Mean-field behavior of percolation above the upper critical dimension

Remco van der Hofstad

TU/e technische universiteit eindhoven

EURANDOM

Mini-course, IHP, October 21-29, 2009

Percolation

Plan of the lectures:

Lecture 1:
Critical percolation on the tree and in high dimensions: results;

Lecture 2:
Critical percolation in high dimensions: the lace expansion;

Lecture 3:
Incipient Infinite Cluster and random walks on it;

Lecture 4:
Critical percolation on high-dimensional tori: finite size scaling;

Lecture 5:
Critical inhomogeneous percolation on the complete graph.

Lecture 1:

Critical percolation on the tree and in high dimensions: results

Percolation on a tree

Let T_r be tree where root has degree $r-1$ and every other vertex has degree r.

Make each edge occupied with probability p, vacant otherwise.
Let $\mathcal{C}(0)$ be cluster of root, i.e.,

$$\mathcal{C}(0) = \{x \in T_r \colon 0 \longleftrightarrow x\},$$

where $0 \longleftrightarrow x$ means that there is a path of occupied bonds connecting 0 and x.

Let

$$\theta(p) = \mathbb{P}_p(|\mathcal{C}(0)| = \infty), \qquad \chi(p) = \mathbb{E}_p[|\mathcal{C}(0)|].$$

Phase transition percolation on a tree

By connection to branching process, we see that

$$\theta(p) = \begin{cases} 0 & p \le 1/(r-1), \\ > 0 & p > 1/(r-1). \end{cases}$$

Also,

$$\chi(p) = \begin{cases} \frac{1}{1-(r-1)p} & p < 1/(r-1), \\ +\infty & p \ge 1/(r-1). \end{cases}$$

Thus, percolation on a tree has phase transition, and $p_c(T_r) = 1/(r-1)$, where

$$p_c(T_r) = \inf\{p\colon \theta(p) > 0\} = \sup\{p\colon \chi(p) < \infty\}.$$

Critical exponents percolation tree

Again using branching process methodology, we see that

$$\theta(p) \sim (p - p_c), \qquad \chi(p) \sim (p_c - p)^{-1}.$$

Moreover,

$$\mathbb{P}_{p_c}(|\mathcal{C}(0)| \geq n) \sim n^{-1/2}, \qquad \mathbb{P}_{p_c}\big(\text{diameter}(|\mathcal{C}(0)|) \geq n\big) \sim n^{-1}.$$

Critical exponents:

$$\beta = 1, \qquad \gamma = 1, \qquad \delta = 2, \qquad \rho_{\text{int}} = 1.$$

High-dimensional percolation

Physics prediction:
Percolation in high-dimensions $(d > d_c = 6)$ behaves as it does on regular infinite tree:

- no infinite critical cluster;
- Critical exponents are same as ones on tree.

Informal reason:
When dimension is high, space is so vast, that faraway pieces of the percolation cluster no longer interact.
Thus, geometry "trivializes", and for most questions answer is same as for percolation on an infinite regular tree.

Goal lectures:
Show how part of these claims can be made rigorous.

Percolation on $\mathbb{Z}^d$

Bonds join x to y for $x, y \in \mathbb{Z}^d$. Make bonds (x, y) independently

occupied	with probability p,
vacant	with probability $1 - p$,

where $p \in [0, 1]$ is percolation parameter.

Key examples:
- nearest-neighbor percolation;
- spread-out percolation, where range of bonds grows proportionally with parameter L, and L is often taken to be large: Bonds between x and y when $0 < \|x\|_\infty \leq L$.

Phase transition

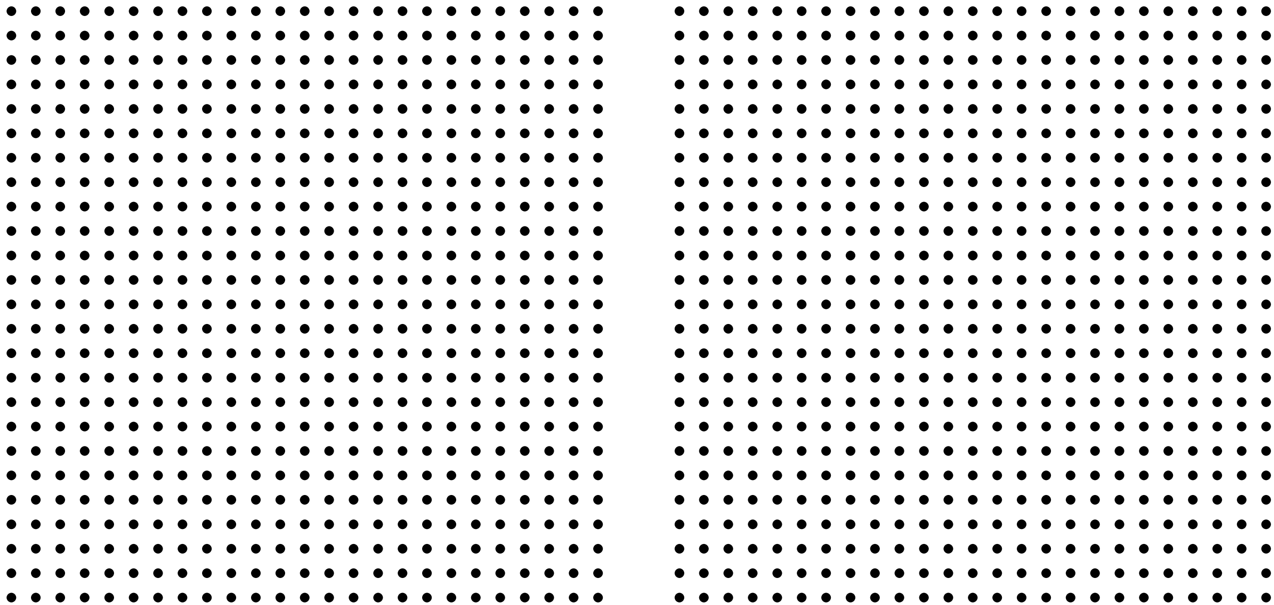
Percolation has a phase transition, i.e, there is a critical probability $p_c = p_c(d, L) \in (0, \infty)$, such that

- For $p < p_c$, a.s. no infinite cluster exists.
- For $p > p_c$, a.s. a (unique) infinite cluster.
- For $p = p_c$, behavior not understood and dimension dependent.

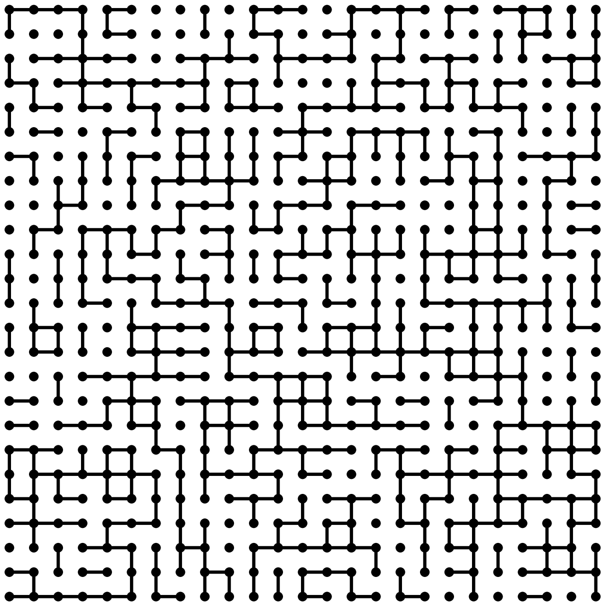
No percolation at criticality for $d = 2$,
and for nn $d \geq 19$ and spread-out model with $d > 6$ (Hara-Slade 90).
Proving continuity percolation function one of main challenges probability and statistical physics.

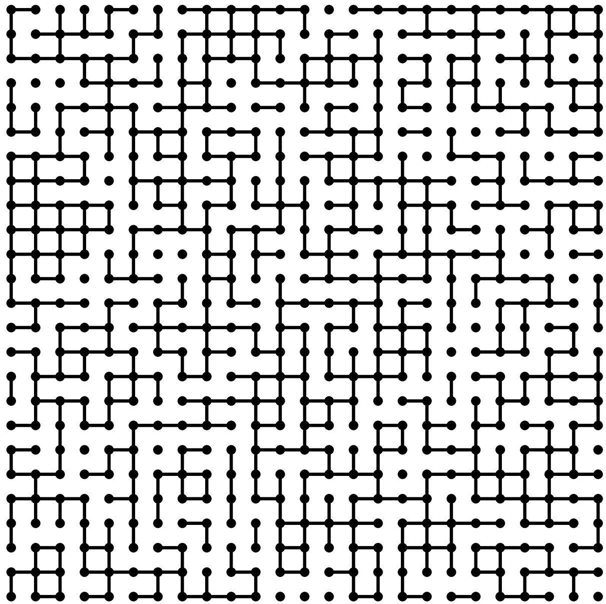
At criticality, large clusters are abundantly present.

