Mixed finite elements for the Richards' equation: linearization procedure

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Abstract. We consider mixed finite element discretization for a class of degenerate parabolic problems including the Richards' equation. After regularization, time discretization is achieved by an Euler implicit scheme, while mixed finite elements are employed for the discretization in space. Based on the results obtained in [17], this paper considers a simple iterative scheme to solve the emerging nonlinear elliptic problems.

Key words. Euler implicit scheme, mixed finite elements, regularization, degenerate parabolic problems, Richards' equation, linearization.

1. Introduction

Water flow in porous media can be modelled by the Richards' equation ([4]), a nonlinear, possibly degenerate, parabolic differential equation. Taking the pressure as a primary unknown, Richards' equation becomes

$$\partial_t \Theta(\psi) - \nabla \cdot K(\Theta)\nabla(\psi + z) = 0 \tag{1}$$

where ψ is the pressure head, Θ the fluid saturation, K the conductivity and z the vertical height. Assuming the air pressure constant (0), we have in the saturated region (where only water is present) $\psi \geq 0$, while $\psi < 0$ in the unsaturated domain. Several retention curves are proposed in the literature to express relations between ψ, K and Θ. Here we are interested in both partially saturated and saturated flow, therefore we retain the pressure ψ as primary unknown.

As suggested in [1], we may expect saturation to be less regular than the pressure. From this point of view the best results are achieved by applying a Kirchhoff transform

$$\mathcal{K} : \mathbb{R} \longrightarrow \mathbb{R}, \qquad \psi \longmapsto \int_0^\psi K(\Theta(s))\, ds. \tag{2}$$

By the properties of K and Θ assumed below, $\mathcal{K}$ is invertible. Taking now $u := \mathcal{K}(\psi)$ as our new unknown, and defining

$$b(u) := \Theta \circ \mathcal{K}^{-1}(u), \qquad k(b(u)) := K \circ \Theta \circ \mathcal{K}^{-1}(u), \tag{3}$$

(1) becomes

$$\partial_t b(u) - \nabla \cdot (\nabla u + k(b(u))\ e_z) = 0 \quad \text{in} \quad (0,T) \times \Omega. \tag{4}$$

Here e_z stands for the vertical unit vector.

Qualitative properties for the Richards' equation are studied in several papers (see, e. g., [7] and [1]). Since explicit solutions can be seldomly obtained, numerical methods are investigated in many articles. Adaptive time stepping is studied, for example, in [18]. Iterative methods are considered for solving the resulting nonlinear equations in case of an implicit method (see, for example, [10], [6], and [9]). For the spatial discretization, mixed finite elements ensure local mass conservation and provide an intrinsec and accurate approximation of the flux ([11]). Hybrid mixed finite elements are studied from an algorithmic point of view in [18].

Convergence results are obtained in [21] (for mixed finite element discretization), [9] (for a relaxation scheme) and [15] (for an implicit scheme in unsaturated regime). This paper is a continuation of [17], where error estimates are obtained for an Euler implicit mixed finite element scheme. It discusses an iterative scheme used for solving the nonlinear problems appearing as a result of the discretization procedure.

2. The numerical scheme

We are interested in solving equation (4) with initial and boundary conditions,

$$\begin{array}{rcll} \partial_t b(u) - \nabla \cdot (\nabla u + k(b(u))e_z) & = & 0 & \text{in} \quad (0,T) \times \Omega, \\ u & = & u^0 & \text{in} \quad 0 \times \Omega, \\ u & = & 0 & \text{on} \quad J \times \Gamma. \end{array} \tag{5}$$

Throughout this paper we make use of the following assumptions:

(A1) $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is bounded with Lipschitz continuous boundary, $T < \infty$.

(A2) $b \in C^1$ is such that $0 \leq b'(u) \leq L_b$ for all reals u.

(A3) $k(b(z))$ is continuous and bounded in z and satisfies, for all $z_1, z_2 \in \mathbb{R}$,

$$| k(b(z_2)) - k(b(z_1)) |^2 \leq C_k(b(z_2) - b(z_1))(z_2 - z_1).$$

(A4) $b(u_0)$ is essentially bounded (by 0 and 1) in Ω and $u_0 \in L^2(\Omega)$.

