Continuous Time Markov Chains (CTMCs)

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ISP
Continuous Time Markov Chains (CTMCs)

In analogy with the definition of a discrete-time Markov chain, given in Chapter 4, we say that the process \( \{ X(t) : t \geq 0 \} \), with state space \( S \), is a continuous-time Markov chain if for all \( s, t \geq 0 \) and nonnegative integers \( i, j, x(u), 0 \leq u < s \)

\[
P[X(t + s) = j | X(s) = i, X(u) = x(u), 0 \leq u < s] = P[X(t + s) = j | X(s) = i]
\]

In other words, a continuous-time Markov chain is a stochastic process having the Markovian property that the conditional distribution of the future \( X(t + s) \) given the present \( X(s) \) and the past \( X(u), 0 \leq u < s \), depends only on the present and is independent of the past.

If, in addition,

\[
P[X(t + s) = j | X(s) = i]
\]

is independent of \( s \), then the continuous-time Markov chain is said to have stationary or homogeneous transition probabilities.
Continuous Time Markov Chains (CTMCs)

To what follows, we will restrict our attention to time-homogeneous Markov processes, i.e., continuous-time Markov chains with the property that, for all $s, t \geq 0$,

$$P[X(s + t) = j \mid X(s) = i] = P[X(t) = j \mid X(0) = i] = P_{ij}(t).$$

The probabilities $P_{ij}(t)$ are called transition probabilities and the $|S| \times |S|$ matrix

$$P(t) = \begin{pmatrix} P_{00}(t) & P_{01}(t) & \ldots \\ P_{10}(t) & P_{11}(t) & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

is called the transition probability matrix.

The matrix $P(t)$ is for all $t$ a stochastic matrix.
Continuous Time Markov Chains (CTMCs)

Memoryless property

Suppose that a continuous-time Markov chain enters state $i$ at some time, say, time 0, and suppose that the process does not leave state $i$ (that is, a transition does not occur) during the next 10 min.

What is the probability that the process will not leave state $i$ during the following 5 min?

Now since the process is in state $i$ at time 10 it follows, by the Markovian property, that the probability that it remains in that state during the interval $[10, 15]$ is just the (unconditional) probability that it stays in state $i$ for at least 5 min. That is, if we let

$$T_i : \text{the amount of time that the process stays in state } i \text{ before making a transition into a different state},$$

then

$$P[T_i > 15 | T_i > 10] = P[T_i > 5].$$
Continuous Time Markov Chains (CTMCs)

Memoryless property

Suppose that a continuous-time Markov chain enters state \( i \) at some time, say, time \( s \), and suppose that the process does not leave state \( i \) (that is, a transition does not occur) during the next \( t \) min.

What is the probability that the process will not leave state \( i \) during the following \( t \) min?

With the same reasoning as before, if we let

\[ T_i : \text{the amount of time that the process stays in state } i \text{ before making a transition into a different state}, \]

then

\[ P[T_i > s + t | T_i > s] = P[T_i > t] \]

for all \( s, t \geq 0 \). Hence, the random variable \( T_i \) is memoryless and must thus (see Section 5.2.2) be exponentially distributed!
Continuous Time Markov Chains (CTMCs)
Memoryless property

In fact, the preceding gives us another way of defining a continuous-time Markov chain. Namely, it is a stochastic process having the properties that each time it enters state $i$

(i) the amount of time it spends in that state before making a transition into a different state is exponentially distributed with mean, say, $E[T_i] = 1/v_i$, and

(ii) when the process leaves state $i$, it next enters state $j$ with some probability, say, $P_{ij}$. Of course, the $P_{ij}$ must satisfy

$$P_{ii} = 0, \text{ all } i$$

$$\sum_j P_{ij} = 1, \text{ all } i.$$
Continuous Time Markov Chains (CTMCs)

Poisson process $i \rightarrow i + 1$

$X(t) = \# \text{ of arrivals in } (0, t]$
State space $\{0, 1, 2, \ldots\}$

\[
\begin{align*}
0 \xrightarrow{\lambda} 1 \xrightarrow{\lambda} 2 \xrightarrow{\lambda} \cdots
\end{align*}
\]

**Figure:** Transition rate diagram of the Poisson Process

$T_i \sim \text{Exp}(\lambda)$, so $\nu_i = \lambda$, $i \geq 0$

$P_{ij} = 1, j = i + i$

$P_{ij} = 0, j \neq i + 1.$
Continuous Time Markov Chains (CTMCs)
Birth-Death Process $i \rightarrow i + 1$ and $i \rightarrow i - 1$