Percolation

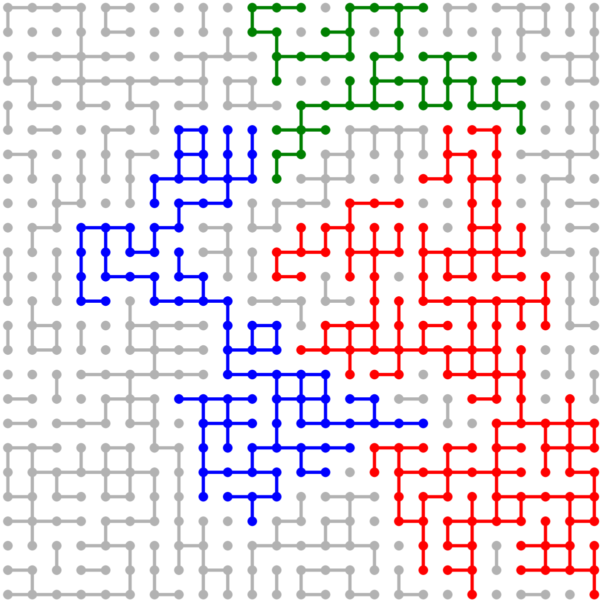


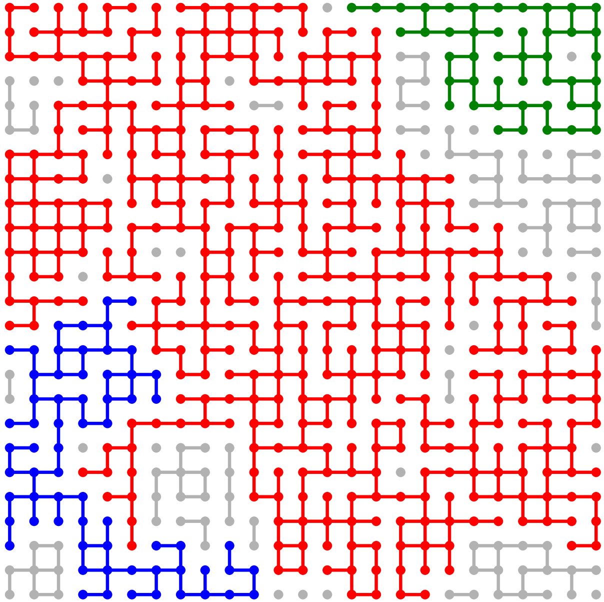
Percolation



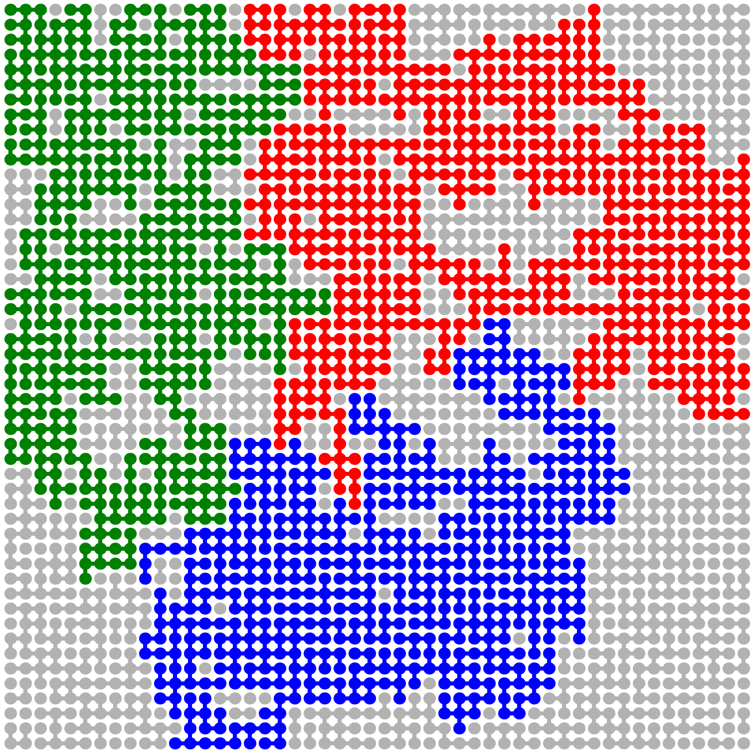


Percolation

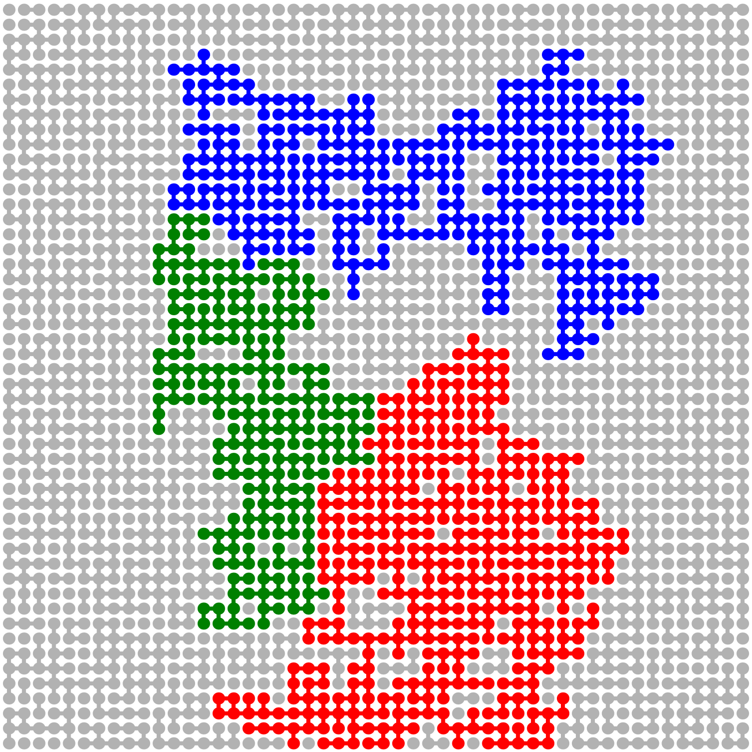




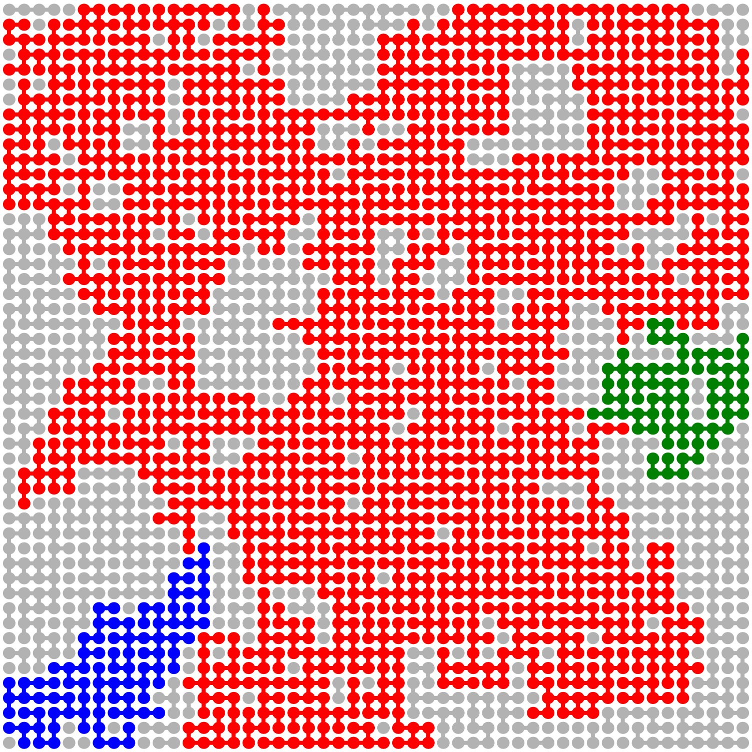
Critical percolation



Critical percolation



Critical percolation



Percolation in high d: critical exponents

Theorem 1. (AN84, BA91, HS90) For spread-out percolation with L sufficiently large and $d > 6$, or nearest-neighbour percolation for d sufficiently large,

$$\theta(p) \asymp (p - p_c), \qquad \chi(p) \asymp (p_c - p)^{-1} \qquad \mathbb{P}_{p_c}(|\mathcal{C}(0)| \geq n) \asymp n^{-1/2},$$

i.e., critical exponents β, γ, δ exist and take on tree values

$$\beta = 1, \qquad \gamma = 1, \qquad \delta = 2.$$

Many more results known in high dimension, shall discuss a few of those later on

Contrast: two-dimensions

Theorem 2. (Schramm00, Smirnov01, SW01, LSW02)
For site percolation on two-dimensional triangular lattice, critical exponents β, γ, δ exist in logarithmic sense, and take on values

$$\beta = \frac{5}{36}, \qquad \gamma = \frac{43}{18}, \qquad \delta = \frac{91}{5}.$$

Proof:
conformal invariance and Schramm/Stochastic Loewner Evolution (SLE).

Shall not discuss this exciting topic further...

Percolation in high d: triangle condition

History of proof:

- Aizenman-Newman (1984) proved that $\gamma = 1, \delta = 2$;
- Barsky-Aizenman (1991) that $\beta = 1$,

both subject to geometric condition called

triangle condition.

Let

$$\tau_{p_c}(x, y) = \mathbb{P}_{p_c}(x \longleftrightarrow y).$$

Triangle condition:

$$\nabla(p_c) = \sum_{x,y} \tau_{p_c}(0, x)\tau_{p_c}(x, y)\tau_{p_c}(0, y) < \infty.$$

Triangle condition was proved by Hara-Slade (1990), using

lace expansion.

Percolation in high d: upper-critical dimension

Triangle condition can be expected to hold only when $d > 6$.

Reason: In high d, clusters look like trees.
Add geometry by embedding trees in $\mathbb{Z}^d$:

Branching Random Walk (BRW).

$$\tau_{p_c}(x,y) \approx \text{expected \# of particles at } y \text{ from tree rooted at } x$$
$$\approx |y-x|^{-(d-2)} = |y-x|^{-(d-2-\eta)},$$

with $\eta = 0$. Alternatively, when $x \longleftrightarrow y$, then

occupied path connecting x and y is like random walk path,

so that $\tau_{p_c}(x,y) \approx G(x,y) \approx |x-y|^{-(d-2)}$: random walk Green 's function.

Percolation in high d: upper-critical dimension

Triangle condition:

$$\nabla(p_c) = \sum_{x,y} \tau_{p_c}(0,x)\tau_{p_c}(x,y)\tau_{p_c}(0,y)$$
$$\approx \sum_{x,y} |x|^{-(d-2)}|y-x|^{-(d-2)}|y|^{-(d-2)} < \infty$$

if and only if $d > 6$.

