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A CHEBYSHEV-GALERKIN METHOD FOR FOURTH ORDER PROBLEMS

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Abstract. We propose an efficient implementation of the Chebyshev Galerkin spectral method for differential equations of the fourth order with Dirichlet boundary conditions. This discretization leads to banded matrices which, compared with other methods of the same type, are better conditioned. The efficiency of the method is illustrated on an eigenvalue model problem, where an improved convergence can be observed and the spurious eigenvalues are removed.

1. Introduction

Spectral methods have been intensively studied in the last two decades because of their good approximation properties. This advantage was shadowed by some difficulties generated by this discretisation. Thus, the matrices which arize in the spectral discretisation of the differential equations are generally full and have also an increased condition number. Therefore, especially for the fourth order problems, stability and numerical accuracy of the computation can be strongly affected when a discretization with a large number of elements is used, so the theoretical accuracy of these methods can be lost. There are several works concerned with the problems mentioned above in any of the three existing types of the spectral methods (see, for example [4], [3] for the tau method, [2], [5] for the collocation one or [7] for the Galerkin approach).

In Section 2 an efficient implementation of the Chebyshev-Galerkin method for fourth-order problems is proposed. This approach leads to better conditioned matrices which, in the case of constant coefficients, are also banded. The effectiveness of the method is shown in Section 3 on a model-eigenvalue problem, where optimal convergence order is obtained.

2. Chebyshev-Galerkin methods for 4^{th} order problems

In this section we describe the Galerkin variant for the Chebyshev spectral method for 4^{th} order problems. Only Dirichlet boundary conditions are considered but similar ideas can be applied also in other cases. Here we denote by $L^2_\omega(-1,1)$

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the space of functions which are square integrable with respect to the weight function $\omega(x) = (1-x^2)^{-\frac{1}{2}}$. By $T_k(x) = \cos(k \arccos(x)), \quad k \in \mathbb{N}$ we denote the k^{th} order Chebyshev polynomial. The following properties are important for the spectral discretization (see, e.g., [1])

$$\begin{aligned} T_{k+2}(x) =& 2xT_{k+1}(x) - T_k(x), \\ T_k(x)T_p(x) =& \frac{1}{2}(T_{k+p} + T_{|k-p|}), \\ T_k(\pm 1) = (\pm 1)^k, \quad & T_k'(\pm 1) = (\pm 1)^k k^2, \\ (T_k, T_p)_{0,\omega} =& \frac{\pi}{2} c_k \delta_{k,p}, \end{aligned} \tag{2.1}$$

where $k, p \in \mathbb{N}$, $\delta_{k,p}$ is the Kronecker symbol and $c_i = \begin{cases} 2, & \text{if } i = 0; \\ 1, & \text{if } i > 0. \end{cases}$

If $u^N = \sum_{k=0}^N a_k T_k$, its 4^{th} order derivative can be expressed in the following form ([6])

$$(u^N)^{(iv)} = \sum_{k=0}^N a_k^{(4)} T_k = \sum_{k=0}^N (D^4 a)_k T_k, \tag{2.2}$$

with $(D^4 a)_k = \sum_{j=0}^N d_{kj}^4 a_j$ and

$$c_i d_{ij}^4 = \begin{cases} j[j^2(j^2-4)^2 - 3i^2 j^4 + 3i^4 j^2 - i^2(i^2-4)^2], & \text{if } j = i+4, i+6, \cdots, \\ 0, & \text{otherwise.} \end{cases}$$

The condition number of this matrix is $O(N^8)$. In the case of homogenous Dirichlet boundary conditions, W. Heinrichs ([3]) proposed the following approach

$$u^N = \sum_{k=0}^N a_k \Psi_k, \text{ where } \Psi_k(x) = (1-x^2)^2 \cdot T_k(x), k = \overline{0, N}. \tag{2.3}$$

This basis satisfies apriori homogenous boundary conditions, but further the Chebyshev coefficients of this expansion were computed and then imposed to be equal with the corresponding ones from the right hand side of the equation. Therefore, this approach is closer to the tau-spectral method.

J. Shen ([7]) considers another expansion within a Galerkin method $u^N = \sum_{k=0}^N a_k \Phi_k$, where

$$\Phi_k(x) = T_k(x) - \frac{2(k+2)}{(k+3)} T_{k+2}(x) + \frac{(k+1)}{(k+3)} T_{k+4}(x), k = \overline{0, N}. \tag{2.4}$$

For both of these basises, an explicit form of the 4^{th} order differentiation matrix can be given. The condition number is reduced up to $O(N^4)$ and the diferentiation matrices remain upper triangular, but not banded.

