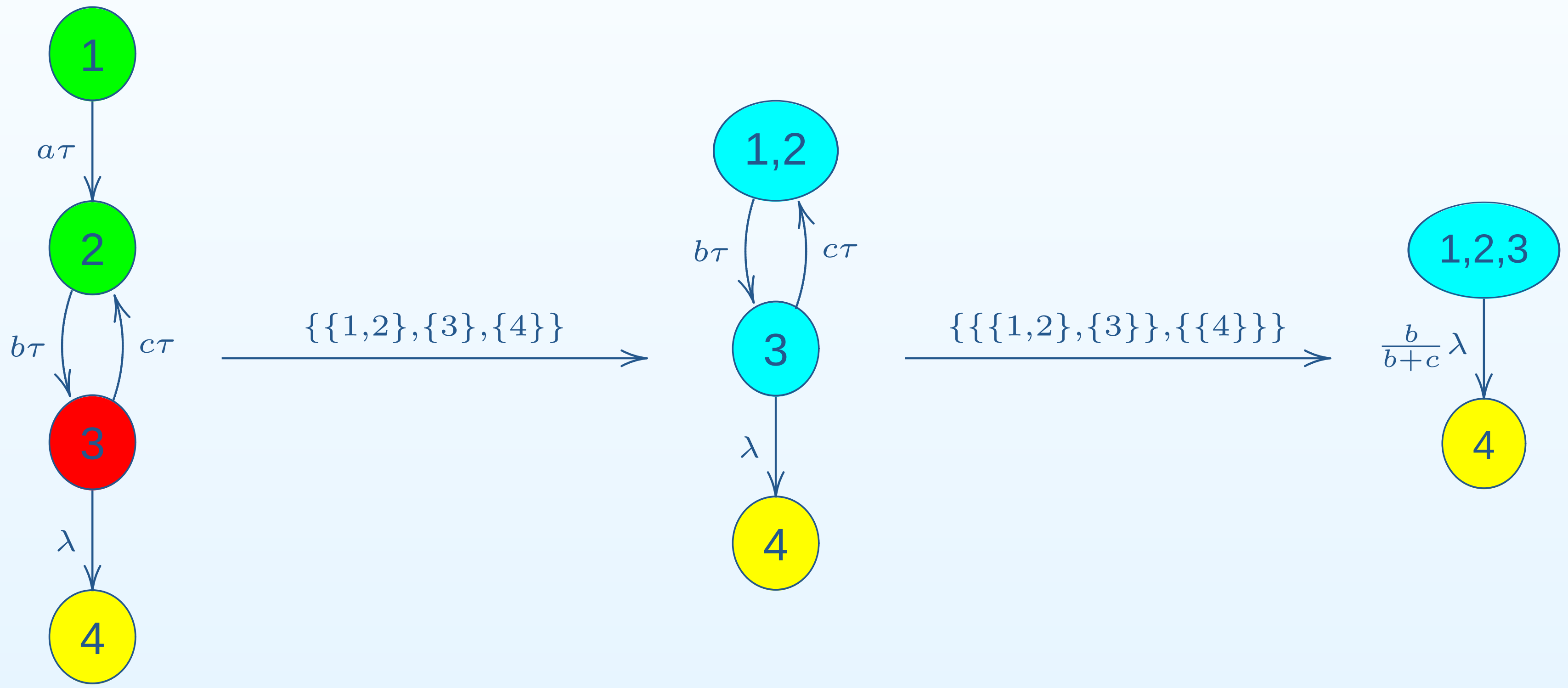

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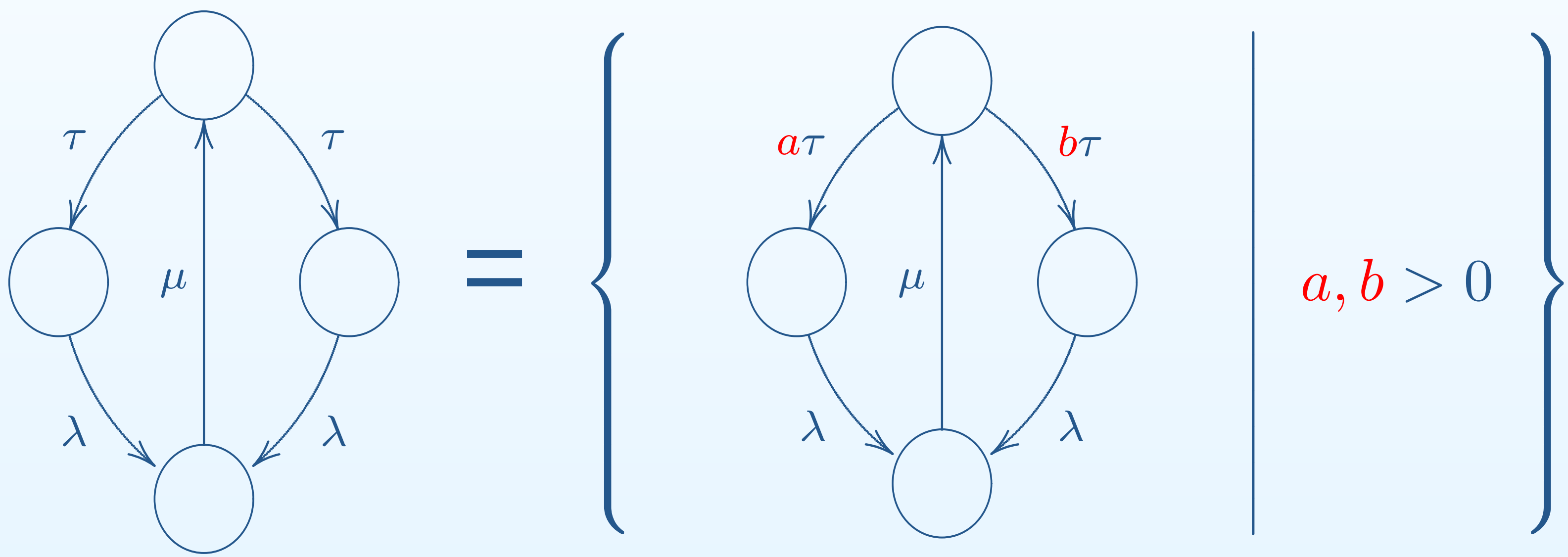