Assuming k bounded is reasonable since, for Richards' equation, k models the medium conductivity of the medium. The same is also valid for (A4), recalling that $b(u_0)$ stands for the initial saturation in the porous medium. The growth condition on $k(b(\cdot))$ (see also [15] and

[14]) relaxes the more often assumed Lipschitz continuity of k (see, for example, [12] or [2]) and ensures uniqueness of a weak solution (see [1]). Source terms satisfying (A3) can also be considered here without any additional difficulty.

Because of the degenerate character, solutions of (5) should be understood in a weak sense. To give the corresponding definition we let $(\cdot,\cdot)$ stand for the inner product on $L^2(\Omega)$ or the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, $\|\cdot\|$ for the norm in $L^2(\Omega)$, $\|\cdot\|_1$ and $\|\cdot\|_{-1}$ for the norms in $H^1(\Omega)$, respectively $H^{-1}(\Omega)$. For vector valued functions we make use of the space $H(div,\Omega) = \{\vec{q} \in (L^2(\Omega))^d / \nabla\cdot\vec{q} \in L^2(\Omega)\}$.

Analogous notations are used for the inner product and corresponding norm on $L^2(0,T;\mathcal{H})$, with $\mathcal{H}$ being either $L^2(\Omega)$, $H^1(\Omega)$, or $H^{-1}(\Omega)$. In addition, we often write u or $u(t)$ instead of $u(t,x)$ and use C to denote a generic positive constant, not depending on the discretization or regularization parameters.

A weak solution for problem (5) by defined as (see [1])

Definition 2.1. *A function u is called a weak solution for equation (5) iff $b(u) \in H^1(0,T;H^{-1}(\Omega))$, $u \in L^2(0,T;H_0^1(\Omega))$, $u(0) = u^0$ (in H^{-1} sense) and for all $\varphi \in L^2(0,T;H_0^1(\Omega))$ it holds*

$$\int_0^T (\partial_t b(u(t)),\varphi(t)) \ + \ (\nabla u(t) + k(b(u(t)))e_z, \nabla\varphi(t))dt = 0. \tag{6}$$

Existence, uniqueness and essential bounds for a weak solution of the above problem is studied in several papers (see, for example, [1], [13] and the references therein).

Our numerical approach employs the lowest order Raviart–Thomas (RT_0) finite elements in space and Euler implicit in time. These schemes are applied after performing a regularization step. Specifically, for an $\epsilon > 0$ we define b_ϵ as an approximation of the original nonlinearity b, satisfying

$$0 < \epsilon \le b_\epsilon'(u) \le C_1 < \infty, \qquad |b(u) - b_\epsilon(u)| \le C_2\epsilon,$$

for any real u. Possible choices are

$$b_\epsilon(u) = b(u) + \epsilon u, \qquad \text{or} \qquad b_\epsilon(u) = \int_0^u \max\{\epsilon, \min\{b'(v), 1/\epsilon\}\}dv. \tag{7}$$

With $N > 0$ integer, set $\tau = T/N$ and let $\mathcal{T}_h$ being a decomposition of Ω into closed d-simplices; h stands for the mesh-size.

Formally, the numerical scheme under consideration reads

$$\begin{aligned} b_\epsilon(p_h^n) + \tau\nabla q_h^n &= b_\epsilon(p_h^{n-1}), \\ q_h^n + \nabla p_h^n + k(b(p_h^n))e_z &= 0, \end{aligned} \tag{8}$$

for $n = \overline{1,N}$; p_h^0 approximates u^0 in the finite dimensional approximation space. By p_h^n we denote a piecewise constant approximation of u and q_h^n is a Raviart-Thomas (RT_0) approximation of the flux -($\nabla u + k(b(u))e_z)$ on $\mathcal{T}_h$, both at $t = n\tau$.