In our approach a Galerkin method is considerred using the Heinrichs basis ($\{\Psi_k, k = 0, \cdots, N\}$) as trial basis and the Shen basis ($\{\Phi_k, k = 0, \cdots, N\}$) for the test one. We have the following

Table 2.1. Condition number for the D^4 discretization matrix in (2.5).

N	$cond(A)$	$cond(A)/N^4$	$cond(PA)$	$cond(PA)/N^2$
16	$3.85 \cdot 10^3$	0.059	$2.12 \cdot 10^1$	0.083
32	$6.35 \cdot 10^4$	0.060	$7.67 \cdot 10^1$	0.075
64	$1.10 \cdot 10^6$	0.065	$2.88 \cdot 10^2$	0.070
128	$1.89 \cdot 10^7$	0.070	$1.11 \cdot 10^3$	0.068
256	$3.19 \cdot 10^8$	0.074	$4.35 \cdot 10^3$	0.066
512	$5.30 \cdot 10^9$	0.077	$1.72 \cdot 10^4$	0.066

Lemma 2.1. *For Ψ_k and Φ_k defined as above we have $(\Phi_i, \Psi_j^{(iv)})_2 = \frac{2}{\pi} a_{ij}$, where*

$$a_{ij} = \begin{cases} c_i(i+1)(i+2)(i+3)(i+4), & \text{if } j = i, \\ -2i(i+1)(i+2)(i+4), & \text{if } j = i+2, \\ i(i+1)^2(i+2), & \text{if } j = i+4, \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

This formula and other similar ones for other operators can be obtained by direct computations using the properties of Chebyshev polynomials. As one could observe, the resulting matrix is tridiagonal. More, it is upper triangular. Its condition number is almost halved, comparing it with the one offered by the Shen method, so this discretisation is more stable. Even the simplest preconditioner, the diagonal one (denoted by P), is effective, generating a condition of order N^2. These statements are supported by the results in Table 2.1 for the fourth order derivative, but similar values are obtained also for other operators.

3. Application

The efficiency of our approach can be illustrated on a model problem, the celebrated Orr-Sommerfeld equation.

$$\Phi^{(iv)} - 2\alpha^2\Phi'' + \alpha^4\Phi = i\alpha Re[(U-\lambda)(\Phi'' - \alpha^2\Phi) - U''\Phi]$$

$$\Phi(\pm 1) = \Phi'(\pm 1) = 0, x \in [-1, 1]$$

The classical tau method generates two spurious eigenvalues in this case (see, e.g [6]). We have compared the results with those obtained by W. Huang and D. M. Sloan ([5]) using a modified pseudospectral method and also with those generated by the classical tau method. As one can see from the tables below, an improved convergence is obtained for the Chebyshev-Galerkin method. This accuracy is not lost when the approximation order is increased (we have tested it up to N=512), this being a consequence of the stabilized treatment of the 4^{th} order derivative. No spurious eigenvalues are obtained. We have achieved an accuracy up to 8 digits. In simpler cases (for example in the computation of the eigenvalues of the 4^{th} order derivative) the machine precision was rapidly achieved.

Conclusion. We have proposed an approach for the Chebyshev-Galerkin spectral method. This discretization generates sparse matrices which are better conditioned. Applied to differential eigenvalue problems, it does not generate spurious eigenvalues.

Table 3.1. First eigenvalue, $\alpha = 1.00, R = 10000$, modified pseudospectral method.

N	$Re(\lambda)$	$Im(\lambda)$	$log_{10}\lvert\lambda - \lambda_{ex}\rvert$
13	0. 36841258081	0.0 6487680791	-0.84
18	0.237 09997536	0.00 441789200	-3.10
23	0.23 563796411	0.00 151181136	-2.53
33	0.237 48476605	0.003 68618134	-4.17
38	0.23752 489827	0.00373 265586	-5.14
43	0.237526 75297	0.003739 21305	-6.27
48	0.237526 51907	0.00373967 171	-7.52

Exact ([6]): $\lambda_{ex} = 0.23752648882 + 0.00373967062i$.

Table 3.2. First eigenvalue, $\alpha = 1.00, R = 10000$, Chebyshev-Galerkin method.

N	$Re(\lambda)$	$Im(\lambda)$	$log_{10}\lvert\lambda - \lambda_{ex}\rvert$
13	0.2 2440097715	0.0 1144196988	-1.82
18	0.23 854356657	0.00 081447143	-2.51
23	0.2375 0868627	0.003 47148295	-3.57
33	0.23752 593252	0.00373 317977	-5.19
38	0.237526 66392	0.003739 84116	-6.61
43	0.237526 52365	0.00373967 138	-7.49
48	0.23752648 754	0.0037396 6968	-8.41

Exact ([6]): $\lambda_{ex} = 0.23752648882 + 0.00373967062i$.

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