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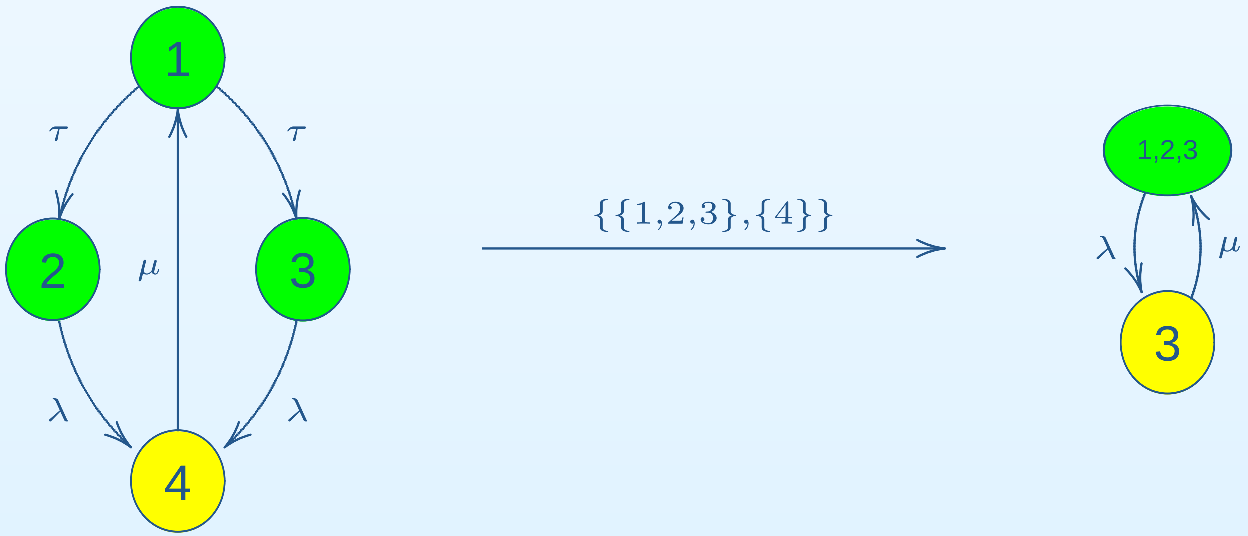

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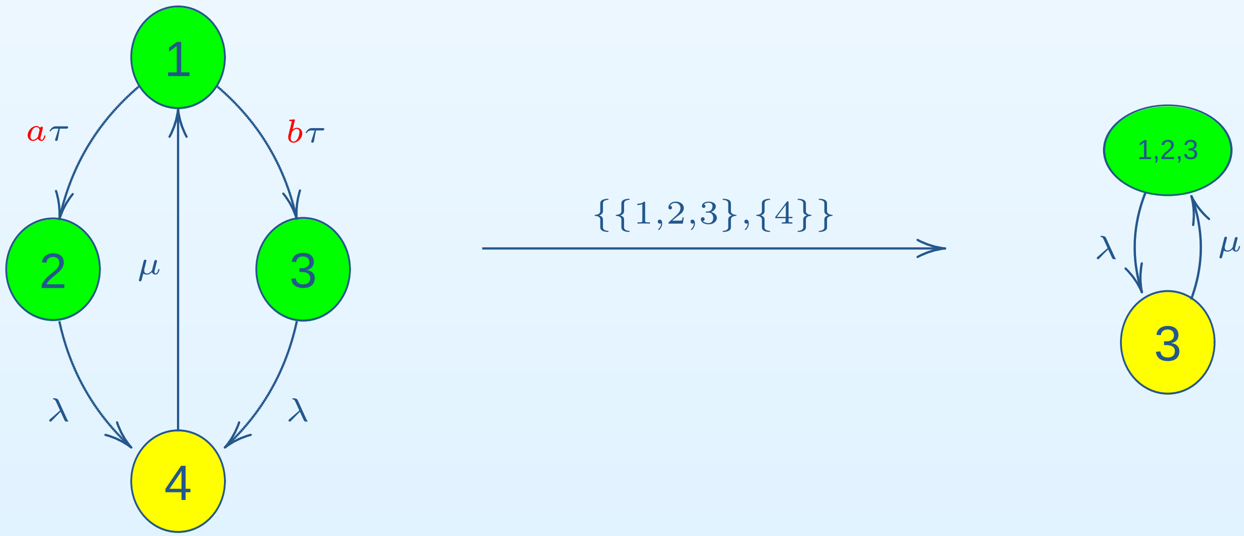
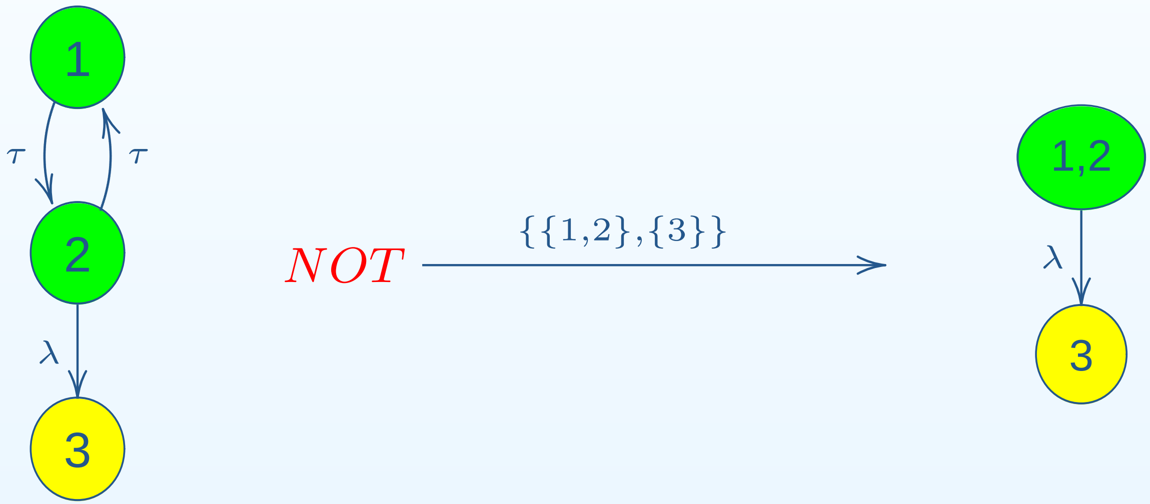