To give a rigorous formulation of (8) we make use of the discrete subspaces $W_h \times V_h \subset L^2(\Omega) \times H(div, \Omega)$ defined as

$$
\begin{aligned}
W_h &:= \{p \in L^2(\Omega)|\ p \text{ is constant on each element } T \in \mathcal{T}_h\}, \\
V_h &:= \{\overrightarrow{q} \in H(div,\Omega)|\ \overrightarrow{q}_{|T} = \overrightarrow{a} + b\overrightarrow{x} \ \text{ for all } T \in \mathcal{T}_h\}.
\end{aligned} \tag{9}
$$

So W_h denotes the space of piecewise constant functions, while V_h is the RT_0 space (see [5]). Now the fully discrete mixed finite element approximation of problem (5) is defined in

Definition 2.2. *Let $n \in \{1, \ldots, N\}$ and $p_h^{n-1} \in W_h$ be given. Find $(p_h^n, q_h^n) \in W_h \times V_h$ such that*

$$(b_\epsilon(p_h^n), w_h) + \tau(\nabla q_h^n, w_h) = (b_\epsilon(p_h^{n-1}), w_h), \tag{10}$$

$$(q_h^n, v_h) - (p_h^n, \nabla v_h) + (k(b(p_h^n))e_z, v_h) = 0, \tag{11}$$

for all $w_h \in W_h$ and $v_h \in V_h$.

Initially we take $p_h^0 = b_\epsilon^{-1}(P_h b_\epsilon(u^0))$, where P_h is the usual L^2 projector

$$P_h : L^2(\Omega) \to W_h, \quad ((P_h w - w), w_h) = 0 \qquad \forall w_h \in W_h. \tag{12}$$

Since $P_h b_\epsilon(u^0)$ is constant on any $T \in \mathcal{T}_h$, the same holds for $b_\epsilon^{-1}(P_h b_\epsilon(u^0))$, so $p_h^0 \in W_h$. Moreover, with this choice we obtain, for all $w_h \in W_h$,

$$(b_\epsilon(p_h^0), w_h) = (b_\epsilon(u^0), w_h).$$

Defining, for $0 \le t \le T$, the time-integrated flux as

$$\tilde{q}(t) = -\int_0^t (\nabla u(s) + k(b(u(s)))e_z) ds,$$

if u is the solution of problem (5) in the sense of Definition 2.1, the following error estimates are proven in [17] (Theorem 4.14, Corollary 4.15, and Remark 4.16):

Theorem 2.1. *Assuming (A1)-(A4), we get*

$$
\begin{aligned}
&\|\textstyle\sum_{n=1}^N \int_{t_{n-1}}^{t_n} (u(t) - p_h^n) dt\|^2 + \|\tilde{q}(T) - \tau \sum_{n=1}^N q_h^n\|^2 \le C(\tau + \epsilon + h^2), \\
&\textstyle\sum_{n=1}^N \int_{t_{n-1}}^{t_n} (b_\epsilon(u(t)) - b_\epsilon(p_h^n), u(t) - p_h^n) dt \le C\left(\tau^{\frac{1}{2}} + \epsilon^{\frac{1}{2}} + h^2/\tau^{\frac{1}{2}}\right).
\end{aligned} \tag{13}
$$

3. Linearization procedure

The numerical scheme under consideration is implicit, leading at each time step to a nonlinear equation. To solve it we consider an iterative procedure. Let $K \ge L_b$ be a real constant. Fix $n \in \{1, \ldots, N\}$ and assume $p_h^{n-1} \in W_h$, $p_h^{n,i-1} \in W_h$, $q_h^{n,i-1} \in V_h$ known for a given $i > 0$. The following iteration is inspired from [20], pp. 96

$$\begin{aligned} Kp_h^{n,i} + \tau \nabla q_h^{n,i} &= Kp_h^{n,i-1} + b_\epsilon(p_h^{n-1}) - b_\epsilon(p_h^{n,i-1}), \\ q_h^{n,i} + \nabla p_h^{n,i} &= -k(b(p_h^{n,i-1}))e_z. \end{aligned} \tag{14}$$

This approach was used in the cited reference and other textbooks in partial differential equations for constructing sub- and super-solutions for nonlinear elliptic problems. In [22] (see also [14] and [16]) the same idea is used to construct iteration schemes for degenerate (slow diffusion) problems, and the proof is based on fixed point arguments in the H^1 space. Nevertheless, in case of slow diffusion problems the iterations are converging slowly, so such schemes are not of practical interest.