Denote one-arm exponent ρ_{ext} by

$$\mathbb{P}_{p_c}(0 \longleftrightarrow B_{\mathrm{Eucl}}(R)) \asymp R^{-1/\rho_{\mathrm{ext}}}.$$

Then, connection to BRW suggests that $\rho_{\mathrm{ext}} = 1/2$.

Then, $\eta = 0$ and $\rho_{\mathrm{ext}} = 1/2$ imply that $d \geq 6$:

upper-critical dimension of percolation is $d_c = 6$.

Mean-field model of percolation on $\mathbb{Z}^d$ is

Percolation on tree or branching random walk.

Russo's Formula

Event E is called increasing when it remains to hold when we make more edges occupied.

A bond b is called pivotal for an increasing event E when
- E occurs when b is made occupied;
- E does not occur when b is made vacant.

The set of pivotal bonds for event E is a random subset of all bonds.

Russo's Formula: For every increasing event E depending on finitely many bonds

$$\frac{d}{dp}\mathbb{P}_p(E) = \sum_b \mathbb{P}_p(b \text{ pivotal for } E).$$

BK-inequality

Event E is called increasing when it remains to hold when we make more edges occupied.

The event $E \circ F$ that two increasing events E and F occur disjointly consist of those bond configurations $\omega \in \{0,1\}^{\mathbb{B}}$ for which there exists a (possibly random) subset K of bonds such that

- $\omega_K \in E$;
- $\omega_{K^c} \in F$;

where, for a subset B of bonds and a configuration ω, the configuration ω_B is defined by $\omega_B(b) = \omega(b)$ for $b \in B$, while $\omega_B(b) = 0$ for $b \notin B$.

van den Berg-Kesten (BK) inequality: For increasing events E, F

$$\mathbb{P}_p(E \circ F) \leq \mathbb{P}_p(E)\mathbb{P}_p(F).$$

Open Problem 1

Open Problem 1: Identify the super-critical nature of percolation above 6 dimensions. For example, prove that

$$\chi^{\mathrm{f}}(p) = \mathbb{E}_p[|\mathcal{C}(0)|\mathbb{1}_{\{|\mathcal{C}(0)|<\infty\}}] \asymp (p-p_c)^{-\gamma'},$$

with $\gamma' = \gamma = 1$.

Literature 1

[1] Aizenman and Newman. Tree graph inequalities and critical behavior in percolation models. *J. Stat. Phys.*, **36**:107–143, (1984).

[2] Barsky and Aizenman. Percolation critical exponents under the triangle condition. *Ann. Probab.*, **19**:1520–1536, (1991).

[3] Hara and Slade. Mean-field critical behaviour for percolation in high dimensions. *Commun. Math. Phys.*, **128**:333–391, (1990).

Lecture 2:

Critical percolation in high dimensions: the lace expansion

Infrared bound

Theorem 3. (Percolation infrared bound, HS90+BCHSS05b+HvdHS08.) For $d \gg 6$ in the nearest-neighbor case, and $d > 6$ and L sufficiently large in spread-out case,

$$\hat{\tau}_p(k) = \frac{1 + O(\beta)}{\chi(p)^{-1} + p\Omega[1 - \hat{D}(k)]}$$

uniformly for $p \in (p_c/2, p_c]$, where $\beta = 1/d$ or $\beta = L^{-d}$, respectively, and Ω is degree of underlying graph, i.e.,

$$\Omega = 2d, \qquad \text{or} \qquad \Omega = (2L+1)^d - 1,$$

and $\hat{D}(k)$ is Fourier transform of bond-set RW transition probability:

$$\hat{D}(k) = \frac{1}{\Omega} \sum_{x:(0,x)\in\mathbb{B}} \mathrm{e}^{ik\cdot x}.$$

Triangle condition using infrared bound

Corollary 1. (Percolation triangle condition.)
Under hypotheses Theorem 3, percolation triangle condition holds. More precisely,

$$\nabla(p_c) = \sum_{x,y} \tau_{p_c}(x)\tau_{p_c}(y-x)\tau_{p_c}(y) \leq 1 + O(\beta).$$

AN84, BA91: Implies existence several critical exponents:

$$\gamma = 1, \qquad \beta = 1, \qquad \delta = 2.$$

Lace expansion analysis

Any proof using the lace expansion uses three main steps:

1. Derivation of expansion;
2. Bounds on lace-expansion coefficients $\hat{\Pi}_p(k)$;
3. Analysis of lace-expansion relation.

Will go over these one by one.

Proof infrared bound: lace expansion

For percolation, Hara and Slade (1990) derived a lace expansion reading that

$$\hat{\tau}_p(k) = \frac{1+\hat{\Pi}_p(k)}{1-p\Omega\hat{D}(k)[1+\hat{\Pi}_p(k)]},$$

for certain lace-expansion coefficients $\hat{\Pi}_p(k)$.

Equation is perturbation of equation:

$$\hat{G}_p(k) = \frac{1}{1-p\Omega\hat{D}(k)},$$

which is random walk Green's function.

Goal: Show that $\hat{\Pi}_p(k)$ is a small perturbation.

Further results in high-dimensions

Theorem 3. (Hara+vdH+Slade 03, Hara 08) Fix $d \geq 7$ and sufficiently spread-out model, or $d \geq 19$ and nearest-neighbor model. Then, there is $A = A(d, L)$ s.t. when $L \geq L_0(d)$ as $|x| \to \infty$

$$\tau_{p_c}(x) = \frac{A}{\sigma^2 |x|^{d-2}} \left[1 + o(1)\right].$$

Result not expected to be true for $d < d_c$!

Aizenman (1997): Largest intersection of percolation cluster and cube of side length r is of size r^4 and r^{d-6} in number. Uses bounds in Theorem.

Further results in high-dimensions

Theorem 4. (Kozma-Nachmias 09) Fix $d > 6$ for sufficiently spread-out model, or $d \geq 19$ for nearest-neighbor model. Then, as $R \to \infty$,

$$\mathbb{P}_{p_c}(0 \longleftrightarrow \partial B(R)) \asymp R^{-2}.$$

Implies that $\rho_{\mathrm{Eucl}} = 1/2$.

Open Problems 2

Open Problem 2(a): Prove that nearest-neighbor model satisfied triangle condition for $d > 6$.

Open Problem 2(b): Identify scaling limit of large critical clusters. (Oriented percolation vdHS03: canonical measure of super-Brownian motion.)

Literature 2

[1] Borgs, Chayes, van der Hofstad, Slade, and Spencer. Random subgraphs of finite graphs. II. The lace expansion and the triangle condition. *Ann. Probab.*, **33**(5):1886–1944, (2005).

[2] Hara and Slade. Mean-field critical behaviour for percolation in high dimensions. *Commun. Math. Phys.*, **128**:333–391, (1990).

[3] Heydenreich, van der Hofstad, and Sakai. Mean-field behavior for long- and finite range Ising model, percolation and self-avoiding walk. *Journ. of Stat. Phys.*, **132**: 1001–1049, (2008).

Lecture 3:

Incipient Infinite Cluster and random walks on it

Large critical clusters

Central question:

What is structure of large critical clusters?

Here we can think of

- Dimension of large clusters;
- Local structure of large clusters.

Go by name of incipient infinite cluster (IIC), which is

infinite cluster that is on verge of appearing at criticality.

Question is how to define IIC.

2d-Incipient Infinite Cluster

Kesten (1986) has constructed IIC for percolation on $\mathbb{Z}^2$.
IIC describes local structure of large critical clusters.

Constructions Kesten:
(a) Condition 0 to be in the infinite component for $p > p_c$, and then take limit as $p \downarrow 0$.
(b) Condition on $0 \longleftrightarrow \partial B_n$ at $p = p_c$, and take limit as $n \to \infty$.

For events E, define

$$\text{(IIC)} \qquad \mathbb{P}_\infty(E) = \lim_{p \downarrow p_c} \mathbb{P}_p(E | 0 \longleftrightarrow \infty).$$

Similar for Construction (b). RSW theory plays an important role.

Other constructions of 2d-IIC

Járai (03, 04) has given several more constructions for IIC:

(c) Take uniform point of critical spanning cluster in $B_n = [-n, n]^d$, and let $n \to \infty$;
(d) Take uniform point in largest critical cluster in B_n, and let $n \to \infty$;
(e) Condition v to be in invasion percolation cluster, and let $|v| \to \infty$.

Results in Kesten (86) show further:
(i) from each point in IIC, there is a.s. no double connection to infinity).
Using also conformal invariance (LSW, Smirnov):
(ii) Dimension IIC is $91/48$;
(iii)

$$\mathbb{P}_\infty(0 \longleftrightarrow y \text{ disjointly from } y \longleftrightarrow \infty) = |y|^{-\lambda+o(1)}.$$

Existence IIC for high-dimensional percolation

For cylinder events E, define

$$\text{(IIC)} \qquad \mathbb{P}_\infty(E) = \lim_{|x|\to\infty} \mathbb{P}_{p_c}(E|0 \longleftrightarrow x).$$

Theorem 1. (vdH-Járai (03)) For spread-out percolation with L sufficiently large and $d > 6$, or nearest-neighbour percolation for d sufficiently large, the above limit exists for every cylinder event E. Moreover, $\mathbb{P}_\infty$ extends to a probability measure on full sigma-algebra of events, and $\mathbb{P}_\infty(|\mathcal{C}(0)| = \infty) = 1$.

Results make essential use of the asymptotics critical two-point function

$$\tau(x) = \mathbb{P}_{p_c}(0 \longleftrightarrow x) \sim |x|^{-(d-2)},$$

proved in HHS(03) and Hara (08).

Properties IIC

Theorem 2. (vdH-Járai (03)) Under conditions of Theorem 3, IIC measure $\mathbb{P}_\infty$ satisfies that:

(i) The IIC has a single end $\mathbb{P}_\infty$-a.s.