Lumping for Markov Chains with Silent Steps

Counterexample 1:



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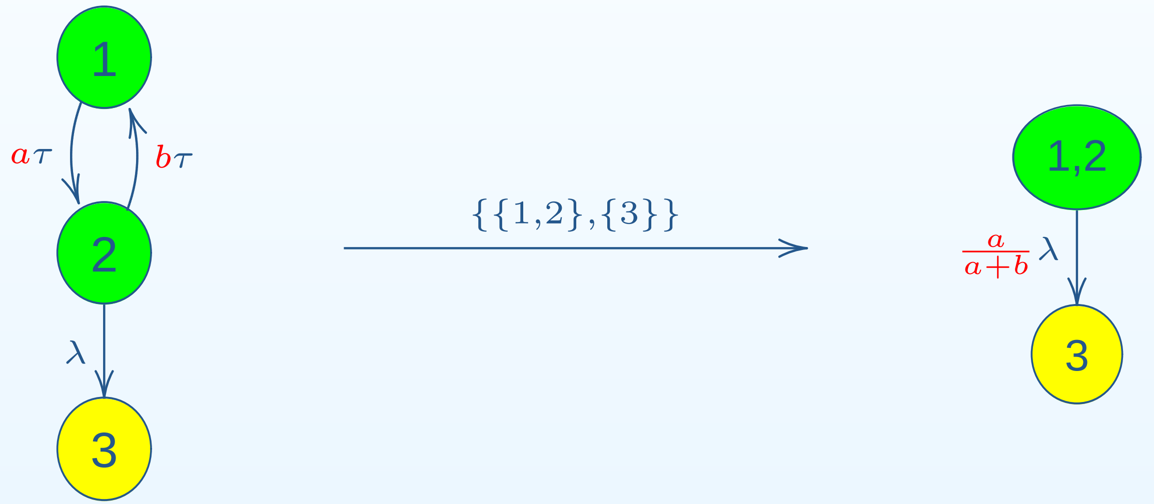
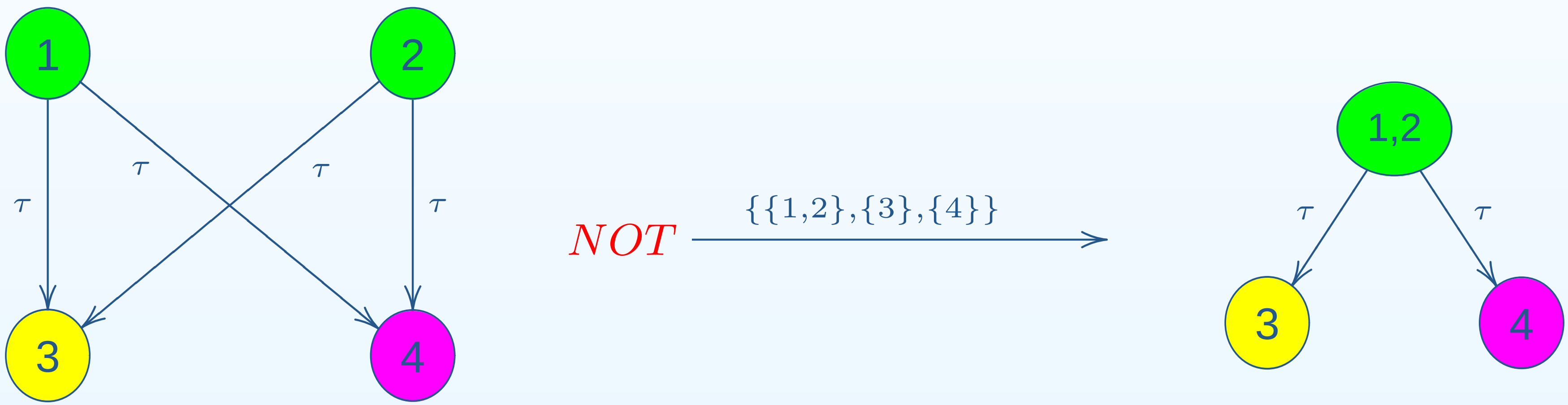


This is because:

- τ-lumping works for any a, b, but the result depends on a, b.