The ideas above are adopted in [19], where the approach is used for constructing an effective iteration scheme for fast diffusion problems. Convection is treated there explicitly, and formulation is conformal. Here we deal with mixed formulation and include an implicit discretization of convection. The scheme in (14) is defined rigorously as

Definition 3.1. *Let $n = \overline{1, N}$, $i > 0$, and $p_h^{n-1} \in W_h$, $p_h^{n,i-1} \in W_h$, $q_h^{n,i-1} \in V_h$ given. Find $(p_h^{n,i}, q_h^{n,i}) \in W_h \times V_h$ such that*

$$(Kp_h^{n,i}, w_h) + \tau(\nabla q_h^{n,i}, w_h) = (Kp_h^{n,i-1} + b_\epsilon(p_h^{n-1}) - b_\epsilon(p_h^{n,i-1}), w_h), \tag{15}$$

$$(q_h^{n,i}, v_h) - (p_h^{n,i}, \nabla v_h) = -(k(b(p_h^{n,i-1}))e_z, v_h), \tag{16}$$

for all $w_h \in W_h$ and $v_h \in V_h$, with $p_h^{n,0} = p_h^{n-1}$.

Remark 3.1. *The setting of Definition 3.1 includes a spatial discretization based on RT_0 elements. Defining the iterative procedure for the spatially continuous case is straightforward, and all the results below apply without any change.*

Remark 3.2. *Convergence is shown below by contraction arguments and holds for any starting point $p_h^{n,0}$. However, the number of iterations required for obtaining a good approximation of p_h^n and q_h^n is influenced by this choice, therefore we take $p_h^{n,0} = p_h^{n-1}$ above, which seems reasonable.*

To show convergence of the iteration scheme we use the following notations

$$e_p^i := p_h^n - p_h^{n,i}, \qquad e_q^i := q_h^n - q_h^{n,i}, \tag{17}$$

where p_h^n and q_h^n are defined in Definition 2.2. The following lemma is proven in [18], pp. 38.

Lemma 3.1. *If the domain Ω is convex, for any $f_h \in W_h$ a $v_h \in V_h$ exists such that*

$$\nabla v_h = f_h, \qquad \textit{and} \qquad \|v_h\| \le C_\Omega \|f_h\|, \tag{18}$$

where the constant $C_\Omega > 0$ does not depend on f_h.

To deal with convection, a mild restriction is imposed for the time step

$$2C_k\tau < 1, \tag{19}$$

with no additional assumption for the regularization parameter ($\epsilon \geq 0$).

Theorem 3.1. *Assuming (19), the scheme in Definition 3.1 converges, and for any $i > 0$ we have*

$$\|e_p^i\| \leq \alpha^{i/2}\ \|e_p^0\|, \ \ and\ \|e_q^i\| \leq \alpha^{i/2}\sqrt{\frac{K+\tau/(2C_\Omega^2)}{\tau}}\|e_p^0\|, \tag{20}$$

with $\alpha := \frac{K-\epsilon(1-2\tau C_k)}{K+\tau/(2C_\Omega^2)} < 1$, with C_Ω from Lemma 3.1 and C_k from (A3).

Proof. We restrict ourselves to proving (20), convergence of $p_h^{n,i}$ to p_h^n, respectively $q_h^{n,i}$ to q_h^n (as $i \nearrow \infty$) following directly since $\alpha < 1$.

Subtracting (15) from (10) and (16) from (11) gives

$$K(e_p^i - e_p^{i-1}, w_h) + \tau(\nabla e_q^i, w_h) + (b_\epsilon(p_h^n) - b_\epsilon(p_h^{n,i-1})), w_h) = 0 \tag{21}$$

$$(e_q^i, v_h) - (e_p^i, \nabla\cdot v_h) + ((k(b(p_h^n)) - k(b(p_h^{n,i-1})))e_z, v_h) = 0 \tag{22}$$

By Lemma 3.1, a $v_h \in V_h$ exists such that $\nabla v_h = e_p^i$ and $\|v_h\| \leq C_\Omega\|\nabla v_h\|$. Using this v_h into (22) together with the inequalities of Cauchy and of means $|ab| \leq \delta a^2 + b^2/(4\delta)$ (for all reals a and b and $\delta > 0$) gives

$$\|e_p^i\|^2 \leq 2C_\Omega^2\{\|e_q^i\|^2 + \|k(b(p_h^n)) - k(b(p_h^{n,i-1}))\|^2\}. \tag{23}$$