$X(t) =$ population size at time $t$
State space $\{0, 1, 2, \ldots\}$

![Transition rate diagram of the Birth-Death Process](image)

**Figure:** Transition rate diagram of the Birth-Death Process

- Time till next ‘birth’: $B_i \sim \text{Exp}(\lambda_i)$, $i \geq 0$
- Time till next ‘death’: $D_i \sim \text{Exp}(\mu_i)$, $i \geq 1$
- Time till next transition: $T_i = \min\{B_i, D_i\} \sim \text{Exp}((\lambda_i + \mu_i) = \nu_i)$, $i \geq 1$
  and $T_0 \sim \text{Exp}(\lambda_0)$

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}, \quad P_{ij} = 0, j \neq i \pm 1, i \geq 1$$

$$P_{01} = 1$$
Birth-Death (B-D) Process: Determine $M(t)$

$X(t) = $ population size at time $t$, state space $\{0, 1, 2, \ldots\}$

![Transition rate diagram of the Birth-Death Process](image)

**Figure:** Transition rate diagram of the Birth-Death Process

Suppose that $X(0) = i$ and let $M(t) = E[X(t)]$

$$M(t + h) = E[X(t + h)] = E[E[X(t + h)|X(t)]]$$

$$X(t + h) = \begin{cases} 
X(t) + 1, & \text{with probability } [\alpha + X(t)\lambda]h + o(h) \\
X(t) - 1, & \text{with probability } X(t)\mu h + o(h) \\
X(t), & \text{otherwise}
\end{cases}$$
Birth-Death (B-D) Process: Determine $M(t)$

$X(t)$ = population size at time $t$, state space $\{0, 1, 2, \ldots\}$

\[
\begin{array}{c c c c c}
0 & \alpha & 1 & \lambda+\alpha & 2 & 2\lambda+\alpha & \ldots \\
\mu & & 2\mu & & 3\mu & & \\
\end{array}
\]

**Figure:** Transition rate diagram of the Birth-Death Process

Suppose that $X(0) = i$ and let $M(t) = E[X(t)]$

\[
M(t + h) = E[E[X(t + h) | X(t)]]
\]

\[
E[X(t + h) | X(t)] = (X(t) + 1)[\alpha h + X(t)\lambda h] + (X(t) - 1)[X(t)\mu h]
\]

\[
+ X(t)[1 - \alpha h - X(t)\lambda h - X(t)\mu h] + o(h)
\]

\[
= X(t) + \alpha h + X(t)\lambda h - X(t)\mu h + o(h)
\]
Birth-Death (B-D) Process : Determine $M(t)$

$X(t) =$ population size at time $t$, state space $\{0, 1, 2, \ldots\}$

Figure: Transition rate diagram of the Birth-Death Process

Suppose that $X(0) = i$ and let $M(t) = E[X(t)]$

$$M(t + h) = E[E[X(t + h)|X(t)]]$$

$$= M(t) + \alpha h + M(t)\lambda h - M(t)\mu h + o(h)$$

$$E[X(t + h)|X(t)] = X(t) + \alpha h + X(t)\lambda h - X(t)\mu h + o(h)$$
Birth-Death (B-D) Process: Determine $M(t)$

$X(t) =$ population size at time $t$, state space $\{0, 1, 2, \ldots\}$

Suppose that $X(0) = i$ and let $M(t) = E[X(t)]$

\[
M(t + h) = E[E[X(t + h)|X(t)]] = M(t) + \alpha h + M(t)\lambda h - M(t)\mu h + o(h)
\]

Taking the limit as $h \to 0$ yields a differential equation

\[
M'(t) = (\lambda - \mu)M(t) + \alpha \implies M(t) = \left(\frac{\alpha}{\lambda - \mu} + i\right)e^{(\lambda - \mu)t} - \frac{\alpha}{\lambda - \mu}
\]
Birth-Death (B-D) Process: First step analysis

Let $T_{ij}$ be the time to reach $j$ for the first time starting from $i$. Then for the B-D process

$$E[T_{i,j}] = \frac{1}{\lambda_i + \mu_i} + P_{i,i+1} \times E[T_{i+1,j}] + P_{i,i-1} \times E[T_{i-1,j}], \ i,j \geq 1$$

This can be solved, starting from $E[T_{0,1}] = 1/\lambda_0$.