Lumping for Markov Chains with Silent Steps

Counterexample 2:



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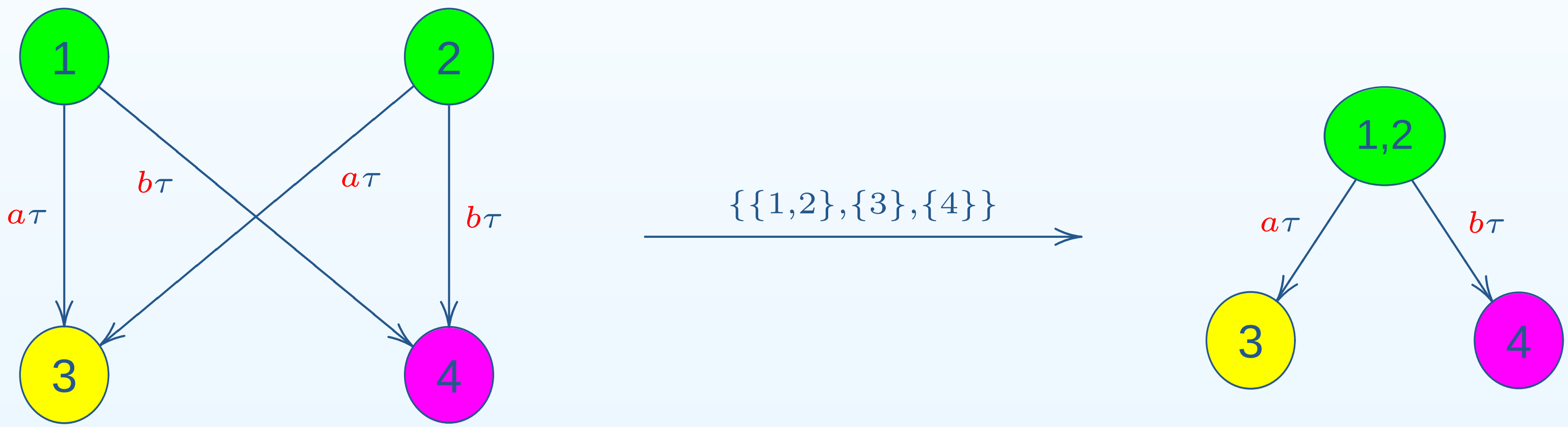

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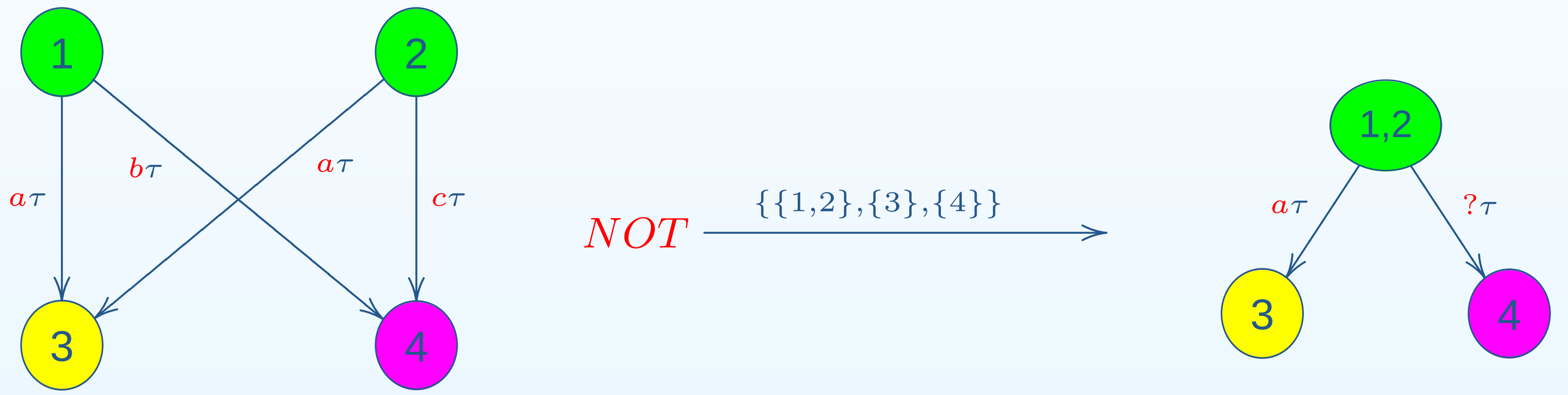

This is because:

- even though it works for arbitrary sequence a, b, a, b,
- it does not work for a, b, a, c.