(ii) There are positive constants $c_1 = c_1(d, L)$ and $c_2 = c_2(d, L)$ such that for $|y| \geq 1$

$$\frac{c_1}{|y|^{d-4}} \leq \mathbb{P}_\infty(0 \longleftrightarrow y) \leq \frac{c_2}{|y|^{d-4}}.$$

(iii) There are positive constants $c_3 = c_3(d, L)$ and $c_4 = c_4(d, L)$ such that for $|y| \geq 1$

$$\frac{c_3}{|y|^{d-2}} \leq \mathbb{P}_\infty(0 \longleftrightarrow y \text{ and } y \longleftrightarrow \infty \text{ disjointly}) \leq \frac{c_4}{|y|^{d-2}}.$$

(Indicates that IIC backbone is 2-, and IIC 4-dimensional).

Alternative definition IIC

For cylinder events E, define

$$\mathbb{Q}_\infty(E) = \lim_{p \uparrow p_c} \frac{1}{\chi(p)} \sum_{x \in \mathbb{Z}^d} \mathbb{P}_p(E \cap \{0 \longleftrightarrow x\}).$$

Theorem 3. (vdH-Járai (03)) Under conditions of Theorem 3, the above limit exists, and $\mathbb{Q}_\infty = \mathbb{P}_\infty$.

IIC is robust and natural object!

Proof

Proof relies on various forms of lace expansion for two-point function $\tau(x) = \mathbb{P}_{p_c}(0 \longleftrightarrow x)$.

Lace expansion can be used to show that when E only depends on bonds within cube of width m,

$$\mathbb{P}_{p_c}(E \cap \{0 \longleftrightarrow x\}) = \psi(E; x) + \sum_y \pi(E; y)\tau_{p_c}(x - y).$$

Lace expansion coefficients $\psi(E; y), \pi(E; y)$ are small when y is large, so we can divide by $\tau_{p_c}(x)$ and take limit using $\tau_{p_c}(x) = c|x|^{-(d-2)}(1 + o(1))$ to get

$$\mathbb{P}_\infty(E) = \lim_{|x|\to\infty} \mathbb{P}_{p_c}(E|0 \longleftrightarrow x) = \sum_y \pi(E; y).$$

Random walks on percolation clusters

Let $(S_n)_{n\geq 0}$ be simple random walk on the super-critical infinite percolation cluster.

Formal construction:
Let $\mathcal{C}_\infty$ be cluster of origin conditioned to be infinite, and let $(S_n)_{n\geq 0}$ be simple random walk on $\mathcal{C}_\infty$.

Theorem 4. (Berger-Biskup 07, Mathieu-Piatnitski 08, Sidoravicius-Sznitman 04)
Fix $d \geq 2$ and $p > p_c$. Then, simple random walk on infinite percolation cluster satisfies an almost sure quenched invariance principle, i.e., for some variance $\sigma^2 = \sigma(p)^2 > 0$ almost surely

$$(S_{\lceil nt \rceil}/\sqrt{n})_{t\geq 0} \xrightarrow{d} (\sigma B_t)_{t\geq 0},$$

where $(B_t)_{t\geq 0}$ is standard Brownian motion.

Random walks on percolation clusters

Theorem 4 can be understood by noting that $\mathcal{C}_\infty$ is full-dimensional.

Proof: Martingale central limit theorem and notion of

corrector

which needs to be added to make random walk martingale.
Prove that corrector is small.

Random walks on high-dimensional IIC

Let $(S_n)_{n\geq 0}$ be simple random walk on IIC, and let

$$p_n(x,y) = \mathbb{P}^x(S_n = y)$$

be probability that random walk started at x is at y at time n.

Spectral dimension:

$$d_s(\mathrm{IIC}) = -2 \lim_{n\to\infty} \frac{\log p_{2n}(x,x)}{\log n}.$$

Volume-growth dimension:

$$d_f(\mathrm{IIC}) = \lim_{r\to\infty} \frac{\log |B_{\mathrm{IIC}}(x,r)|}{\log r},$$

where $B_{\mathrm{IIC}}(x,r)$ consists of vertices in IIC at graph distance at most r.

Random walks on IIC on tree

IIC on tree has been constructed by Kesten 86.

Consists of
- unique infinite line of descent (immortal particle);
- critical clusters attached at every vertex on infinite line.

Kesten proved

$$d_s = 4/3, \qquad d_g = 2.$$

"Random walk trap model."

IIC is not full dimensional, i.e., expect $d_s(\mathrm{IIC}), d_f(\mathrm{IIC}) < d$:

anomalous diffusion.

Random walks on high-dimensional IIC

Theorem 5. (Kozma-Nachmias 08) Fix $d > 6$ and sufficiently spread-out model, or $d \geq 19$ and nearest-neighbor model. Then,

$$d_s(\mathrm{IIC}) = 4/3, \qquad d_f(\mathrm{IIC}) = 2,$$

and, with τ_r hitting-time of ball $B_{\mathrm{IIC}}(0,r)$ and W_n range of random walk, i.e., number of distinct vertices visited at time n, and in probability,

$$\lim_{r\to\infty} \frac{\log E^0[\tau_r]}{\log r} = 3, \qquad \lim_{n\to\infty} \frac{\log |W_n|}{\log n} = 2/3,$$

where E^0 denotes conditional law of RW on IIC.

Proof: two main ingredients

• General theorem implying Theorem 5 by Barlow, Jarái, Kumagai, Slade 08, assuming appropriate bounds on effective resistances and volume growth. BJKS used results to study
(a) random walk on oriented percolation IIC above 6 dimensions;
(b) random walk on IIC for percolation on tree;
(c) random walk on invasion percolation cluster on tree.
Flexible result, Taylor made for applications in various settings.

• Verification of conditions in KN 08:
Proof of $d_f(\mathrm{IIC}) = 2$ and effective resistance bounds.
These bounds are reduced to two main bounds for critical percolation:

$$\mathbb{E}_{p_c}[|B_{p_c}(0,r)|] \asymp r, \qquad \mathbb{P}_{p_c}(\partial B_{p_c}(0,r) \neq \varnothing) \asymp 1/r,$$

alike on tree. Second bound: $\rho_{\mathrm{Int}} = 1$.

A consistent picture

Reasonable to assume that there exists $\alpha > 1$ such that

$$E^0[d_{\mathrm{IIC}}(0, S_n)] \asymp n^{1/\alpha}.$$

Then,

$$E^0[\tau_r] \asymp r^\alpha, \quad E^0[|W_n|] \asymp |B_{\mathrm{IIC}}(0, n^{1/\alpha})|, \quad p_{2n}(0,0) \asymp 1/|B_{\mathrm{IIC}}(0, n^{1/\alpha})|.$$

Assumption implies

$$\mathbb{E}_{p_c}[|B_{p_c}(0,r)| \mid \partial B_{p_c}(0,r) \neq \varnothing] \approx \mathbb{E}_{\mathrm{IIC}}[|B_{\mathrm{IIC}}(0,r)|] \asymp r^2.$$

Thus,

$$d_s(\mathrm{IIC}) = 4/\alpha, \qquad \lim_{r\to\infty} \frac{\log E^0[\tau_r]}{\log r} = \alpha, \qquad \lim_{n\to\infty} \frac{\log |W_n|}{\log n} = 2/\alpha.$$

Remains to determine α.

Effective resistance

Proof make essential use of relation

random walks and electrical networks.

Define quadratic form

$$\mathcal{E}(f,g) = \frac{1}{2}\sum_{b\in\mathbb{B}}(f(\bar{b}) - f(\underline{b}))(g(\bar{b}) - g(\underline{b})).$$

Then, for $A, B \subseteq \mathbb{Z}^d$, effective resistance between A and B is

$$R_{\text{eff}}(A,B)^{-1} = \inf\{\mathcal{E}(f,f)\colon\ f|_A = 1, f|_B = 0\}.$$

$R_{\text{eff}}(A,B)$ satisfies series and parallel laws in electricity. Implies, on any graph G,

$$R_{\text{eff}}(A,B) \le d_G(A,B).$$

Equality when there is unique path between A and B.

Assumptions BJKS 08

Let $J(\lambda)$ be set of $r \geq 1$ such that

$$\lambda^{-1}r^2 \leq |B_{\mathrm{IIC}}(0,r)| \leq \lambda r^2, \qquad \lambda^{-1}r \leq R_{\mathrm{eff}}(0, B_{\mathrm{IIC}}(0,r)^c) \leq \lambda r.$$

Assumption BJKS 08: There exists $r^* \geq 1$ and $c_1, q_0 > 0$ s.t.

$$\mathbb{P}_G(r \in J(\lambda)) \geq 1 - c_1 \lambda^{-q_0}.$$

Implies that volume balls grow as radius squared, and effective resistance grows as radius, like it does on tree.

Effective resistance

Estimate

$$E^0[\tau_r] \asymp R_{\text{eff}}(0, B_{\text{IIC}}(0,r)^c)|B_{\text{IIC}}(0,r)| \asymp r \cdot r^2 = r^3,$$

so that $\alpha = 3$.

- Upper bound always valid;
- Lower bound when path between 0 and $B_{\text{IIC}}(0,r)^c$ is essentially unique.

Implies results Theorem 5, subject to assumptions

- volume growth critical balls; and
- intrinsic one-arm exponent.

Open Problem 3

Open Problem 3(a): Investigate $\sigma(p)^2$ as $p \downarrow p_c$ in high dimensions.
Conjecture: exists $\zeta > 0$ such that

$$\sigma(p)^2 \asymp (p - p_c)^\zeta,$$

where $\zeta = 2$ in high-dimensions.

Open Problem 3(b): Prove more refined assumptions in BJKS 08, yielding more precise convergence results.

Open Problem 3(c): Identify scaling limit of random walk on high-dimensional IIC.