Example

For the B-D process with $\lambda_i = \lambda$ and $\mu_i = \mu$ we obtain recursively

$$E[T_{0,1}] = \frac{1}{\lambda}$$

$$E[T_{1,2}] = \frac{1}{\lambda + \mu} + P_{1,2} \times 0 + P_{1,0} \times E[T_{0,2}]$$

$$= \frac{1}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} (E[T_{0,1}] + E[T_{1,2}]) = \frac{1}{\lambda} \left[ 1 + \frac{\mu}{\lambda} \right]$$

$$E[T_{i,i+1}] = \frac{1}{\lambda} \left[ 1 + \frac{\mu}{\lambda} + \frac{\mu^2}{\lambda^2} + \cdots + \frac{\mu^i}{\lambda^i} \right]$$
Birth-Death Process: First step analysis

Let $T_{ij}$ be the time to reach $j$ for the first time starting from $i$. Then for the B-D process

$$E[e^{-sT_{ij}}] = \frac{\lambda_i + \mu_i}{\lambda_i + \mu_i + s} [P_{i,i+1} \times E[e^{-sT_{i+1,j}}] + P_{i,i-1} \times E[e^{-sT_{i-1,j}}]], \quad i,j \geq 1$$

This can be solved, starting from $E[e^{-sT_{0,1}}] = \lambda_0 / (\lambda_0 + s)$.

**Example**

For the B-D process with $\lambda_i = \lambda$ and $\mu_i = \mu$ we are interested in $T_{1,0}$ (time to extinction):

$$E[e^{-sT_{i,0}}] = \frac{1}{\lambda + \mu + s} [\lambda E[e^{-sT_{i+1,0}}] + \mu E[e^{-sT_{i-1,0}}]], \quad i \geq 1$$

and $E[e^{-sT_{0,0}}] = 1$. For a fixed $s$ this equation can be seen as a second order difference equation yielding

$$E[e^{-sT_{i,0}}] = \frac{1}{(2\lambda)^i} [\lambda + \mu + s - \sqrt{(\lambda + \mu + s)^2 - 4\lambda\mu}]^i, \quad i \geq 1.$$
The Transition Probability Function $P_{ij}(t)$

Let

$$P_{ij}(t) = P[X(t + s) = j | X(s) = i]$$

denote the transition probabilities of the continuous-time Markov chain. How do we calculate them?

**Example**

For the Poisson process with rate $\lambda$

$$P_{ij}(t) = P[j - i \text{ jumps in } (0, t)|X(0) = i]$$
$$= P[j - i \text{ jumps in } (0, t)]$$
$$= e^{-\lambda t} \frac{(\lambda t)^{j-i}}{(j-i)!}.$$
The Transition Probability Function $P_{ij}(t)$

Let

$$P_{ij}(t) = P[X(t + s) = j | X(s) = i]$$

denote the transition probabilities of the continuous-time Markov chain. How do we calculate them?

**Example**

*For the Birth process with rates $\lambda_i$, $i \geq 0$*

$$P_{ij}(t) = P[X(t) = j | X(0) = i]$$

$$= P[X(t) \leq j | X(0) = i] - P[X(t) \leq j - 1 | X(0) = i]$$

$$= P[X_i + \cdots + X_j > t] - P[X_i + \cdots + X_{j-1} > t],$$

with $X_k \sim \text{Exp}(\lambda_k)$, $k=1,2,\ldots$. Hence, for $\lambda_k$ all different

$$P[X_i + \cdots + X_j > t] = \sum_{k=i}^{j} e^{-\lambda_k t} \prod_{r \neq k} \frac{\lambda_r}{\lambda_r - \lambda_k}.$$
The Transition Probability Function $P_{ij}(t)$

We shall derive a set of differential equations that the transition probabilities $P_{ij}(t)$ satisfy in a general continuous-time Markov chain. First we need a definition and a pair of lemmas.

**Definition**

For any pair of states $i$ and $j$, let

$$q_{ij} = v_i P_{ij}$$

Since $v_i$ is the rate at which the process makes a transition when in state $i$ and $P_{ij}$ is the probability that this transition is into state $j$, it follows that $q_{ij}$ is the rate, when in state $i$, at which the process makes a transition into state $j$. The quantities $q_{ij}$ are called the **instantaneous transition rates**.

$$v_i = \sum_j v_i P_{ij} = \sum_j q_{ij}$$

$$P_{ij} = \frac{q_{ij}}{v_i} = \frac{q_{ij}}{\sum_j q_{ij}}$$
The Transition Probability Function $P_{ij}(t)$

Transition Rates

We shall derive a set of differential equations that the transition probabilities $P_{ij}(t)$ satisfy in a general continuous-time Markov chain. First we need a definition and a pair of lemmas.