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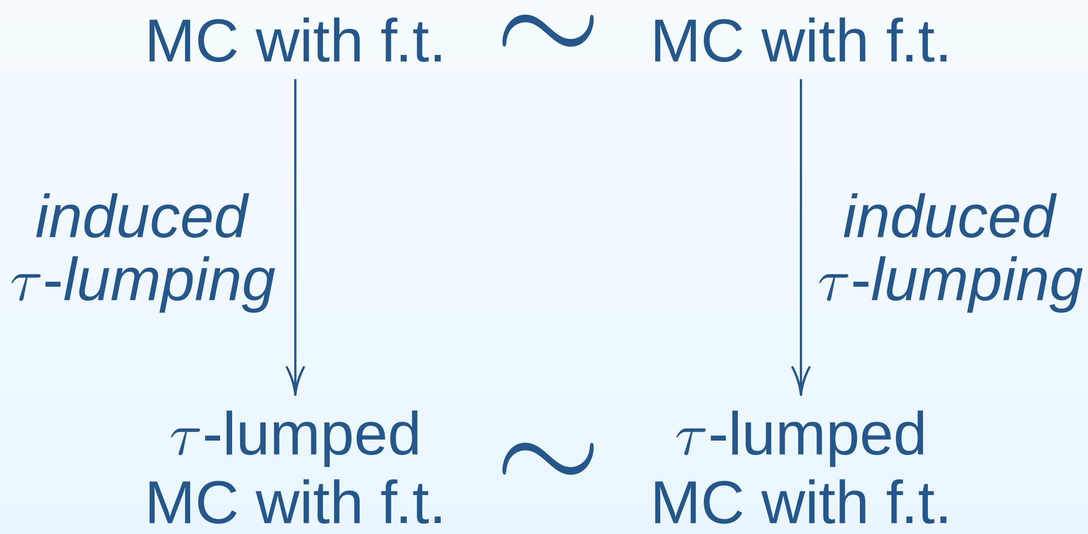
   (c) The class contains only transient states and only one state can reach another class by doing a silent step.

   and
2. All ergodic states from one class reach other classes with the same accumulative rate.

$\tau_{\sim}$-lumping Is Sound

Theorem:



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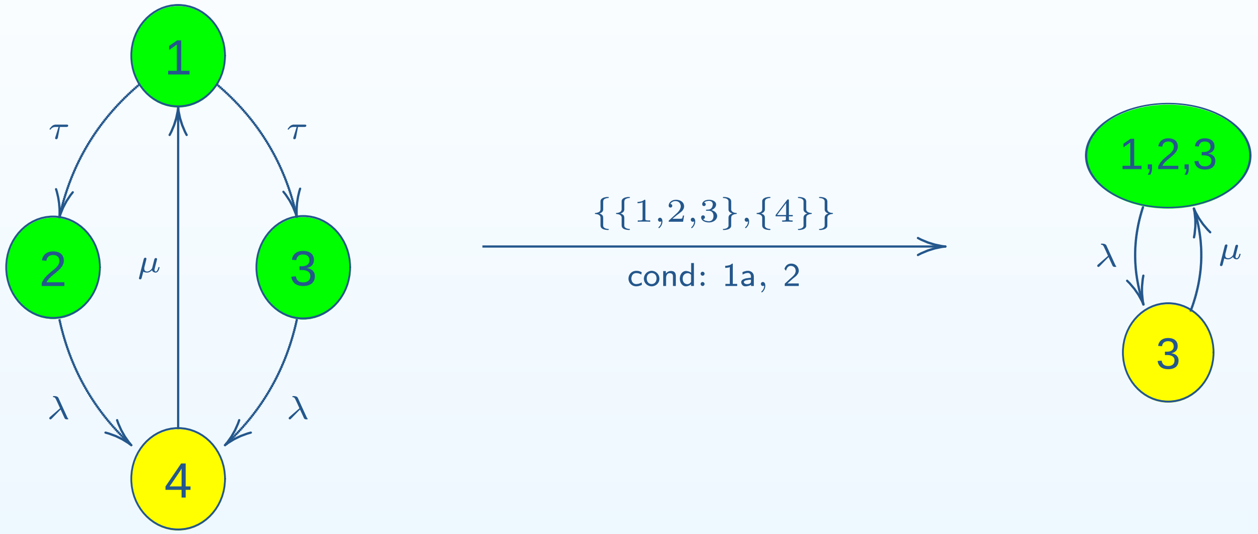

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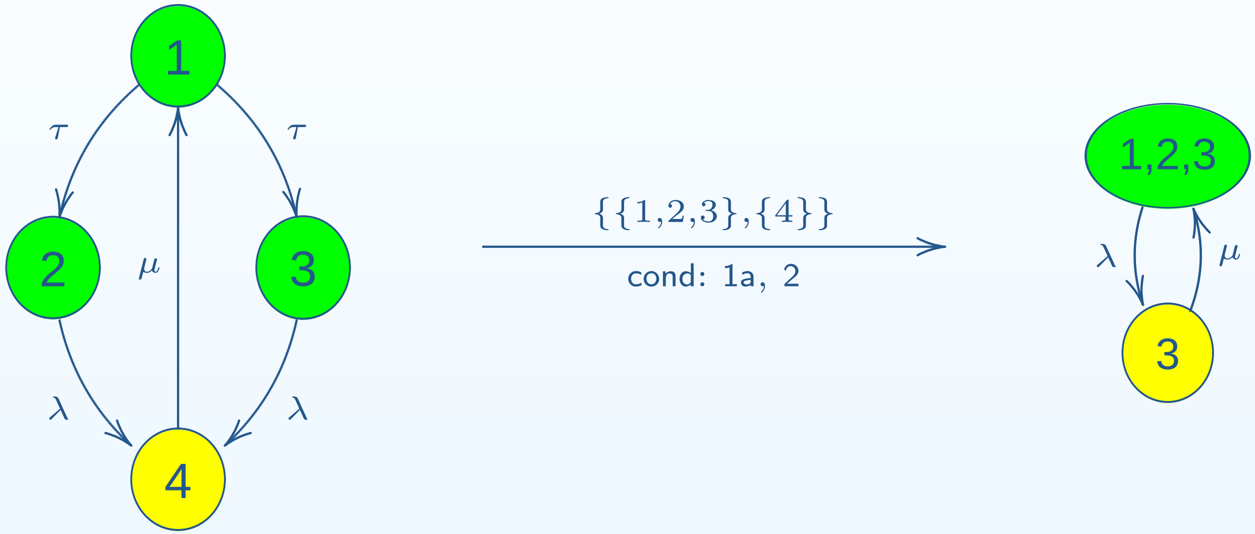