Literature 3

[1] Barlow, Járai, Kumagai, and Slade. Random walk on the incipient infinite cluster for oriented percolation in high dimensions. *Comm. Math. Phys.*, **278**(2):385–431, (2008).

[2] van der Hofstad and Járai. The incipient infinite cluster for high-dimensional unoriented percolation. *J. Statist. Phys.*, **114**(3-4):625–663, (2004).

[3] Kozma and Nachmias. The Alexander-Orbach conjecture holds in high dimensions. Preprint (2008).

Lecture 4:

Critical percolation on high-dimensional tori: finite-size scaling

Percolation on tori

Random subgraph of finite tori

$$\mathbb{T}_{r,d} = (\mathbb{V}, \mathbb{B}) \qquad \text{where} \qquad \mathbb{V} = \{0, \ldots, r-1\}^d.$$

Two extreme examples:

- High-dimensional tori with finite-range bonds above 6 dimensions;
- Hypercube $\mathbb{Q}_d = \{0,1\}^d$.

For d-cube, *bond* (u,v) is pair $u,v \in \mathbb{Q}_d$ satisfying $\#\{i : u_i \neq v_i\} = 1$.
Bonds independently

occupied	with probab. p,
vacant	with probab. $1-p$.

Goal: Study structure largest critical clusters as number of vertices grows large.

Phase transition on $\mathbb{Z}^d$

Critical value $p_c(\mathbb{Z}^d)$ for percolation on $\mathbb{Z}^d$ is

$$p_c(\mathbb{Z}^d) = \inf\{p : \mathbb{P}_p(0 \longleftrightarrow \infty) > 0\}$$

Let $\mathcal{C}(x)$ be the cluster of x, and $\chi(p) = \mathbb{E}_p|\mathcal{C}(0)|$.
Aizenman and Barsky (87) or Menshikov (86):

$$p_c(\mathbb{Z}^d) = \sup\{p : \chi(p) < \infty\}$$

Central question:

What is structure of large critical clusters?

Mean-field model percolation on finite graph is

percolation on complete graph.

Erdős-Rényi random graph

Erdős-Rényi random graph is random subgraph of complete graph on V vertices where each of $\binom{V}{2}$ edges is occupied with probab. p.

Phase transition: (Erdős and Rényi (60))
For $p = (1+\varepsilon)/V$, largest component is
(a) $\Theta_{\mathbb{P}}(\log V)$ for $\varepsilon < 0$;
(b) $\Theta_{\mathbb{P}}(V)$ for $\varepsilon > 0$;

Scaling window: (Bollobás (84) and Łuczak (90))
For $p = (1/V)(1+\lambda/V^{1/3})$, largest component is $\Theta_{\mathbb{P}}(V^{2/3})$, with expected cluster size $\Theta(V^{1/3})$.

Extensions:
- Aldous (97): Weak convergence of ordered clusters;
- Nachmias and Peres (05): Diameter of large clusters is $\Theta_{\mathbb{P}}(V^{1/3})$;
- Addario-Berry, Broutin, Goldschmidt (09): Weak convergence of clusters as graphs.

High-dimensional tori

Torus in $\mathbb{Z}^d$ and $d > 6$:
Aizenman (1997), in combination with Hara(03), HHS(01), show that largest intersection of critical cluster with cube of width r is r^4.

Conjecture Aizenman:
Changes to $r^{2d/3} = V^{2/3}$ for periodic boundary conditions (= percolation on torus).

Conclusion: Scaling largest critical component is sensitive to boundary conditions.

Volume largest critical cluster

Define largest connected component for percolation on torus by

$$|\mathcal{C}_{\max}| = \max_{v} |\mathcal{C}_{\mathbb{T}}(v)|,$$

where $|\mathcal{C}_{\mathbb{T}}(v)|$ is size connected component of v, and $V = r^d$ is volume torus of width r.

Theorem 1. (Heydenreich+vdH 07+09) For $d \gg 6$ and nearest-neighbor bonds, or for $d > 6$ and sufficiently spread-out bonds, there exists $b > 0$ such that for all $\omega \geq 1$

$$\mathbb{P}_{p_c(\mathbb{Z}^d)}\Big(\frac{1}{\omega}V^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega V^{2/3}\Big) \geq 1 - \frac{b}{\omega} \qquad \text{as } r \to \infty.$$

Moreover, $|\mathcal{C}_{\max}|V^{-2/3}$ is not concentrated.

Non-concentration of largest cluster is hallmark of critical behavior.

Related results

Theorem 2. (Heydenreich+vdH (09)) Under conditions of Theorem 1, for every $m = 1, 2, \ldots$ there exist constants $b_1, \ldots, b_m > 0$, s.t. for all $\omega \geq 1$, and all $i = 1, \ldots, m$,

$$\mathbb{P}_{p_c(\mathbb{Z}^d)}\Big(\omega^{-1} V^{2/3} \leq |\mathcal{C}_{(i)}| \leq \omega V^{2/3}\Big) \geq 1 - \frac{b_i}{\omega} \qquad \text{as } r \to \infty.$$

Theorem 3. (Heydenreich+vdH (09)) Under conditions of Theorem 1, for every $m = 1, 2, \ldots$ there exist constants $c_1, \ldots, c_m > 0$, s.t. for all $\omega \geq 1$, and all $i = 1, \ldots, m$,

$$\mathbb{P}_{p_c(\mathbb{Z}^d)}\Big(\omega^{-1} V^{1/3} \leq \mathrm{diam}(\mathcal{C}_{(i)}) \leq \omega V^{1/3}\Big) \geq 1 - \frac{c_i}{\omega^{1/3}} \qquad \text{as } r \to \infty.$$

Lazy random walks mixing time

Transition probabilities lazy simple random walk on graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

$$p(x,y) = \begin{cases} 1/2 & \text{if } x = y; \\ \frac{1}{2\,\deg(x)} & \text{if } (x,y) \in \mathcal{E}; \\ 0 & \text{otherwise.} \end{cases}$$

Stationary distribution on finite graph is $\pi(x) = \deg(x)/(2|\mathcal{E}|)$.
Mixing time of lazy simple random walk is

$$T_{\mathrm{mix}}(\mathcal{G}) = \min\left\{n\colon \|\mathsf{p}^n(x,\cdot) - \pi(\cdot)\|_{\mathrm{TV}} \leq 1/4 \text{ for all } x \in \mathcal{V}\right\},$$

Theorem 4. (Heydenreich+vdH (09)) Under conditions of Theorem 1,

$$\mathbb{P}_{p_c(\mathbb{Z}^d)}\Big(\omega^{-1}V \leq T_{\mathrm{mix}}(\mathcal{C}_{(i)}) \leq \omega V\Big) \geq 1 - \frac{d_i}{\omega^{1/34}} \qquad \text{as } r \to \infty.$$

A consistent picture (revisited)

$$|\mathcal{C}_{(i)}| \asymp (\mathrm{diam}(\mathcal{C}_{(i)}))^2, \qquad T_{\mathrm{mix}}(\mathcal{C}_{(i)}) \asymp (\mathrm{diam}(\mathcal{C}_{(i)}))^3.$$

Similar to scaling on IIC. Should think of IIC describing

local picture large critical clusters.

Random graph asymptotics

Results confirm that large critical percolation clusters on high-dimensional tori behave similarly as those on the Erdős-Rényi random graph:

random graph asymptotics.

Investigated similarities: cluster sizes, diameter, mixing times.

Results apply more generally, e.g., to other high-dimensional tori as studied in (BCHSS05a,b).
Gives yet another motivation for choice of critical value in (BCHSS05a,b).

Key ingredients in proof

(1) Coupling relating $|\mathcal{C}(v)|$, size of cluster of v, on torus and on $\mathbb{Z}^d$.

(2) Lace expansion results for high-dimensional tori in BCHSS05a,b, which establish

• random graph scaling around internal critical value on torus defined by

$$\chi_{\mathbb{T}}(p_c(\mathbb{T}_{r,d})) = \lambda V^{1/3},$$

where $\chi_{\mathbb{T}}(p)$ is the expected cluster size on torus.

• an excellent control over various quantities (such as expected cluster size) for subcritical p.

(3) Proof completed using mean-field results for high-dimensional percolation on $\mathbb{Z}^d$ in HS90, Hara 95, HHS03, Hara 06, proved using lace expansion.

Need results on expected cluster size $\gamma = 1$, correlation length $\nu = 1/2$, and critical two-point function $\eta = 0$.

Coupling of clusters on torus and $\mathbb{Z}^d$

Proposition 1. (Heydenreich+vdH (07) There exists a coupling $\mathbb{P}_{\mathbb{Z},\mathbb{T}}$ such that $\mathbb{P}_{\mathbb{Z},\mathbb{T}}$-almost surely,

$$|\mathcal{C}_{\mathbb{T}}(0)| \leq |\mathcal{C}_{\mathbb{Z}}(0)|.$$

Consequently, $\chi_{\mathbb{T}}(p) \leq \chi_{\mathbb{Z}}(p)$.

Also some form of lower bound is given.

Exploration cluster

Law of connected component is described by an exploration process.

- Bonds can be Occupied, Vacant and Unexplored.

Status vertices	meaning
Active	Incident to ≥ 1 vacant and ≥ 1 unexplored bond
Inactive	Incident to occupied and vacant bonds only
Neutral	Incident to unexplored bonds only

START Order vertices in arbitrary way.
At time $t = 0$, only vertex 0 is active, rest neutral.
All bonds are unexplored.

REPEAT Explore bonds incident to smallest active vertex.
Update status vertices according to above rules.

STOP when no more active vertices.

$\mathcal{C}(0)$, cluster or connected component of 0, is set of inactive vertices at end of exploration.

Exploration cluster on torus

Construct $\mathcal{C}_{\mathbb{T}}(0)$ and $\mathcal{C}_{\mathbb{Z}}(0)$ simultaneously from single percolation configuration on $\mathbb{Z}^d$ by unwrapping torus.

