Transition Systems and Bisimulations in Matrix Theory

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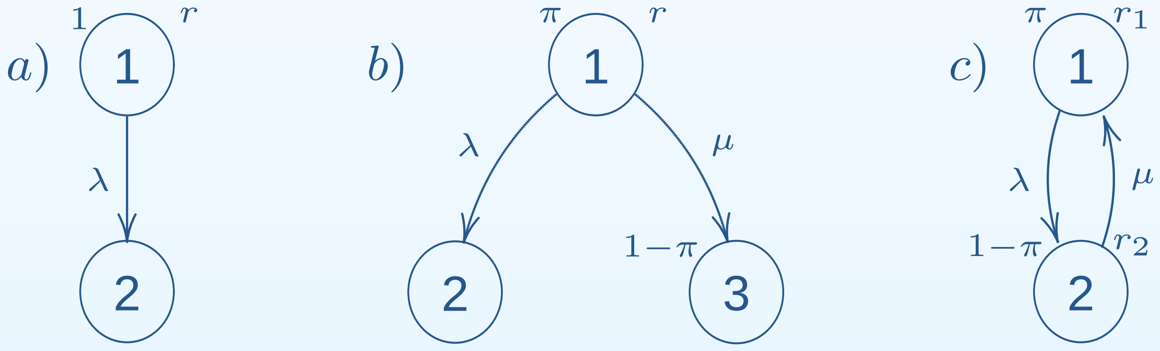
Overview

- Markov reward chains
- Labeled transition systems
- Boolean matrix theory
- Transition systems as systems of boolean matrices
- Strong bisimulation = Ordinary lumping (Kemeny-Snell 1960)
- Weak bisimulation = τ-lumping (Markovski, Trčka, 2005)
- ? = τ-reduction (Doeblin 1938,Delebecque-Quadrat 1981, Coderch 1983)
- Branching bisimulation = ?

Continuous Time Markov Reward Chains

- Stochastic process with the history-less property (future independent of the past).
- Widely popular for quantitative analysis.
- Examples:

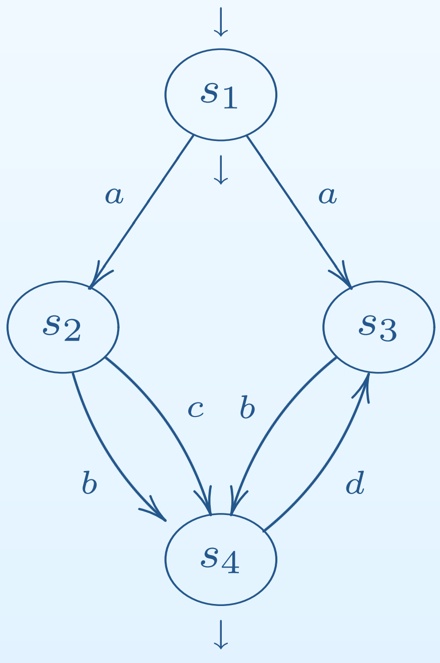


- Almost always explained in terms of matrix theory
- The Markov reward chain in c) is the triple (σ, Q, ρ) where

$$\sigma = \begin{pmatrix} \pi & 1-\pi \end{pmatrix}, \quad Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}, \quad \text{and} \quad \rho = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

Labeled Transition Systems

- Finite state
- One starting state
- Successful termination
- Example:



A $= \{a, b, c, d\}$ – set of actions

$\mathcal{S} = \{s_1, s_2, s_3, s_4\}$ – set of states

s_1 – initial state

s_1, s_4 – terminating states

Boolean Set algebra

- S – arbitrary set
- $\mathbb{P}(S) = (\mathcal{P}(S), \cup, \cap, {}^-, \emptyset, S)$ – boolean algebra
- Different symbols (to emphasize connection with standard matrix theory):

$$\begin{aligned} + &\overset{\text{def}}{=} \cup \\ \cdot &\overset{\text{def}}{=} \cap \\ 0 &\overset{\text{def}}{=} \emptyset \\ 1 &\overset{\text{def}}{=} S. \end{aligned}$$

- Some axioms (for $\alpha \subseteq S$):

$$\begin{aligned} &\alpha + \alpha = \alpha \cdot \alpha = \alpha \\ &\alpha + 1 = 1 \\ &\alpha \cdot 1 = \alpha \\ &\alpha + 0 = \alpha \\ &\alpha \cdot 0 = 0. \end{aligned}$$

- Order: $\alpha \leq \beta \quad$ iff $\quad \alpha + \beta = \beta$.