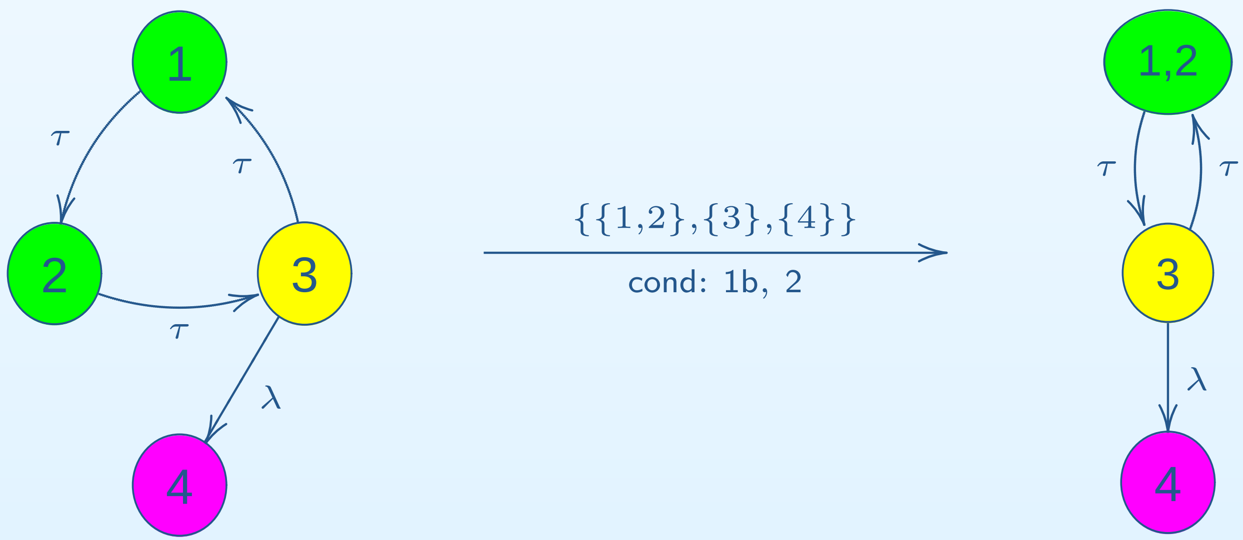

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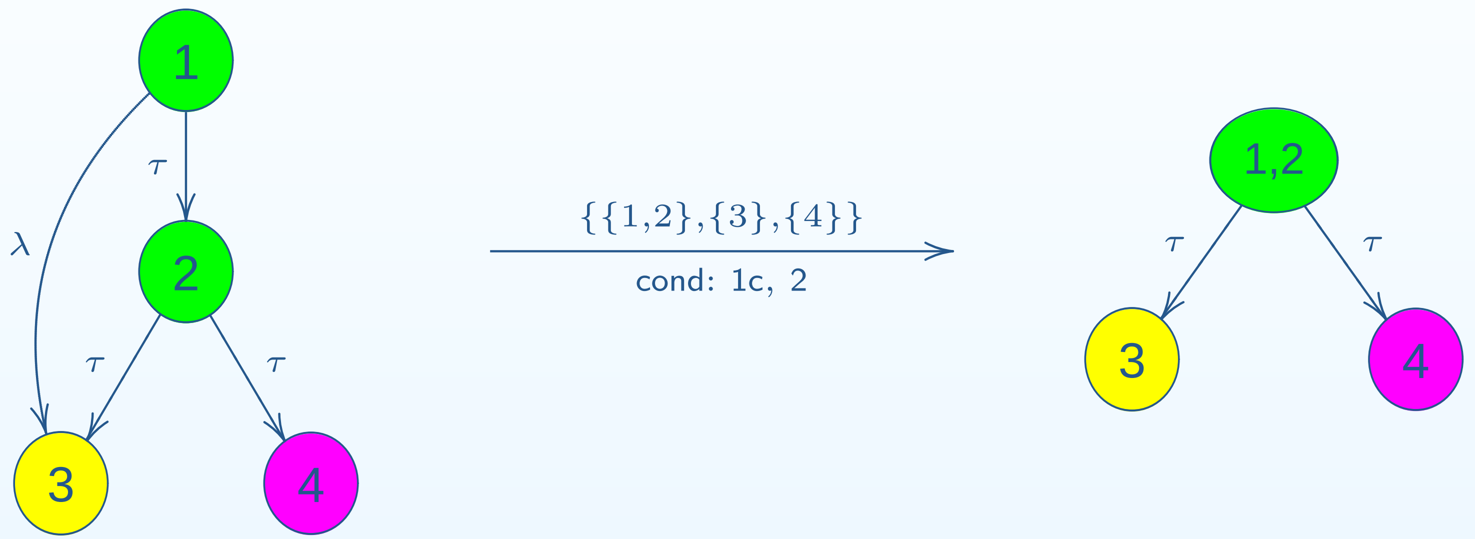