Two bonds b_1, b_2 are equivalent when $b_1 = b_2 + zr$ for some $z \in \mathbb{Z}^d$. Distinction percolation torus and $\mathbb{Z}^d$ is that equivalent bonds have same status on torus, while they have independent status on $\mathbb{Z}^d$.

Extra bond status: RED when bond unexplored, but one of its equivalent bonds has been explored. Adapt statuses vertices as follows:

Status vertices	meaning
Active	Incident to ≥ 1 vacant and ≥ 1 unexplored bond
Inactive	Incident to occupied, vacant and RED bonds only
Neutral	Incident to unexplored and RED bonds only

The coupling conclusion

Exploration cluster as before, BUT after exploring bond, we make all its equivalent bonds RED. Apart from this, rules exploration process remain unchanged.

$\mathcal{C}_{\mathbb{T}}(0)$ is set of vertices in torus that are equivalent to a vertex that is part of occupied edge $\mathbb{Z}^d$ at end of this exploration.

To construct $\mathcal{C}_{\mathbb{Z}}(0)$, turn status RED bonds to Unexplored and complete exploration process as before (but now without RED bonds). $\mathcal{C}_{\mathbb{Z}}(0)$ is set of inactive vertices at end.

Conclusion: $|\mathcal{C}_{\mathbb{T}}(0)| \leq |\mathcal{C}_{\mathbb{Z}}(0)|$.

Proof upper bound Theorem 1

Let $Z_{\geq k}$ denote number of vertices in clusters of size at least $k,$ i.e.,

$$Z_{\geq k} = \sum_{v \in \mathbb{T}_{r,d}} \mathbb{1}_{\{|\mathcal{C}(v)| \geq k\}}.$$

Intimate relation between $Z_{\geq k}$ and $|\mathcal{C}_{\max}|$:

$$\{|\mathcal{C}_{\max}| \geq k\} = \{Z_{\geq k} \geq k\}.$$

Proofs BCHSS05a,b, HvdH07,09 center around analysis of $k \mapsto Z_{\geq k}$.

Lace expansion results in high dimensions

Hara-Slade (90) and Aizenman-Newman (84) show, under conditions Theorem 1, that $\delta = 2$, i.e.,

$$\mathbb{P}_{p_c(\mathbb{Z}^d)}(|\mathcal{C}_{\mathbb{Z}}(v)| \geq k) \leq bk^{-1/2}.$$

Then,

$$\mathbb{P}_p(|\mathcal{C}_{\max}| \geq k) = \mathbb{P}_p(Z_{\geq k} \geq k) \leq \frac{\mathbb{E}_p[Z_{\geq k}]}{k} = \frac{V}{k}\mathbb{P}_p(|\mathcal{C}_{\mathbb{T}}(v)| \geq k).$$

By Proposition 1 and $\delta = 2$,

$$\mathbb{P}_{p_c(\mathbb{Z}^d)}(|\mathcal{C}_{\max}| \geq \omega V^{2/3}) \leq \frac{V}{\omega V^{2/3}}\mathbb{P}_{p_c(\mathbb{Z}^d)}(|\mathcal{C}_{\mathbb{Z}}(v)| \geq \omega V^{2/3}) \leq \frac{b}{\omega^{3/2}},$$

which proves (slightly better) upper bound Theorem 1.

A variance estimate

$$\mathbb{P}_{p_c(\mathbb{Z}^d)}(|\mathcal{C}_{\max}| < k) = \mathbb{P}_p(Z_{\geq k} = 0) \leq \frac{\mathrm{Var}_p(Z_{\geq k})}{\mathbb{E}_p[Z_{\geq k}]^2}.$$

Proposition 2. (BCHSS05a) $\mathrm{Var}_p(Z_{\geq k}) \leq V\chi_{\mathbb{T}}(p)$.
Proof relies on BK-inequality.

BCHSS05a,b, HvdH (09):

$$\chi_{\mathbb{T}}(p_c(\mathbb{Z}^d)) \leq CV^{1/3}, \qquad \mathbb{P}_{p_c(\mathbb{Z}^d)}(|\mathcal{C}_{\mathbb{T}}(v)| \geq k) \geq ck^{-1/2},$$

the latter bound valid when $k \leq V^{2/3}$. Conclude that

$$\mathbb{E}_{p_c(\mathbb{Z}^d)}[Z_{\geq \omega^{-1}V^{2/3}}] = V\mathbb{P}_{p_c(\mathbb{Z}^d)}(|\mathcal{C}_{\mathbb{T}}(v)| \geq \omega^{-1}V^{2/3}) \geq c^2V^{4/3}\omega,$$

so that

$$\mathbb{P}_{p_c(\mathbb{Z}^d)}(|\mathcal{C}_{\max}| < \omega^{-1}V^{2/3}) \leq \frac{V\chi_{\mathbb{T}}(p_c(\mathbb{Z}^d))}{\mathbb{E}_{p_c(\mathbb{Z}^d)}[Z_{\geq \omega^{-1}V^{2/3}}]^2} \leq \frac{CV^{4/3}}{c^2V^{4/3}\omega} \leq \frac{b}{\omega}.$$

More general high-dimensional tori

Prove our results in more general setting of high-dimensional tori $\mathbb{T}_{r,n}$ satisfying

high-dimensionality condition.

(Condition excludes cases where scaling is believed to be different.)

Condition in terms of random walk on the graph, and is similar to percolation triangle condition, but simpler to verify.

Fourier theory plays an important role in analysis, so that we specialize to torus.

In this generality, all results are valid, except for super-critical lower bounds and asymptotic expansion for $p_c(n)$, which we have only proved for n-cube.

BCHSS approach to criticality

Recall $\mathbb{E}_p|\mathcal{C}(0)| = \Theta(V^{1/3})$ for random graph and $p = \frac{1}{V}$, where V is volume graph.

Suggests us to define critical threshold as

$$p_c(\mathbb{T}_{r,d}) = \min\{p : \chi_{\mathbb{T}}(p) \geq \lambda V^{1/3}\},$$

for some $\lambda > 0$, where $\chi_{\mathbb{T}}(p) = \mathbb{E}_p|\mathcal{C}_{\mathbb{T}}(0)|$ is expected cluster size on torus.

Does this bring us closer to critical value?
Is this indeed natural?

HvdH (07+09):

$$|p_c(\mathbb{T}_{r,d}) - p_c(\mathbb{Z}^d)| = O(V^{-1/3}).$$

Example: n-cube

Write $\mathcal{C}_{\max}$ for largest connected component, and take $p = (1+\varepsilon)/n$.

Ajtai, Komlós and Szemerédi (82):

$$|\mathcal{C}_{\max}| = O(\varepsilon 2^n) \quad \text{for } \varepsilon > 0, \qquad |\mathcal{C}_{\max}| = o(2^n) \quad \textbf{a.a.s.} \quad \text{for } \varepsilon < 0.$$

Bollobás, Kohayakawa and Łuczak (92):

$$|\mathcal{C}_{\max}| = (2\log 2)\frac{n}{\varepsilon^2} \quad \textbf{a.a.s.} \quad \text{for } \varepsilon \le -(\log n)^2/(\log\log n)\sqrt{n},$$

while

$$|\mathcal{C}_{\max}| = 2\varepsilon 2^n\big(1 + o(1)\big) \quad \textbf{a.a.s.} \quad \text{for } \varepsilon \ge 60(\log n)^3/n.$$

Transition is extremely sharp for n large, critical value close to $1/n$.

Question BKL: Is critical value equal to $1/(n-1)$?

A hierarchy of phase transitions on n-cube

Theorem 5. (BCHSS, HS05,06) There exist rational numbers a_i with $a_1 = a_2 = 1, a_3 = 7/2$, s.t., for every $s \geq 1$, if

$$p = \sum_{i=1}^{s} a_i n^{-i} + \delta n^{-s},$$

with $\delta < 0$, then, as $n \to \infty$,

$$|\mathcal{C}_{\max}| \leq (2 \log 2) n^{2s-1} |\delta|^{-2} [1 + o(1)] \qquad \text{a.s.},$$

while if $\delta > 0$, then, as $n \to \infty$, there exist $0 < c_1 < c_2 < \infty$, such that

$$c_1 \delta^{-1} n^{-(s-1)} 2^n \leq |\mathcal{C}_{\max}| \leq c_2 \delta^{-1} n^{-(s-1)} 2^n \qquad \text{a.a.s.}$$

Extension of AKS to all powers of $\frac{1}{n}$. Exponential jump for all s!

Critical window

Theorem 6 (Scaling window).
If $p = p_c(1+\varepsilon)$, with $|\varepsilon| \leq \Lambda V^{-1/3}$, then there exist $b_1 = b_1(\Lambda) > 0$ such that

$$\mathbb{P}_p\Big(\omega^{-1} V^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega V^{2/3}\Big) \geq 1 - \frac{b_1}{\omega}.$$

Tightness of scaled largest cluster size when $p = p_c(1 + O(V^{-1/3}))$.

H+vdH (07+09): $|\mathcal{C}_{\max}| V^{-2/3}$ is non-degenerate.
Non-degeneracy is hallmark of critical behavior, so this gives (yet another) justification for choice $p_c(\mathbb{T}_{r,n})$.

Alternative approach by Nachmias 07

$p_t(u,v)$ is probability that non-backtracking random walk starting at u is at time t at v. Graph is transitive with degree Ω.
Nachmias' mean-field assumption:

$$\limsup_{V\to\infty} V^{1/3} \sum_{t=1}^{V^{1/3}} t p_t(v,v) < \infty.$$

For $p = (1+\varepsilon)/(\Omega-1)$ and $\varepsilon = \lambda V^{-1/3}$, and ω sufficiently large,

$$\mathbb{P}_p(\omega^{-1}V^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega V^{2/3}) \geq 1 - o(1).$$

Interestingly, for $p = (1+\varepsilon)/(\Omega-1)$ with $\varepsilon \gg V^{-1/3}$, under related supercriticality assumption,

$$|\mathcal{C}_{\max}| \geq \delta\varepsilon n(\log(n\varepsilon^3))^{-3} \qquad \textbf{whp}.$$

Nachmias and Peres (2007): subcritical regime for Ω-regular graphs.