Boolean Matrix Theory

- Elements of $\mathbb{P}(S)^{n\times m}$
- Operations:

sum:	$(A+B)_{ij}$	$\stackrel{\text{def}}{=}$	$A_{ij}+B_{ij}$
product:	$(A\cdot B)_{ij}$	$\stackrel{\text{def}}{=}$	$\sum_k A_{ik}\cdot B_{kj}$
scalar product:	$(\alpha\cdot A)_{ij}$	$\stackrel{\text{def}}{=}$	$\alpha\cdot A_{ij}$
complement:	$\bar{A}_{ij}$	$\stackrel{\text{def}}{=}$	$\overline{A_{ij}}$
intersection:	$(A\sqcap B)_{ij}$	$\stackrel{\text{def}}{=}$	$A_{ij}\cdot B_{ij}$
transpose:	$(A^{\mathsf{T}})_{ij}$	$\stackrel{\text{def}}{=}$	A_{ji}

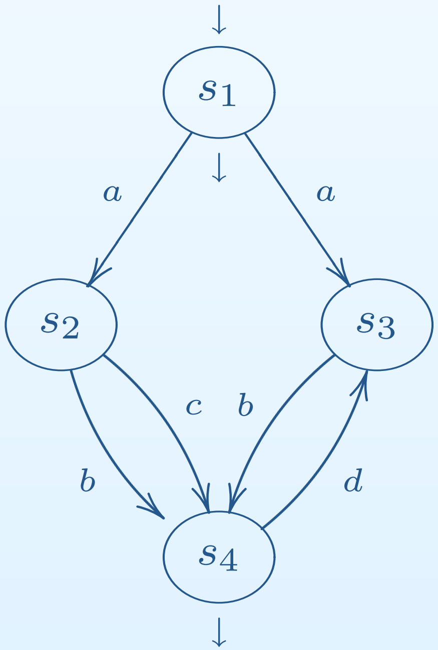
- Order:

$$A\leq B \quad \text{iff} \quad A+B=B \quad \text{iff} \quad A_{ij}\leq B_{ij}$$

- Constants: I**– identity matrix,** $\mathbf{0}$**– zero matrix**
- 0–1 matrices = elements of $\{0,1\}^{n\times m}$ (sometimes called relations)

Transition Systems as Systems of Matrices

- A – set of actions
- **DEF:** A transition system is a triple $\langle \sigma, A, \rho \rangle$ where:
  - $\sigma \in \{0,1\}^{1\times n}$ is the *initial vector*; its exactly one entry is 1,
  - $A \in \mathbb{P}(\mathsf{A})^{n\times n}$ is the *transition matrix*, and
  - $\rho \in \{0,1\}^{n\times 1}$ is the *termination vector*.
- Example:



$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & \{a\} & \{a\} & 0 \\ 0 & 0 & 0 & \{b,c\} \\ 0 & 0 & 0 & \{b\} \\ 0 & 0 & \{d\} & 0 \end{pmatrix} \quad \rho = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Strong Bisimulation in Matrix Terms

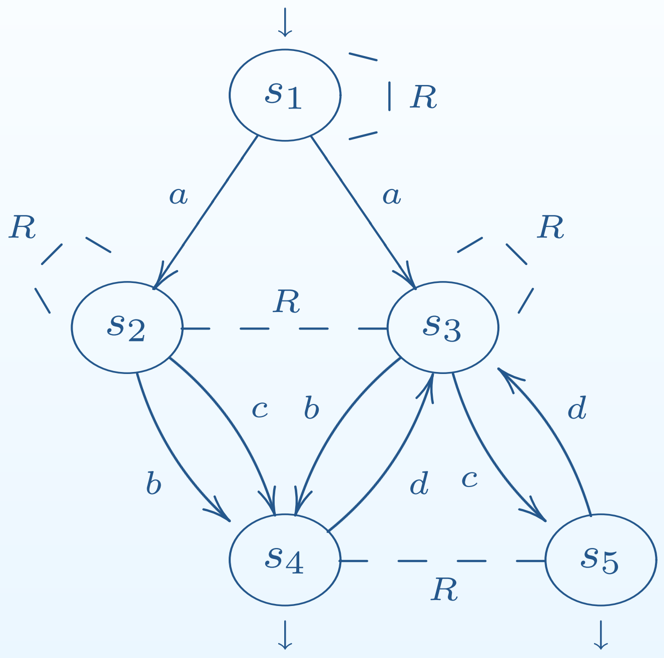
- Standard definition:
  - R – symmetric relation
  - $s_0 R s_0$ – initial state related to itself
  - **Termination condition:** if sRt and $s{\downarrow}$, then $t{\downarrow}$
  - **Transfer condition:** if sRt and $s \xrightarrow{a} s'$, then $t \xrightarrow{a} t'$ and $s' R t'$.
- Matrix definition:
  - $R \in \{0,1\}^{n \times n}$, $R = R^{\mathsf{T}}$
  - $\sigma \leq \sigma R$ – initial state related to itself
  - **Transfer condition:** $RA \leq AR$

$$\begin{array}{ccc} s_i \overset{R}{\text{———}} s_k & & s_i \\ \quad\quad\quad\; {\scriptstyle a}\big\downarrow & \text{implies} & \big\downarrow{\scriptstyle a} \\ \quad\quad\quad\; s_j & & s_\ell \underset{R}{\text{———}} s_j. \end{array}$$