Taking now $w_h = e_p^i \in W_h$ into (21), respectively $v_h = \tau e_q^i$ into (22), and adding the resulting, by the identity $(x-y,x) = [x^2-y^2+(x-y)^2]/2$ we get

$$\frac{K}{2}\left(\|e_p^i\|^2 - \|e_p^{i-1}\|^2 + \|e_p^i - e_p^{i-1}\|^2\right) + \tau\|e_q^i\|^2 + (b_\epsilon(p_h^n) - b_\epsilon(p_h^{n,i-1}), e_p^{i-1})$$
$$= -(b_\epsilon(p_h^n) - b_\epsilon(p_h^{n,i-1}), e_p^i - e_p^{i-1}) - \tau((k(b(p_h^n)) - k(b(p_h^{n,i-1})))e_z, e_q^i).$$

Applying again Cauchy's inequality and the inequality of mean, after multiplying the above by 2 we obtain

$$\begin{aligned} & K\|e_p^i\|^2 - K\|e_p^{i-1}\|^2 + \tau\|e_q^i\|^2 + 2(b_\epsilon(p_h^n) - b_\epsilon(p_h^{n,i-1}), e_p^{i-1}) \\ \leq\ & \tfrac{1}{K}\|b_\epsilon(p_h^n) - b_\epsilon(p_h^{n,i-1})\|^2 + \tau\|k(b(p_h^n)) - k(b(p_h^{n,i-1}))\|^2. \end{aligned} \tag{24}$$

Adding $\tau\|k(b(p_h^n)) - k(b(p_h^{n,i-1}))\|^2$ on both sides of (24), using (23), (A2) and (A3) gives

$$\left(K + \frac{\tau}{2C_\Omega^2}\right)\|e_p^i\|^2 \leq K\|e_p^{i-1}\|^2 + \left(\frac{L_b}{K} + 2\tau C_k - 2\right)(b_\epsilon(p_h^n) - b_\epsilon(p_h^{n,i-1}), e_p^{i-1}).$$

Since b_ϵ is monotone, the scalar product on the right is positive. In fact we have

$$\epsilon\|e_p^{i-1}\|^2 \leq (b_\epsilon(p_h^n) - b_\epsilon(p_h^{n,i-1}), e_p^{i-1}) \leq L_b\|e_p^{i-1}\|^2.$$

By (19), because $K \geq L_b$ we have $L_b/K + 2\tau C_k - 2 \leq 2\tau C_k - 1 < 0$, so we end up with

$$\left(K + \frac{\tau}{2C_\Omega^2}\right) \|e_p^i\|^2 \leq (K - \epsilon\,(1 - 2\tau C_k))\,\|e_p^{i-1}\|^2, \tag{25}$$

and the first part of (20) follows immediately.

For estimating the flux error we use (24) again and proceed as before to get

$$K\|e_p^i\|^2 + \tau\|e_q^i\|^2 \leq (K - \epsilon(1 - 2\tau C_k))\|e_p^{i-1}\|^2.$$

Using now the estimates in the pressure gives the second part of (20). □ □

Remark 3.3. *Theorems 3.2 and 3.1 show that the linearization in (14) is convergent in both pressure and flux. This is consistent with previous results, where H^1 estimates are obtained in the conformal formulation.*

The convergence rate becomes squared if convection is 0, as follows from below. Moreover, as in [22] or [19], the scheme converges for $\tau > 0$ and $\epsilon \geq 0$.

Theorem 3.2. *In the absence of convection ($k = 0$), the scheme given in Definition 3.1 is convergent and for any $i > 0$ we have*

$$\|e_p^i\| \leq \alpha^i\ \|e_p^0\|, \quad \textit{and} \quad \|e_q^i\| \leq \alpha^i\ \frac{K + \tau/C_\Omega}{2K\tau^{1/2}}\|e_p^0\|, \tag{26}$$

with $\alpha := \frac{K-\epsilon}{K+\tau/C_\Omega} < 1$.