Open Problem 4

Open Problem 4(a): Identify necessary conditions for random graph supercritical behavior on high-dimensional tori:

What is role of geometry in supercritical phase?

Open Problem 4(b): Identify scaling limit of the largest critical clusters on high-dimensional tori, and prove that it agrees with that on Erdős-Rényi random graph (Aldous 97).

Literature 4

[1] Borgs, Chayes, van der Hofstad, Slade, and Spencer. Random subgraphs of finite graphs. I. The scaling window under the triangle condition. *RSA*, **27**(2):137–184, (2005)

[2] Borgs, Chayes, van der Hofstad, Slade, and Spencer. Random subgraphs of finite graphs. II. The lace expansion and the triangle condition. *AoP*, **33**(5):1886–1944, (2005).

[3] Borgs, Chayes, van der Hofstad, Slade, and Spencer. Random subgraphs of finite graphs. III. The phase transition for the n-cube. *Combinatorica*, **26**(4):395–410, (2006).

[4] Heydenreich and van der Hofstad. Random graph asymptotics on high-dimensional tori. *CMP*, **270**(2):335–358, (2007).

Preprints 4

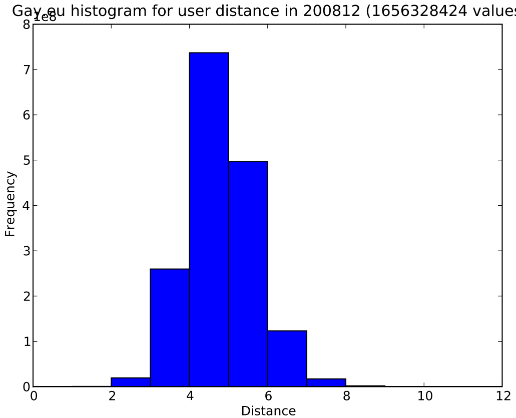
[1] Heydenreich and van der Hofstad. Random graph asymptotics on high-dimensional tori II. Volume, diameter and mixing time. To appear in *Prob. Theory Rel. Fields*.

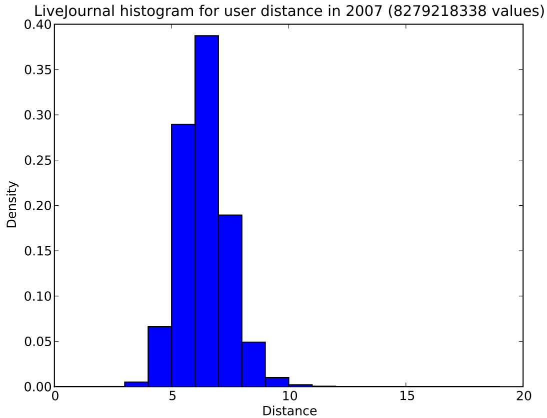
[2] Nachmias. Mean-field conditions for percolation in finite graphs. Preprint 2007. To appear in *Geometric and Functional Analysis*.

Lecture 5:

Critical inhomogeneous percolation on complete graph

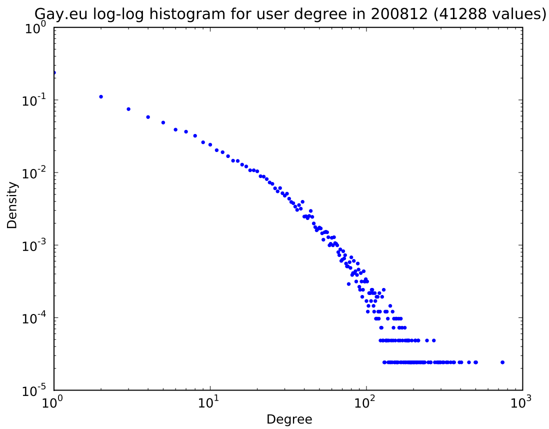
Small-world paradigm

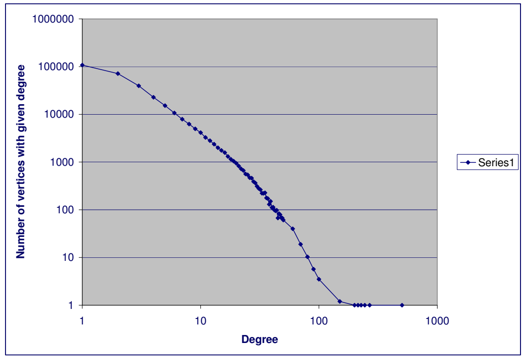




Distances in social networks `gay.eu` on December 2008 and `livejournal` in 2007.

Scale-free paradigm





Loglog plot of degree sequences in `gay.eu`
and in the collaboration graph among mathematicians
(`http://www.oakland.edu/enp`)

Network functions

Internet: e-mail

WWW: Information gathering

Friendship networks: gossiping, spread of information and disease

Power grids: reliability

Network functions

Internet: e-mail
Routing on networks, congestion, network failure

WWW: Information gathering
Crawling networks, motion on networks

Friendship networks: gossiping, spread of information and disease
Spread of diseases, motion on networks, consensus reaching

Power grids: reliability
Robustness to (random and deliberate) attacks

Network functions

Internet: e-mail
Routing on networks, congestion, network failure

WWW: Information gathering
Crawling networks, motion on networks

Friendship networks: gossiping, spread of information and disease
Spread of diseases, motion on networks, consensus reaching

Power grids: reliability
Robustness to (random and deliberate) attacks

Processes on networks!

Network functions

Internet: e-mail
Routing on networks, congestion, network failure

WWW: Information gathering
Crawling networks, motion on networks

Friendship networks: gossiping, spread of information and disease
Spread of diseases, motion on networks, consensus reaching

Power grids: reliability
Robustness to (random and deliberate) attacks

Processes on networks!

Modeling real networks

• Inhomogeneous Random Graphs:
Static random graph, independent edges with inhomogeneous edge occupation probabilities, possibly yielding scale-free graphs.

• Configuration Model:
Static random graph with prescribed degree sequence.

• Preferential Attachment Model:
Dynamic random graph, attachment proportional to degree plus constant.

Erdős-Rényi random graph

Vertex set $[n] := \{1, 2, \ldots, n\}$.

Erdős-Rényi random graph is random subgraph of complete graph on $[n]$ where each of $\binom{n}{2}$ edges is occupied with probab. p.

Simplest imaginable model of a random graph.

- Attracted tremendous attention since introduction 1959, mainly in combinatorics community.

Probabilistic method (Erdős et al).

Egalitarian: Every vertex has equal probability of being connected to. Misses hub-like structure of real networks.

Rank-1 inhomogeneous random graphs

Attach edge with probability p_{ij} between vertices i and j, where

$$p_{ij} = \frac{w_i w_j}{l_n},$$

and

$$l_n = \sum_{i=1}^{n} w_i,$$

and different edges are independent.

Here w_i is expected degree vertex i.
When $w_i = \lambda$, we retrieve Erdős-Rényi random graph with $p = \lambda/n$.

Assume throughout talk $w_i^2/l_n \leq 1$ for all $i \in [n]$.

Choice of weights

Take $\boldsymbol{w} = (w_1, \ldots, w_n)$ as

$$w_i = [1 - F]^{-1}(i/n),$$

where $F(x)$ is distribution function.
Interpretation: proportion of vertices i with $w_i \leq x$ is close to $F(x)$.

Simple example:

$$F(x) = \begin{cases} 0 & \text{for } x < a, \\ 1 - (a/x)^{\tau-1} & \text{for } x \geq a, \end{cases}$$

in which case

$$[1 - F]^{-1}(u) = a(1/u)^{-1/(\tau-1)}, \qquad \text{so that} \qquad w_j = a(n/j)^{1/(\tau-1)}.$$

Degree structure graph

Denote proportion of vertices with degree k by

$$P_k^{(n)} = \frac{1}{n} \sum_{i \in [n]} \mathbb{1}_{\{D_i = k\}},$$

where D_i denotes degree of vertex i.

Model is sparse, i.e., there exists probability distribution $\{p_k\}_{k=0}^{\infty}$ s.t.

$$P_k^{(n)} \xrightarrow{\mathbb{P}} p_k.$$

For $w_i = [1 - F]^{-1}(i/n)$, with W having distribution function F,

$$p_k = \mathbb{E}\Big[\mathrm{e}^{-W} \frac{W^k}{k!}\Big].$$

In particular, $p_k \sim ck^{-\tau}$ precisely when $\mathbb{P}(W \geq k) \sim ck^{-\tau}$.

Critical value

Bollobás-Janson-Riordan (2007): Let $W \sim F$, then

- largest component $\sim \rho n$ with $\rho \in (0,1)$ for $\nu = \mathbb{E}[W^2]/\mathbb{E}[W] > 1$;
- largest component $o(n)$ for $\nu = \mathbb{E}[W^2]/\mathbb{E}[W] \leq 1$.

Identifies critical value IRG as

$$\nu = 1,$$

where ν is asymptotic expected number of forward neighbors.

In simple example $F(x) = 1 - (a/x)^{\tau-1}$ for $x \geq a$

$$\mathbb{E}[W] = \frac{a(\tau-1)}{\tau-2}, \qquad \mathbb{E}[W^2] = \frac{a^2(\tau-1)}{\tau-3},$$

so that critical case arises when $a = (\tau-3)/(\tau-2)$.