**Lemma (6.2)**

a) $\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i$

b) $\lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij}$, when $i \neq j$.

**Lemma (6.3 – Chapman-Kolmogorov equations)**

For all $s, t \geq 0$

$$P_{ij}(t + s) = \sum_{k \in S} P_{ik}(t)P_{kj}(s)$$

$$\mathbf{P}(t + s) = \mathbf{P}(t)\mathbf{P}(s)$$
The Transition Probability Function $P_{ij}(t)$

Transition Rates

We shall derive a set of differential equations that the transition probabilities $P_{ij}(t)$ satisfy in a general continuous-time Markov chain.

First we need a definition and a pair of lemmas.

**Theorem (6.1 – Kolmogorov’s Backward equations)**

*For all states $i, j$ and times $t \geq 0$*

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

with initial conditions $P_{ii}(0) = 1$ and $P_{ij}(0) = 0$ for all $j \neq i$.

**Theorem (6.2 – Kolmogorov’s Forward equations)**

*Under suitable regularity conditions*

$$P'_{ij}(t) = \sum_{k \neq j} P_{ik}(t)q_{kj} - v_j P_{ij}(t)$$
CTMC and Limiting probabilities

Assuming that the \( \lim_{t \to \infty} P_{ij}(t) \) exists

(a) all states communicate

(b) the MC is positive recurrent (finite mean return time)

we can define \( \lim_{t \to \infty} P_{ij}(t) = P_j \) and then by taking properly the limit as \( t \to \infty \) in the Kolmogorov forward equations yields

\[
v_j P_j = \sum_{k \neq j} q_{kj} P_k
\]

This set of equations together with the normalization equation \( \sum_j P_j = 1 \) can be solved to obtain the limiting probabilities.

Interpretations of \( P_j \):

- Limiting distribution
- Long run fraction of time spent in \( j \)
- Stationary distribution
CTMC and Limiting probabilities

Assuming that the $\lim_{t \to \infty} P_{ij}(t)$ exists

(a) all states communicate

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we can define $\lim_{t \to \infty} P_{ij}(t) = P_j$ and then by taking properly the limit as $t \to \infty$ in the Kolmogorov forward equations yields

$$v_j P_j = \sum_{k \neq j} q_{kj} P_k$$

This set of equations together with the normalization equation $\sum_j P_j = 1$ can be solved to obtain the limiting probabilities.

When the limiting probabilities exist we say that the MC is ergodic.

The $P_j$ are sometimes called stationary probabilities since it can be shown that if the initial state is chosen according to the distribution $\{P_j\}$, then the probability of being in state $j$ at time $t$ is $P_j$, for all $t$. 
CTMC and Limiting probabilities

Assuming that the $\lim_{t \to \infty} P_{ij}(t)$ exists

(a) all states communicate
(b) the MC is positive recurrent (finite mean return time)

we can define the vector $\mathbf{P} = (\lim_{t \to \infty} P_{ij}(t))$ and then by taking properly
the limit as $t \to \infty$ the Kolmogorov forward equations yield

$$\mathbf{PQ} = 0$$

This set of equations together with the normalization equation $\sum_j P_j = 1$
can be solved to obtain the limiting probabilities.
The matrix $\mathbf{Q}$ is called *generator matrix* and contains all rate transitions:

$$\mathbf{Q} = \begin{pmatrix}
  -\nu_0 & q_{01} & \cdots \\
  q_{10} & -\nu_1 & \cdots \\
  \vdots & \vdots & \ddots
\end{pmatrix}$$

The sum of all elements in each row is 0.
Birth-Death (B-D) Process and Limiting probabilities

\[ X(t) = \text{population size at time } t, \text{ state space } \{0, 1, 2, \ldots\} \]

![Transition rate diagram of the Birth-Death Process](image)