  - **Termination condition:** $R\rho \leq \rho$

$$s_i \overset{R}{\text{———}} s_j{\downarrow} \qquad \text{implies} \qquad s_i{\downarrow}.$$

Strong Bisimulation – Example



$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \end{pmatrix}^{\mathsf{T}}$$

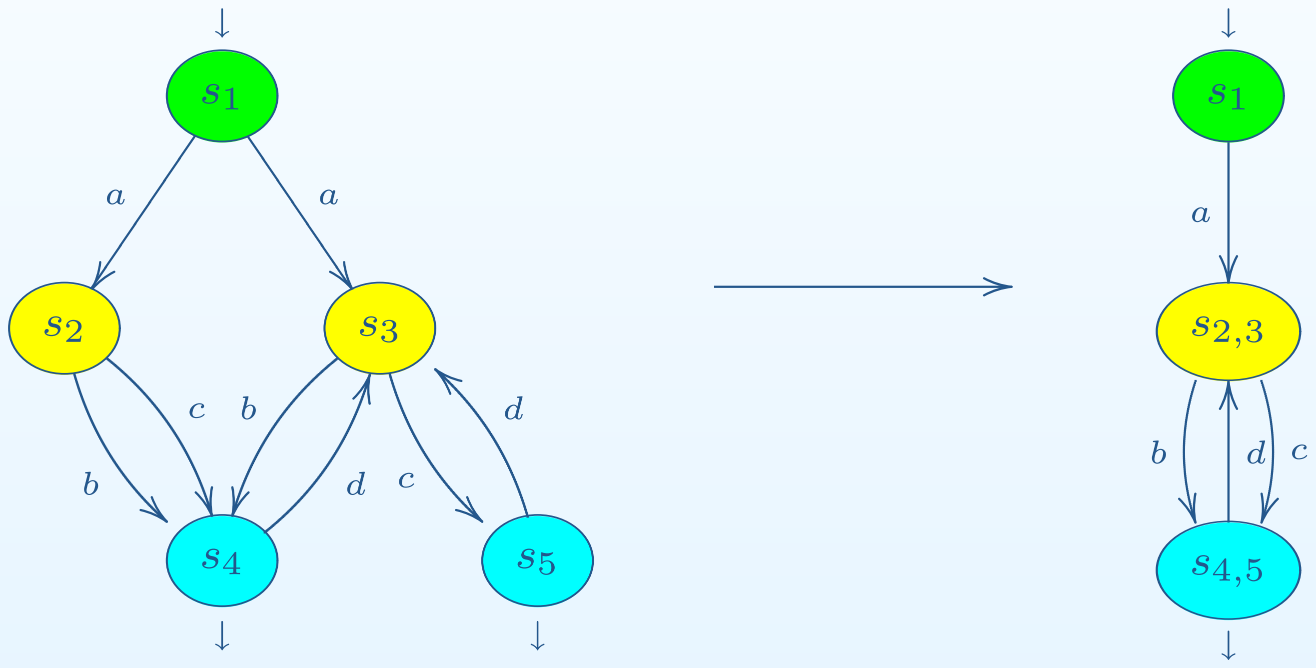
$$A = \begin{pmatrix} 0 & \{a\} & \{a\} & 0 & 0 \\ 0 & 0 & 0 & \{b,c\} & 0 \\ 0 & 0 & 0 & \{b\} & \{c\} \\ 0 & 0 & \{d\} & 0 & 0 \\ 0 & 0 & \{d\} & 0 & 0 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$RA = \begin{pmatrix} 0 & \{a\} & \{a\} & 0 & 0 \\ 0 & 0 & 0 & \{b,c\} & \{c\} \\ 0 & 0 & 0 & \{b,c\} & \{c\} \\ 0 & 0 & \{d\} & 0 & 0 \\ 0 & 0 & \{d\} & 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 0 & \{a\} & \{a\} & 0 & 0 \\ 0 & 0 & 0 & \{b,c\} & \{b,c\} \\ 0 & 0 & 0 & \{b,c\} & \{b,c\} \\ 0 & \{d\} & \{d\} & 0 & 0 \\ 0 & \{d\} & \{d\} & 0 & 0 \end{pmatrix} = AR$$

Strong Lumping

- Aggregation modulo strong bisimulation equivalence
- Example:



Strong Lumping in Matrix Terms

- R is an equivalence relation: $I \leq R$, $R = R^T$, $R^2 \leq R$
- Unique decomposition: $R = VV^\mathsf{T}$

$$R = \begin{pmatrix} 1&0&0&0&0 \\ 0&1&1&0&0 \\ 0&1&1&0&0 \\ 0&0&0&1&1 \\ 0&0&0&1&1 \end{pmatrix} = \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&1&0 \\ 0&0&1 \\ 0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0&0 \\ 0&1&1&0&0 \\ 0&0&0&1&1 \end{pmatrix} = VV^\mathsf{T}$$