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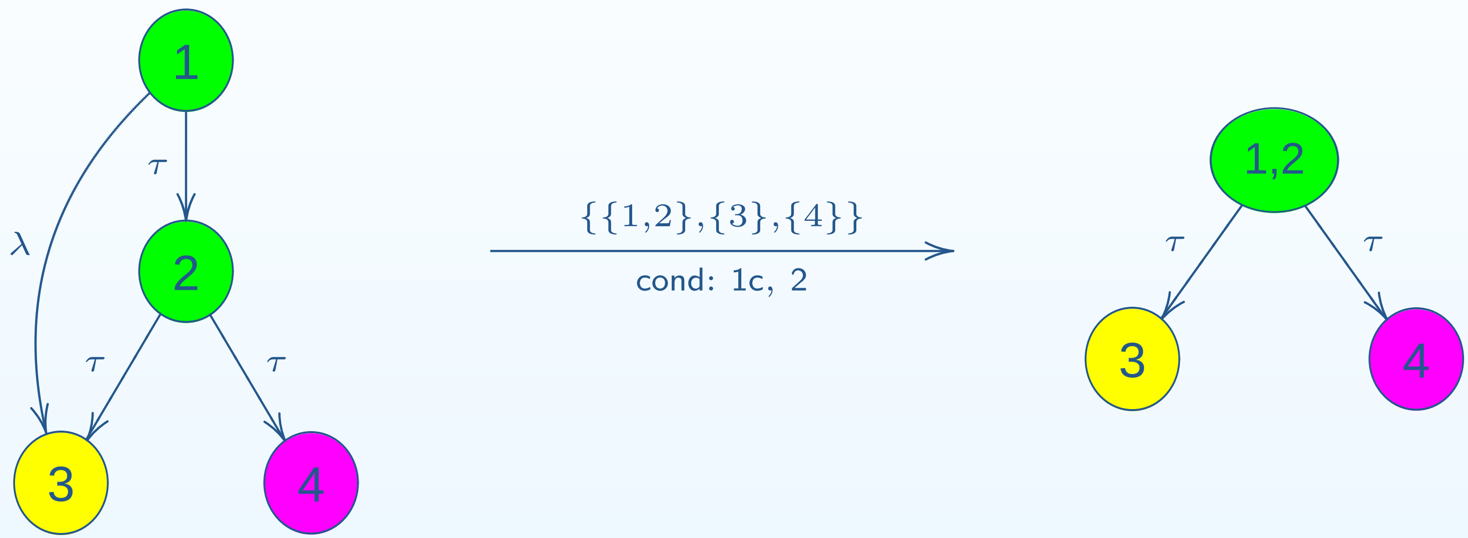

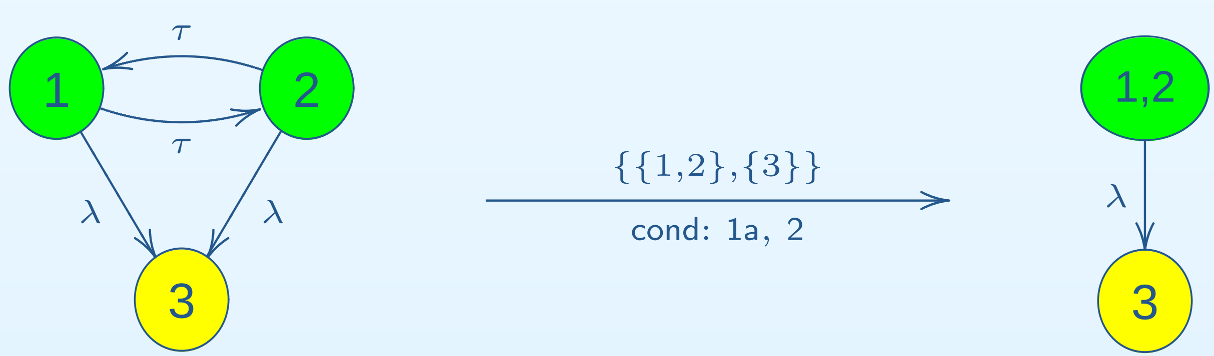

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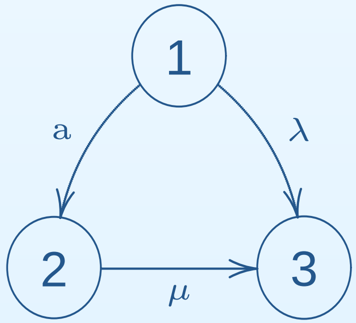
IMC
Open system