**Figure**: Transition rate diagram of the Birth-Death Process

From the balance argument: “Outflow state \( j = \text{Inflow state } j \)”, we obtain the following system of equations for the limiting probabilities:

\[
\begin{align*}
P_0 \lambda_0 &= P_1 \mu_1, \\
P_1(\lambda_1 + \mu_1) &= P_0 \lambda_0 + P_2 \mu_2, \\
& \vdots \\
P_i(\lambda_i + \mu_i) &= P_{i-1} \lambda_{i-1} + P_{i+1} \mu_{i+1}, \quad i = 2, 3, \ldots
\end{align*}
\]
Birth-Death (B-D) Process and Limiting probabilities

\[ X(t) = \text{population size at time } t, \text{ state space } \{0, 1, 2, \ldots\} \]

\[
\begin{array}{c}
0 \\
\lambda_0 \\
\mu_1 \\
1 \\
\lambda_1 \\
\mu_2 \\
2 \\
\lambda_2 \\
\mu_3 \\
\vdots \\
\end{array}
\]

Figure: Transition rate diagram of the Birth-Death Process

By adding to each equation the equation preceding it, we obtain

\[ P_0 \lambda_0 = P_1 \mu_1, \]
\[ P_1 \lambda_1 = P_2 \mu_2, \]
\[ \vdots \]
\[ P_i \lambda_i = P_{i+1} \mu_{i+1}, \quad i = 2, 3, \ldots \]

Solving yields

\[ P_i = \frac{\lambda_{i-1} \lambda_{i-2} \cdots \lambda_0}{\mu_i \mu_{i-1} \cdots \mu_1} P_0, \quad i = 1, 2, \ldots \]
Birth-Death (B-D) Process and Limiting probabilities

\[ X(t) = \text{population size at time } t, \text{ state space } \{0, 1, 2, \ldots\} \]

![Transition rate diagram of the Birth-Death Process](image)

**Figure:** Transition rate diagram of the Birth-Death Process

\[ P_i = \frac{\lambda_{i-1}\lambda_{i-2}\cdots\lambda_0}{\mu_i\mu_{i-1}\cdots\mu_1} P_0, \ i = 1, 2, \ldots \]

And by using the fact that \( \sum_{j=0}^{\infty} P_j = 1 \) we can solve for \( P_0 \)

\[ P_0 = (1 + \sum_{i=1}^{\infty} \frac{\lambda_{i-1}\lambda_{i-2}\cdots\lambda_0}{\mu_i\mu_{i-1}\cdots\mu_1})^{-1}, \ i = 1, 2, \ldots \]
Birth-Death (B-D) Process and Limiting probabilities

$$X(t) = \text{population size at time } t, \text{ state space } \{0, 1, 2, \ldots\}$$

![Transition rate diagram of the Birth-Death Process](image)

**Figure:** Transition rate diagram of the Birth-Death Process

$$P_i = \frac{\lambda_{i-1} \lambda_{i-2} \cdots \lambda_0}{\mu_i \mu_{i-1} \cdots \mu_1} P_0, \quad i = 1, 2, \ldots$$

$$P_0 = (1 + \sum_{i=1}^{\infty} \frac{\lambda_{i-1} \lambda_{i-2} \cdots \lambda_0}{\mu_i \mu_{i-1} \cdots \mu_1})^{-1}, \quad i = 1, 2, \ldots$$

It is **sufficient** and **necessary** for the limiting probabilities to exists

$$\sum_{i=1}^{\infty} \frac{\lambda_{i-1} \lambda_{i-2} \cdots \lambda_0}{\mu_i \mu_{i-1} \cdots \mu_1} < \infty$$

Otherwise, the MC is transient or null-recurrent; no limiting distribution.
CTMC and Limiting probabilities: Balance equations

From the balance argument: “Outflow state \( j = \) Inflow state \( j \)”, we obtain the following system of equations for the limiting probabilities, called balance equations:

\[
v_j P_j = \sum_{k \neq j} q_{kj} P_k
\]

In many cases we can reduce the original system of balance equations by selecting appropriately a set \( A \) and summing over all states \( i \in A \)

\[
\sum_{j \in A} P_j \sum_{i \notin A} q_{ji} = \sum_{i \notin A} P_i \sum_{j \in A} q_{ij}
\]
Embedded Markov chain

Let \( \{X(t), t \geq 0\} \) be some irreducible and positive recurrent CTMC with parameters \( v_i \) and \( P_{ij} = q_{ij}/v_i \). Let \( Y_n \) be the state of \( X(t) \) after \( n \) transitions. Then the stochastic process \( \{Y_n, n = 0, 1, \ldots\} \) is DTMC with transition probabilities \( P_{ij} \) (\( P_{ij} = q_{ij}/v_i \) if \( i \neq j \), and \( P_{jj} = 0 \)). This process is called Embedded chain.