- $V \in \mathbb{R}^{n \times N}$ – called *collector* matrix
- $V^\mathsf{T} V = I$
- Lumping condition is the requirement that $R = VV^\mathsf{T}$ is a strong bisimulation:

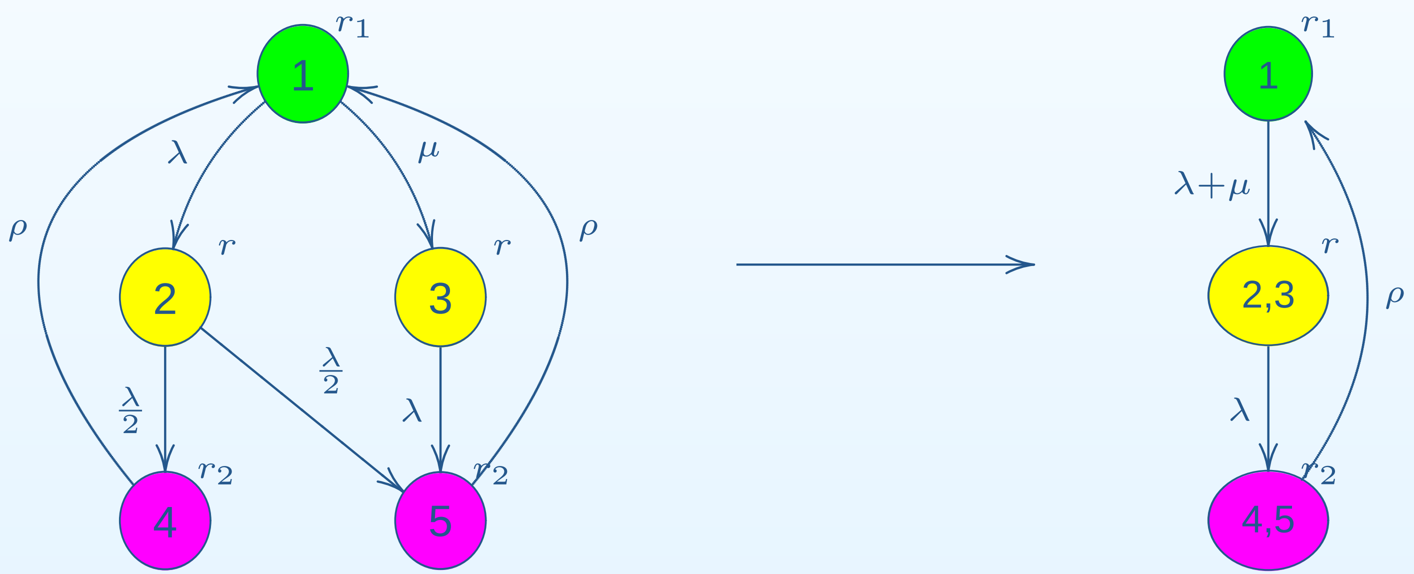
$$VUAV = AV \quad \text{and} \quad VU\rho = \rho, \qquad \text{for any } U \text{ such that } UV = I$$

- No condition on σ because $\sigma = \sigma I \leq \sigma R$
- Lumped system $(\hat{\sigma}, \hat{A}, \hat{\rho})$ is defined by:

$$\hat{\sigma} = \sigma V, \quad \hat{A} = UAV \quad \text{and} \quad \hat{\rho} = U\rho$$

Strong Lumping = Ordinary Lumping for Markov chains

- Ordinary lumping was defined by Kemeny, Snell (1960)
- The most common way of aggregation
- All relevant properties preserved
- Example:



- Lumping condition:

$$VUQV = QV \quad \text{and} \quad VU\rho = \rho$$

- Lumped process:

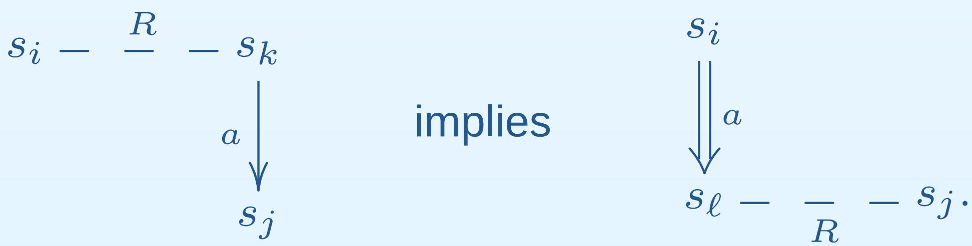
$$\hat{Q} = UQV \quad \text{and} \quad \hat{\rho} = U\rho$$

