An elementary proof of the hitting time theorem

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December 21, 2006

In this note, we give an elementary proof of the random walk hitting time theorem, which states that, for a left-continuous random walk on $\mathbb{Z}$ starting at a nonnegative integer k, the conditional probability that the walk hits the origin for the first time at time n, given that it does hit zero at time n, is equal to k/n.

We start by introducing some notation. Let $\mathbb{P}_k$ denote the law of a random walk starting in $k \geq 0$, let $\{Y_i\}_{i=1}^{\infty}$ be the i.i.d. steps of the random walk, let $S_n = k + Y_1 + \cdots + Y_n$ be the position of the random walk starting in k after n steps, and let

$$T_0 = \inf\{n : S_n = 0\} \tag{1}$$

denote the walk's first hitting time of the origin. Clearly, $T_0 = 0$ $\mathbb{P}_0-$a.s., that is, $T_0 = 0$ a.s. when the walker starts in the origin. Then, the hitting time theorem is the following result:

Theorem 1 (Hitting time theorem). *For a random walk starting in $k \geq 1$ with i.i.d. steps $\{Y_i\}_{i=1}^{\infty}$ satisfying that $Y_i \geq -1$ almost surely, the distribution of T_0 under $\mathbb{P}_k$ is given by*

$$\mathbb{P}_k(T_0 = n) = \frac{k}{n}\mathbb{P}_k(S_n = 0). \tag{2}$$

Theorem 1 gives the remarkable conclusion that, conditionally on the event $\{S_n = 0\}$, and regardless of the precise distribution of the steps of the walk $\{Y_i\}_{i=1}^{\infty}$ as long as $Y_i \geq -1$ a.s., the probability of the walk to be at 0 *for the first time* is equal to $\frac{k}{n}$. Theorem 1 has particular importance in the context of branching processes, where, for $k = 1$, the law of T_0 is the same as the total progeny of a branching process with offspring distribution equal to the law of $Y_i + 1$ (see [4] or [9, Problem 12, page 234] for this connection, and [1, Section 10.4] for a modern application in random graph theory).

The first proofs of Theorem 1 and the related result for $k = 1$ can be found in [8]. The extension of $k \geq 2$ is in [6], or in [4] using a result in [3]. Most of these proofs make unnecessary use of generating functions, in particular, the Lagrange inversion formula, which the simple proof below does not employ. See also [5, Page 165-167] for a more recent version of the generating function proof. In [10], various proofs of the Hitting time theorem are given, including a combinatorial proof making use of a relation in [2]. A proof for random walks making only steps of size ± 1 using the reflection principle can for example be found in [5, Page 79].

The hitting time theorem is closely related to the ballot theorem, which has a long history dating back to Bertrand in 1887 (see [7] for an excellent overview of the history and literature). The version of the ballot theorem in [7] states that, for a random walk $\{S_n\}_{n=0}^{\infty}$ starting at 0, with exchangeable, non-negative steps, the probability that $S_m < m$ for all $m = 1, \ldots, n$, conditionally on $S_n = k$, equals k/n. This proof borrows upon queueing theory methodology, and is related to, yet slightly different from, our proof below.

Proof. We prove (2) for all $k \geq 0$ by induction in n. When $n = 1$, both sides are equal to 0 when $k > 1$ and $k = 0$, and are equal to $\mathbb{P}(Y_1 = -1)$ when $k = 1$. This initializes the induction.

To advance the induction, we take $n \geq 2$, and note that both sides are equal to 0 when $k = 0$. Thus, we may assume that $k \geq 1$. We condition on the first step to obtain

$$\mathbb{P}_k(T_0 = n) = \sum_{s=-1}^{\infty} \mathbb{P}_k(T_0 = n \big| Y_1 = s)\mathbb{P}(Y_1 = s). \tag{3}$$

By the random walk Markov property,

$$\mathbb{P}_k(T_0 = n \big| Y_1 = s) = \mathbb{P}_{k+s}(T_0 = n-1) = \frac{k+s}{n-1}\mathbb{P}_{k+s}(S_{n-1} = 0), \tag{4}$$

where in the last equality we have used the induction hypothesis, which is allowed since $k \geq 1$ and $s \geq -1$, so that $k + s \geq 0$. This leads to

$$\mathbb{P}_k(T_0 = n) = \sum_{s=-1}^{\infty} \frac{k+s}{n-1}\mathbb{P}_{k+s}(S_{n-1} = 0)\mathbb{P}(Y_1 = s). \tag{5}$$

We undo the law of total probability, noting that

$$\sum_{s=-1}^{\infty} (k+s)\mathbb{P}_{k+s}(S_{n-1} = 0 \big| Y_1 = s)\mathbb{P}(Y_1 = s) = \mathbb{P}_k(S_n = 0)\Big(k + \mathbb{E}_k[Y_1|S_n = 0]\Big), \tag{6}$$

where $\mathbb{E}_k[Y_1|S_n = 0]$ is the conditional expectation of Y_1 given that $S_n = 0$ occurs. We next note that the conditional expectation of $\mathbb{E}_k[Y_i|S_n = 0]$ is independent of i, so that

$$\mathbb{E}_k[Y_1|S_n = 0] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_k[Y_i|S_n = 0] = \frac{1}{n}\mathbb{E}_k\Big[\sum_{i=1}^{n} Y_i \big| S_n = 0\Big] = -\frac{k}{n}, \tag{7}$$

since $\sum_{i=1}^{n} Y_i = S_n - k = -k$ when $S_n = 0$. Therefore, we arrive at

$$\mathbb{P}_k(T_0 = n) = \frac{1}{n-1}\Big[k - \frac{k}{n}\Big]\mathbb{P}_k(S_n = 0) = \frac{k}{n}\mathbb{P}_k(S_n = 0). \tag{8}$$

This advances the induction, and completes the proof of Theorem 1. □

We close this paper by deriving a slightly stronger result than Theorem 1. We first introduce some notation. We write $\vec{m} = (m_{-1}, m_0, m_1, \ldots)$ where m_i are non-negative integers for all $i \geq -1$. Further, for $\vec{m}$ and $n \geq 1$, let $A_{\vec{m}}(n)$ denote the event that $S_n = 0$, and the random walk $\{S_i\}_{i=0}^{\infty}$ has made m_{-1} steps of size -1, m_0 steps of size 0, m_1 steps of size 1, etc. In other words, we let

$$A_{\vec{m}}(n) = \{S_n = 0\} \cap \{\#\{i \in \{1, \ldots, n\} : Y_i = s\} = m_s \forall s \geq -1\}. \tag{9}$$

Naturally, $\sum_{i=-1}^{\infty} m_i = n$, and, since $S_n = 0$, we must have that $\sum_{i=-1}^{\infty} im_i = -k$. Then we obtain the following extension of Theorem 1:

Theorem 2. *Under the assumptions in Theorem 1, for all $k \geq 1$, $\vec{m}$ and $n \geq 1$,*

$$\mathbb{P}_k(\{T_0 = n\} \cap A_{\vec{m}}(n)) = \frac{k}{n}\mathbb{P}_k(A_{\vec{m}}(n)). \tag{10}$$

The proof of Theorem 2 is a minor modification of that of Theorem 1, again using that, conditionally on $A_{\vec{m}}(n)$, the steps $(Y_1, \ldots, Y_n)$ are exchangeable. We omit further details.

ACKNOWLEDGEMENT. The work of RvdH was supported in part by Netherlands Organisation for Scientific Research (NWO).

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