The equilibrium probabilities \( \pi_i \) of this embedded Markov chain satisfy

\[
\pi_i = \sum_{j \in S} \pi_j P_{ji}.
\]

Then the equilibrium probabilities of the Markov process can be computed by multiplying the equilibrium probabilities of the embedded chain by the mean times spent in the various states. This leads to,

\[
P_i = C \frac{\pi_i}{v_i}
\]

where the constant \( C \) is determined by the normalization condition.
Uniformization

Consider a continuous-time Markov chain in which the mean time spent in a state is the same for all states.

That is, suppose that \( v_i = \nu \), for all states \( i \). In this case since the amount of time spent in each state during a visit is exponentially distributed with rate \( \nu \), it follows that if we let

\[
N(t) \text{ denote the number of state transitions by time } t,
\]

then \( \{N(t), t \geq 0\} \) will be a Poisson process with rate \( \nu \).

To compute the transition probabilities \( P_{ij}(t) \), we can condition on \( N(t) \):

\[
P_{ij}(t) = P[X(t) = j | X(0) = i] = \sum_{n=0}^{\infty} P[X(t) = j | X(0) = i, N(t) = n]P[N(t) = n | X(0) = i]
\]

\[
= \sum_{n=0}^{\infty} P[X(t) = j | X(0) = i, N(t) = n] \left( \frac{(\nu t)^n}{n!} \right) = \sum_{n=0}^{\infty} P_{ij}^n e^{-\nu t} \left( \frac{(\nu t)^n}{n!} \right)
\]
Uniformization

Consider a continuous-time Markov chain in which the mean time spent in a state is the same for all states.

Given that $v_i$ are bounded, set $\nu \geq \max\{v_i\}$.

Now when in state $i$, the process actually leaves at rate $v_i$, but this is equivalent to supposing that transitions occur at rate $\nu$, but only the fraction $v_i/\nu$ of transitions are real ones and the remaining fraction $1 - v_i/\nu$ are fictitious transitions which leave the process in state $i$. 
Uniformization

Consider a continuous-time Markov chain in which the mean time spent in a state is the same for all states.

Given that $v_i$ are bounded, set $v \geq \max\{v_i\}$.

Any Markov chain with $v_i$ bounded, can be thought of as being a process that spends an exponential amount of time with rate $v$ in state $i$ and then makes a transition to $j$ with probability

$$P_{ij}^* = \begin{cases} 1 - \frac{v_i}{v}, & j = i \\ \frac{v_i}{v} P_{ij}, & j \neq i \end{cases}$$

Hence, the transition probabilities can be computed by

$$P_{ij}(t) = \sum_{n=0}^{\infty} P_{ij}^* e^{-vt} \frac{(vt)^n}{n!}$$

This technique of uniformizing the rate in which a transition occurs from each state by introducing transitions from a state to itself is known as uniformization.
Summary CTMC

1. Memoryless property
2. Poisson process
3. Birth-Death Process
   - Determine $M(t) = E[X(t)]$
   - First step analysis
4. The Transition Probability Function $P_{ij}(t)$
   - Instantaneous Transition Rates
5. Limiting probabilities
   - Balance equations
6. Embedded Markov chain
7. Uniformization
Exercises

*Introduction to Probability Models*
Sheldon M. Ross

Chapter 6

Sections 6.1, 6.2, 6.3, 6.4, 6.5 and 6.7

Exercises: 1, 3, 5, 6(a)-(b), 8, 10 (but you do not have to verify that the transition probabilities satisfy the forward and backward equations), 12, 13, 15, 17, 18, 22, 23+
Exercises

1. Consider a Markov process with states 0, 1 and 2 and with the following transition rate matrix \( Q \):

\[
Q = \begin{bmatrix}
-\lambda & \lambda & 0 \\
\mu & -/(\lambda + \mu) & \lambda \\
\mu & 0 & -\mu
\end{bmatrix}
\]

where \( \lambda, \mu > 0 \).

a. Derive the parameters \( \nu_i \) and \( P_{ij} \) for this Markov process.
b. Determine the expected time to go from state 1 to state 0.

2. Consider the following queueing model: customers arrive at a service station according to a Poisson process with rate \( \lambda \). There are \( c \) servers; the service times are exponential with rate \( \mu \). If an arriving customer finds \( c \) servers busy, then he leaves the system immediately.

a. Model this system as a birth and death process.
b. Suppose now that there are infinitely many servers (\( c = \infty \)). Again model this system as a birth and death process.