Weak Bisimulation

- Standard definition:
  - **Termination condition:** if sRt and $s{\downarrow}$, then $t \Rightarrow t'$ and $t'{\downarrow}$
  - **Transfer condition:** if sRt and $s \xrightarrow{a} s'$, then $t \overset{a}{\Rightarrow} t'$ and $s'Rt'$ ($\overset{a}{\Rightarrow}$ is $\Rightarrow \xrightarrow{a} \Rightarrow$).
- Matrix definition:
  - Unique representation of the transition matrix T
  - $T = A + \{\tau\} \cdot S$ where $S \in \{0,1\}^{n \times n}$ and $\{\tau\} \cdot A = \mathbf{0}$
  - Example:

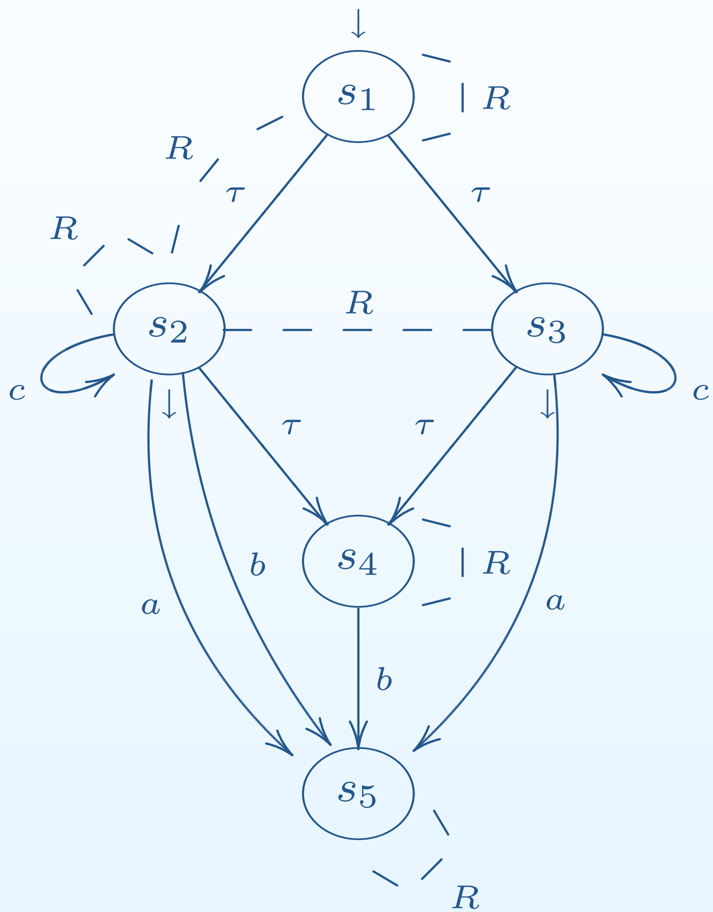
$$T = \begin{pmatrix} \{a\} & \{b,\tau\} \\ \{\tau\} & 0 \end{pmatrix} = \begin{pmatrix} \{a\} & \{b\} \\ 0 & 0 \end{pmatrix} + \{\tau\} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A + \{\tau\} \cdot S$$

  - **Transfer condition for τ transitions:** $RS \leq S^* R$
  - **Transfer condition for other actions:** $RA \leq S^* A S^* R$



  - **Termination condition:** $R\rho \leq S^* \rho$

Weak Bisimulation – Example



$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \end{pmatrix}^{\mathsf{T}}$$

$$S = \begin{pmatrix} 0&1&1&0&0\\ 0&0&0&1&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \quad A = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&\{a,b\}\\ 0&0&\{c\}&0&\{a\}\\ 0&0&0&\{c\}&\{b\}\\ 0&0&0&0&0 \end{pmatrix}$$