Proof. We use similar arguments as for proving Theorem 3.1. Subtracting (15) from (10) and taking $w_h = e_p^i \in W_h$ yields,

$$K\|e_p^i\|^2 + \tau(\nabla e_q^i, e_p^i) = (Ke_p^{i-1} - (b_\epsilon(p_h^n) - b_\epsilon(p_h^{n,i-1})), e_p^i). \tag{27}$$

Similarly, from (16) and (11) we get

$$(e_q^i, v_h) - (e_p^i, \nabla v_h) = 0, \tag{28}$$

for all $v_h \in V_h$. Testing with $v_h = \tau e_q^i \in V_h$ and adding the result to (27) gets

$$K\|e_p^i\|^2 + \tau\|e_q^i\|^2 = (Ke_p^{i-1} - (b_\epsilon(p_h^n) - b_\epsilon(p_h^{n,i-1})), e_p^i). \tag{29}$$

By Lemma 3.1, a $v_h \in V_h$ exists such that $\nabla v_h = e_p^i \in W_h$ and $\|v_h\| \leq C_\Omega\|e_p^i\|$. Using this as a test function into (28) gives

$$\|e_p^i\|^2 = (e_p^i, \nabla v_h) = (e_q^i, v_h) \leq \|v_h\|\|e_q^i\| \leq C_\Omega\|e_p^i\|\|e_q^i\|,$$

implying

$$\|e_p^i\| \leq C_\Omega\|e_q^i\|. \tag{30}$$

Since $\epsilon \leq b'_\epsilon \leq L_b \leq K$, it follows that

$$|Ke_p^{i-1} - (b_\epsilon(p_h^n) - b_\epsilon(p_h^{n,i-1}))| \leq (K - \epsilon)|e_p^{i-1}| \tag{31}$$

almost everywhere in Ω. Using Cauchy's inequality, (30) and (31) we end up with

$$\left(K + \frac{\tau}{C_\Omega}\right) \|e_p^i\|^2 \le (K - \epsilon)\|e_p^{i-1}\|\|e_p^i\|,$$

so $\|e_p^i\| \le \alpha \|e_p^{i-1}\|$ and the first part of (26) is proven. For the second estimate we use (29) again and the mean inequality to obtain

$$K\|e_p^i\|^2 + \tau\|e_q^i\|^2 \le (K - \epsilon)\|e_p^{i-1}\|\|e_p^i\| \le K\|e_p^i\|^2 + \frac{(K-\epsilon)^2}{4K^2}\|e_p^{i-1}\|^2,$$

and the rest of the proof is straightforward. □ □

Remark 3.4. *The same result is obtained if convection is discretized explicitly (see also* [19]*).*

The theoretical estimates are validated on a test problem proposed in [8]. After performing the Kirchhoff transformation we end up with (4), where $k \equiv 0$ and

$$b(u) = \begin{cases} \frac{\pi^2}{2} - \frac{u^2}{2} & \text{for} \quad u \le 0 \\ \frac{\pi^2}{2} & \text{for} \quad u > 0. \end{cases}$$

With appropriate initial and (Dirichlet) boundary conditions, an exact solution can be given

$$u_{\text{ex}}(t, x, y) = \begin{cases} \frac{-2(e^s - 1)}{e^s + 1} & \text{for} \quad s \ge 0 \\ -s & \text{for} \quad s < 0 \end{cases}$$

where $s = x - y - t$. The equation is solved in the unit square $(0,1)^2$, and the final time is $T = 1.0$.

We consider a hybrid implementation of the mixed finite element method. The emerging algebraic system of equations is solved by adding Lagrange multipliers. Within an iteration step we first eliminate the flux on each element and then continuity equation is solved locally for pressure. The algorithm is implemented in *UG* (version 3.8, see also [3]) on a SUN workstation.

Computations are performed on a uniform triangular mesh. Initially we take $h = 0.25$ and $\tau = 0.04$. Then τ and h^2 are successively halved, up to $\tau = 0.000625$ and $h = 0.03125$. For this particular problem we took $\epsilon = 0$ and $K = 2$. Few iterations (5 to 8) were sufficient to obtain solutions differing by at most 10^{-10}. Similar results are provided by the Newton method developed in [18].

The order of convergence is estimated by the logarithm of the ratio between two squared errors, obtained for different parameters. Results are given in Table 1, confirming the theoretical estimates in Theorem 2.1.

Table 1. Numerical results.

N	τ	h	Squared error	Convergence order
1	0.04	0.25	7.084860e-06	—
2	0.02	0.176	3.620121e-06	0.81
3	0.01	0.125	2.057357e-06	0.82
4	0.005	0.088	9.574648e-07	1.10
5	0.0025	0.0625	5.362180e-07	0.84
6	0.00125	0.044	2.431789e-07	1.14
7	0.000625	0.03125	1.355457e-07	0.84

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