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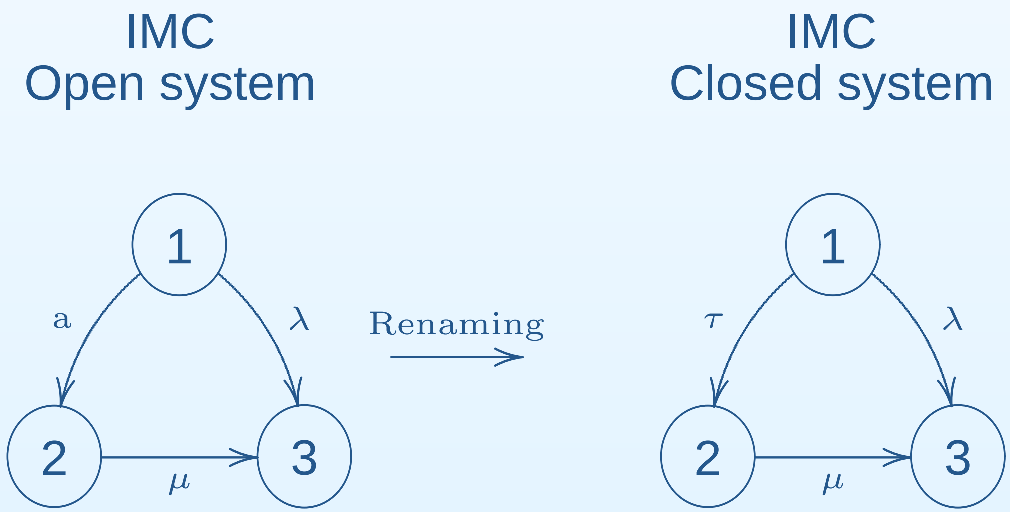

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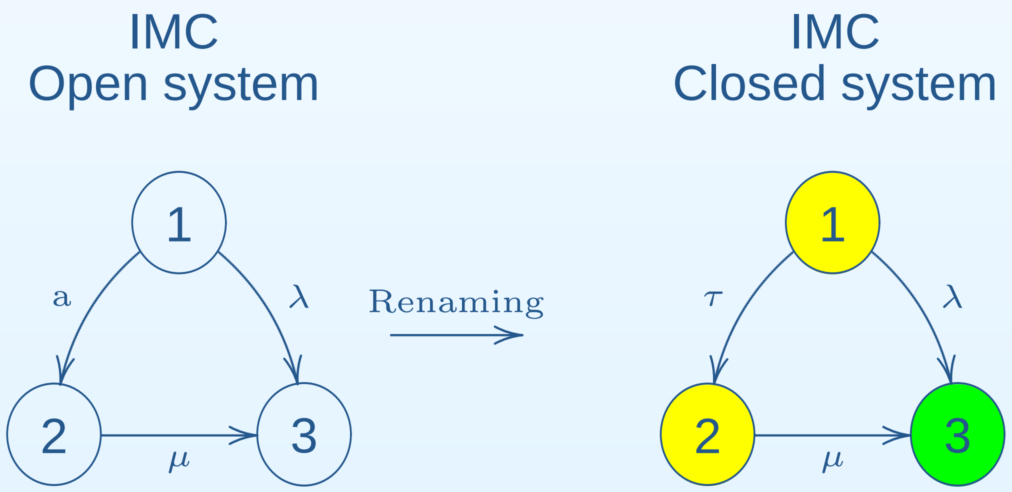

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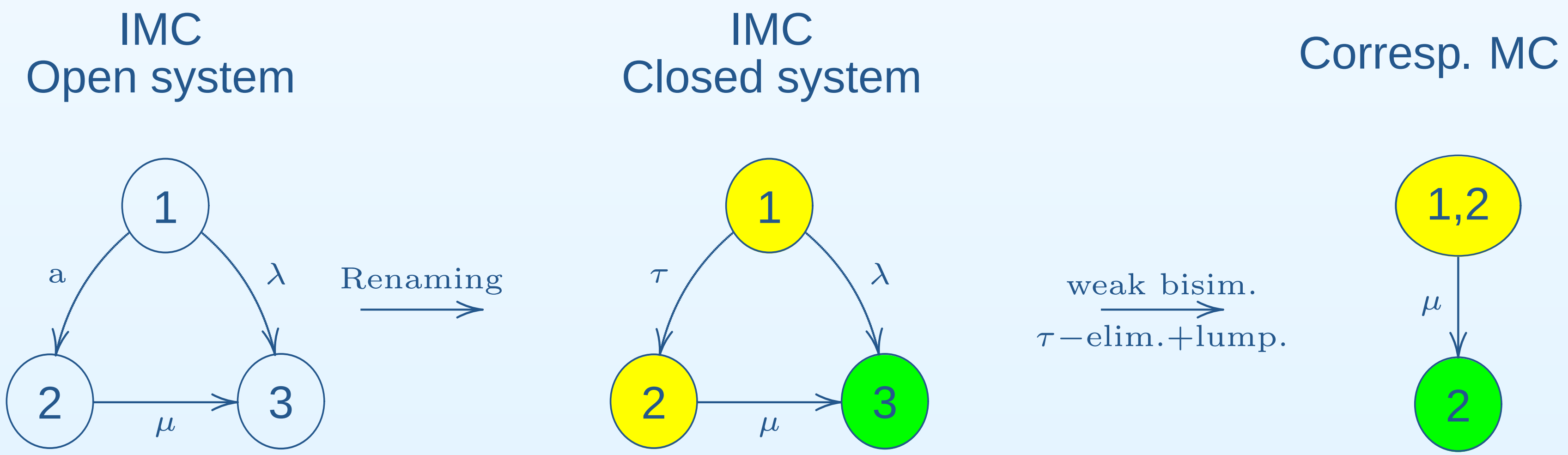

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Markovian weak bsm.

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Present and Future Work

- Lumping with rewards (problem when rewards also depend on τ).
- Introduction of probabilistic choice.
- Addition of actions and parallel composition.
- Algorithms.