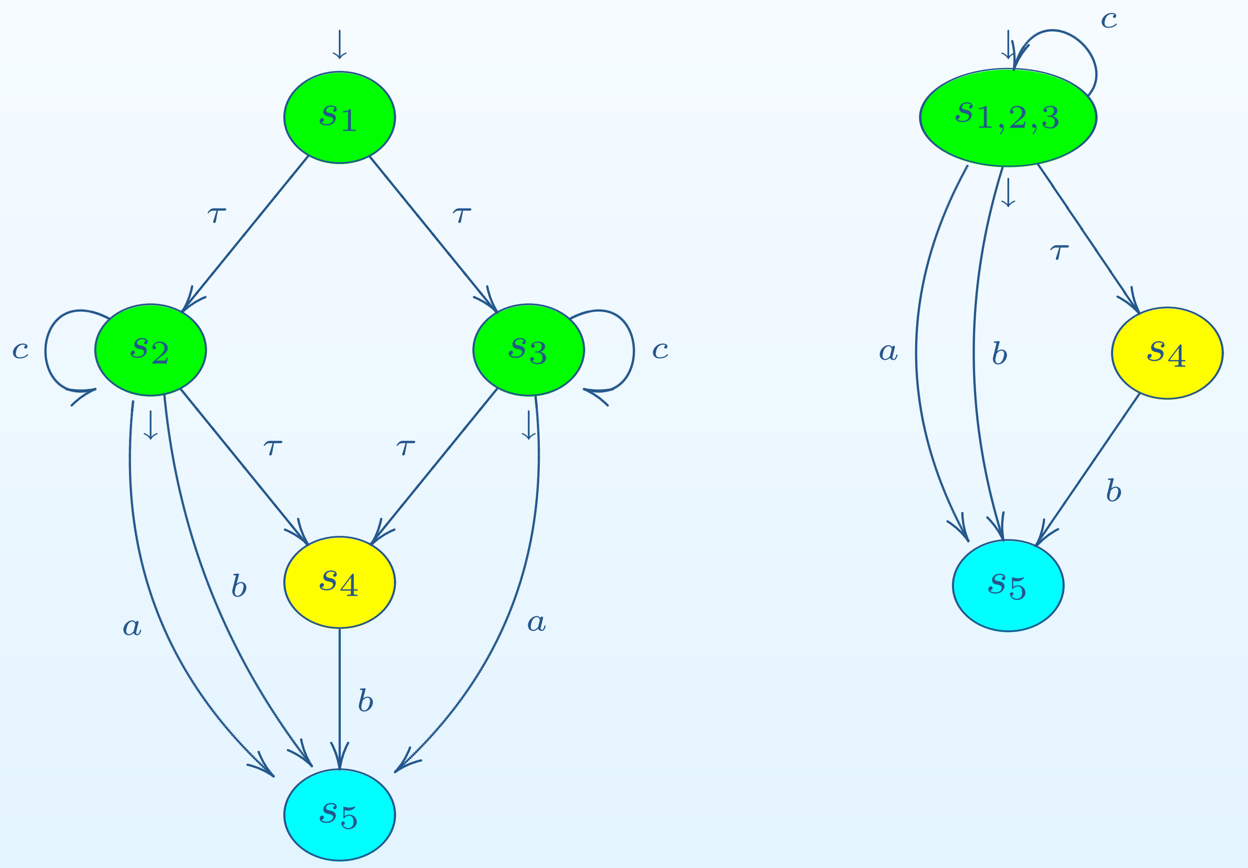
$$S^* = \begin{pmatrix} 1&1&1&1&0\\ 0&1&0&1&0\\ 0&0&1&1&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix} \quad S^*AS^* = \begin{pmatrix} 0&0&0&0&\{a,b\}\\ 0&0&0&0&\{a,b\}\\ 0&0&0&0&\{a,b\}\\ 0&0&0&0&\{b\}\\ 0&0&0&0&0 \end{pmatrix}$$

$$R = \begin{pmatrix} 1&1&0&0&0\\ 1&1&1&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}$$

$$RA = \begin{pmatrix} 0&\{c\}&0&0&\{a,b\}\\ 0&\{c\}&\{c\}&0&\{a,b\}\\ 0&\{c\}&0&0&\{a,b\}\\ 0&0&0&0&\{b\}\\ 0&0&0&0&0 \end{pmatrix} \leq \begin{pmatrix} \{c\}&\{c\}&\{c\}&\{c\}&\{a,b\}\\ \{c\}&\{c\}&\{c\}&\{c\}&\{a,b\}\\ 0&\{c\}&0&\{c\}&\{a,b\}\\ 0&0&0&0&\{b\}\\ 0&0&0&0&0 \end{pmatrix} = S^*AS^*R.$$

Weak Lumping

- Aggregation modulo weak bisimulation equivalence
- Example:



Weak Lumping in Matrix Terms

- $I \leq R$, $R = R^T$, $R^2 \leq R$
- Decomposition: $R = VV^{\mathsf{T}}$
- Lumping condition is the requirement that $R = VV^{\mathsf{T}}$ is a weak bisimulation:

$$VUS^*V = S^*V, \quad VUS^*AS^*V = S^*AS^*V, \quad \text{and} \quad VUS^*\rho = S^*\rho \qquad (UV = I)$$

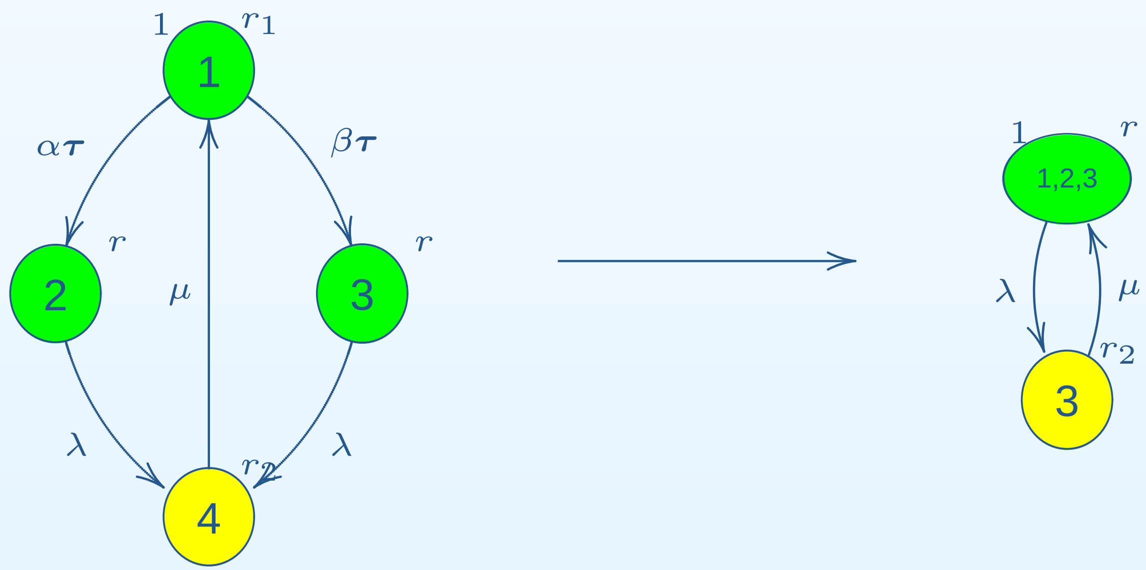
- No condition on σ because $\sigma = \sigma I \leq \sigma R$
- Lumped system is defined by:

$$\hat{\sigma} = \sigma V, \quad \hat{S} = V^{\mathsf{T}}SV, \quad \hat{A} = V^{\mathsf{T}}AV \quad \text{and} \quad \hat{\rho} = V^{\mathsf{T}}\rho$$

- We cannot use arbitrary U but only V^{T}

τ-lumping for Markov Reward Chains

- Introduced by Markovski, Trčka (2005)
- Lumping method that abstracts from unobservable behavior
- Unobservable behavior = Instantaneous probabilistic behavior
- Example:



τ-lumping = Weak Bisimulation

- Follows from matrix representation of τ-lumping
- Unique decomposition of the generator: $Q = Q_s + \tau \cdot Q_f$
  - Q_s corresponds to A
  - Q_f corresponds to S
- $\Pi = \lim_{t\to\infty} e^{Q_f t}$ – ergodic projection
  - Π corresponds to S^* (they satisfy the same properties)
- Lumping condition:

$$VU\Pi V = \Pi V, \quad VU\Pi Q_s \Pi V = \Pi Q_s \Pi V, \quad \text{and} \quad VU\Pi\rho = \Pi\rho$$

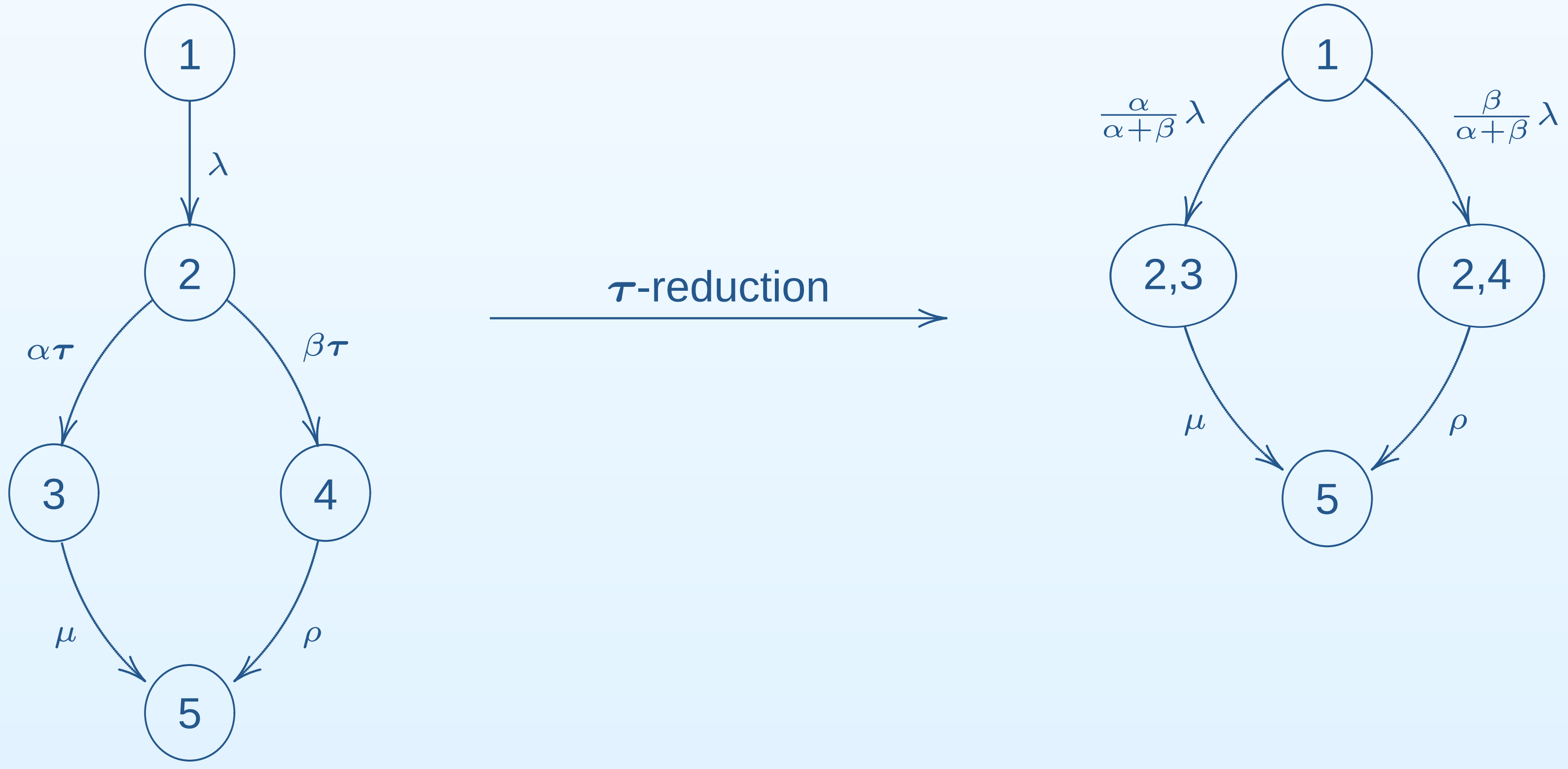
- Lumped process:

$$\hat{Q}_s = WQ_sV, \quad \hat{Q}_f = WQ_fV, \quad \text{and} \quad \hat{\rho} = W\rho \qquad (W - \text{special})$$

  - W corresponds to V^T (they satisfy the same properties)

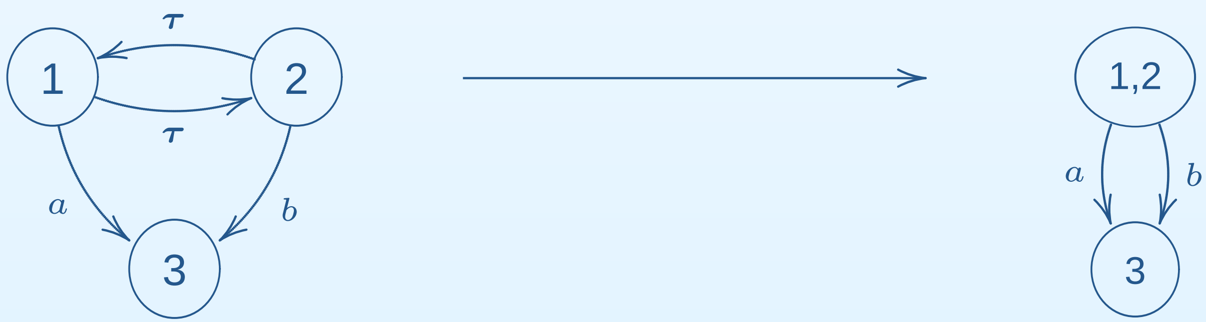
τ-reduction for Markov Chains

- Delebecque-Quadrat (1981), Coderch (1983); has roots in works of Doeblin (1938)
- Abstracts from unobservable behavior
- Can split states
- Example:



τ-reduction = ? for Transition Systems

- Matrix definition gives us:



Branching bisimulation

- Standard definition:
  - **Termination condition:** if sRt and $s{\downarrow}$, then $t \Rightarrow t'$, sRt', and $t'{\downarrow}$
  - **Transfer condition:** if sRt and $s \xrightarrow{a} s'$, then $t \Rightarrow t'' \xrightarrow{a} t'$, sRt'' and $s'Rt'$
  - Cannot be directly expressed in matrix terms. Alternative: tRt'' instead of sRt''
- Matrix definition:
  - $T = A + \{\tau\} \cdot S$ where $S \in \{0,1\}^{n \times n}$ and $\{\tau\} \cdot A = \mathbf{0}$
  - **Transfer condition for τ transitions:** $RS \leq (S^* \sqcap R)(I + S)R$
  - **Transfer condition for other actions:** $RA \leq (S^* \sqcap R)AR$

  - **Termination condition:** $R\rho \leq (S^* \sqcap R)\rho$

Branching bisimulation = ? for Markov chains

- Problem: How to interpret $\sqcap$
- Π is stochastic matrix but $\Pi \sqcap R$ is not. Must be normalized somehow.
- Does it make sense?
- Is it stronger than τ-lumping?