Robustness of networks

Above has important implications for robustness network under various attacks:

Random attack: Remove vertices uniformly at random with probability p. Then, again obtain rank-1 IRG where now probability of edge ij between kept vertices equals

$$\frac{w_i w_j}{l_n},$$

and otherwise equals 0.
Giant component exists whenever

$$(1-p)\nu > 1.$$

Robustness of networks

Above has important implications for robustness network under various attacks:

Random attack: Remove vertices uniformly at random with probability p.
Then, again obtain rank-1 IRG where now probability of edge ij between kept vertices equals

$$\frac{w_i w_j}{l_n},$$

and otherwise equals 0.
Giant component exists whenever

$$(1-p)\nu > 1.$$

In particular, when $\nu = \infty$, always giant component:

Robust to random failure.

Robustness of networks

Above has important implications for robustness network under various attacks:

Deliberate attack: Remove proportion p of vertices with highest weight. Now, again obtain rank-1 IRG where probability of edge ij for $i, j > np$ equals

$$\frac{w_i w_j}{l_n},$$

while otherwise probability equals 0.
Thus, giant component exists whenever

$$\frac{\sum_{i>np} w_i^2}{l_n} > 1.$$

Robustness of networks

Above has important implications for robustness network under various attacks:

Deliberate attack: Remove proportion p of vertices with highest weight. Now, again obtain rank-1 IRG where probability of edge ij for $i, j > np$ equals

$$\frac{w_i w_j}{l_n},$$

while otherwise probability equals 0.
Thus, giant component exists whenever

$$\frac{\sum_{i>np} w_i^2}{l_n} > 1.$$

In particular, even when $\nu = \infty$, for p large, no giant component:

Fragile to deliberate attacks.

Critical behavior Erdős-Rényi random graph

Double jump (Erdős and Rényi (60))
For $p = (1+\varepsilon)/n$, largest component is
(a) $\Theta_{\mathbb{P}}(\log n)$ for $\varepsilon < 0$;
(b) $\Theta_{\mathbb{P}}(n)$ for $\varepsilon > 0$;
(c) $\Theta_{\mathbb{P}}(n^{2/3})$ for $\varepsilon = 0$.

Scaling window: (Bollobás (84) and Łuczak (90))
For $p = (1/n)(1+\lambda n^{-1/3})$, largest component is $\Theta_{\mathbb{P}}(n^{2/3})$.

Extension: Aldous (97): Weak convergence of ordered clusters.

Key question:
How much remains valid when we let go of homogeneity vertices?

Critical behavior

Let

$$1 - F(x) \sim cx^{-(\tau-1)} \qquad \text{for } x \text{ sufficiently large.}$$

Further, let $|\mathcal{C}_{\max}|$ denote largest connected component.

Theorem 1. (vdH 09) Assume that $\nu = 1$.

(a) Let $\tau > 4$. Then, there exists $b > 0$ such that for all $\omega \geq 1$

$$\mathbb{P}\Big(\frac{1}{\omega}n^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega n^{2/3}\Big) \geq 1 - \frac{b}{\omega} \qquad \text{as } n \to \infty.$$

(b) Let $\tau \in (3,4)$. Then, there exists $b > 0$ such that for all $\omega \geq 1$

$$\mathbb{P}\Big(\frac{1}{\omega}n^{(\tau-2)/(\tau-1)} \leq |\mathcal{C}_{\max}| \leq \omega n^{(\tau-2)/(\tau-1)}\Big) \geq 1 - \frac{b}{\omega} \qquad \text{as } n \to \infty.$$

Scaling limit for $\tau > 4$

Let $\mu = \mathbb{E}[W], \sigma^2 = \mathbb{E}[W^3]/\mathbb{E}[W]$. Consider

$$B_s^\lambda = \sigma B_s + s\lambda - s^2\sigma^2/(2\mu),$$

where B is standard Brownian motion. Let

$$R_s^\lambda = B_s^\lambda - \min_{0\leq u\leq s} B_s^\lambda.$$

Aldous (1997): Excursions of R^λ can be ranked in increasing order as $\gamma_1(\lambda) > \gamma_2(\lambda) > \ldots$.

Let $|\mathcal{C}_{(1)}(\lambda)| \geq |\mathcal{C}_{(2)}(\lambda)| \geq |\mathcal{C}_{(3)}(\lambda)| \ldots$ denote sizes of components with weights $\tilde{w}_i = (1 + \lambda n^{-1/3})w_i$ arranged in increasing order.

Theorem 2. (BvdHvL 09a) Assume that $\nu = 1$, and $\mathbb{E}[W^3] < \infty$. Then

$$\left(n^{-2/3}|\mathcal{C}_{(i)}(\lambda)|\right)_{i\geq 1} \xrightarrow{d} \left(\gamma_i(\lambda)\right)_{i\geq 1}.$$

Scaling limit for $\tau \in (3,4)$

Let $|\mathcal{C}_{(1)}(\lambda)| \geq |\mathcal{C}_{(2)}(\lambda)| \geq |\mathcal{C}_{(3)}(\lambda)| \ldots$ denote sizes of components with weights $\tilde{w}_i = (1 + \lambda n^{-(\tau-3)/(\tau-1)}) w_i$ arranged in increasing order.

Theorem 3. (BvdHvL 09b) Assume that $\nu = 1$, and $\tau \in (3,4)$. Then,

$$\big(n^{-(\tau-2)/(\tau-1)}|\mathcal{C}_{(i)}(\lambda)|\big)_{i\geq 1} \xrightarrow{d} \big(H_i(\lambda)\big)_{i\geq 1}.$$

Moreover, for every i, j fixed

$$\mathbb{P}(i \longleftrightarrow j) \to q_{ij}(\lambda) \in (0,1).$$

Limits $H_i(\lambda)$ correspond to ordered hitting times of 0 of a certain fascinating 'thinned' Lévy process.

Multiplicative coalescents

Multiplicative coalescent is continuous-time Markov process $\lambda \mapsto \mathbf{X}(\lambda)$, where

$$\mathbf{X}(\lambda) \in \{\mathbf{x} = (x_i)_{i \geq 1} : x_i \geq x_{i+1}\},$$

where x_i corresponds to mass of i^{th} largest particle, and where particles with masses x_i and x_j merge to particle of mass $x_i + x_j$ at rate

$$x_i x_j.$$

Process describes evolution of masses where particles coalesce at rate equal to product of their masses.

Theorem 4. (Aldous97, BvdHvL 09b) As function of $\lambda \in \mathbb{R}$, processes $\lambda \mapsto \big(\gamma_i(\lambda)\big)_{i \geq 1}$ and $\lambda \mapsto \big(H_i(\lambda)\big)_{i \geq 1}$ are

multiplicative coalescents.

Distinction between $\tau > 4$ and $\tau \in (3, 4)$ arises through

entrance boundary at $\lambda = -\infty$.

Proof: weak convergence stochastic processes

Proof relies on three main ingredients:

(1) subsequent exploration of clusters;

(2) removal of possible further neighbors due to their exploration:

depletion of points effect;

(3) in critical window, these effects play at same scale, and

cluster exploration process weakly converges;

Cluster sizes correspond to excursion lengths limiting process having an increasingly negative drift.

• $\tau > 4$: exploration process has finite variance steps, so that Brownian motion appears in limit, and $\mathbb{P}(1 \in \mathcal{C}_{\max}) \to 0$: 'power to the masses!'
• $\tau \in (3, 4)$: exploration process is dominated by vertices with high weights, and $\mathbb{P}(1 \in \mathcal{C}_{\max}) \to q_1(\lambda) \in (0, 1)$: 'power to the wealthy!'

Cluster exploration for $\tau > 4$

Take $\lambda = 0$.

- For all ordered pairs of vertices (i, j), let $U(i, j)$ be i.i.d. $U(0, 1)$.
- Choose vertex $v(1)$ with probability proportional to $\boldsymbol{w}$, so that

$$\mathbb{P}(v(1) = i) = w_i / l_n.$$

Children of $v(1)$ are those vertices j for which

$$U(v(1), j) \leq \frac{w_{v(1)} w_j}{l_n}.$$

Label children of $v(1)$ as $v(2), v(3), \ldots v(c(1) + 1)$ in increasing order of their $U(v(1), \cdot)$ values.

- Move to $v(2)$, explore all of its children, and label them as before.

Once we finish exploring one component, move onto next component by choosing starting vertex in size-biased manner amongst remaining vertices.

Size-biased reordering

Size-biased order $v^*(1), v^*(2), \ldots, v^*(n)$ is random reordering of vertex set $[n]$ where

- $v^*(1) = i$ with prob. w_i/l_n;
- given $v^*(1), \ldots, v^*(i-1)$, $v^*(i) = j \in [n] \setminus \{v^*(1)\}$ with prob. proportional to w_j.

Key ingredient proof:

$(v(i))_{i\in[n]}$ is size-biased reordering.

Number of new neighbors $c(i)$ of $v(i)$ is close to

$$c(i) = \mathrm{Poi}\Big(w_{v(i)} \sum_{j\in[n]\setminus\{v(1),\ldots,v(i)\}} w_j/l_n\Big).$$

Connected components

Recall number of new neighbors of $v(i)$ is close to

$$c(i) = \mathrm{Poi}\Big(w_{v(i)} \sum_{j\in[n]\setminus\{v(1),\ldots,v(i)\}} w_j/l_n\Big).$$

Denote cluster exploration process Z_n by $Z_n(0) = 0$ and

$$Z_n(i) = Z_n(i-1) + c(i) - 1.$$

Denote first hitting time of $-j$ by

$$\eta(j) = \min\{i : Z_n(i) = -j\}.$$

Then, all connected component sizes are given by successive excursions from past minima

$$\mathcal{C}^*(j) = \eta(j) - \eta(j-1).$$

Scaling limit of cluster exploration

Process $t \mapsto n^{-1/3} Z_n(sn^{2/3})$ is close to Brownian motion with changing drift given by

$$\mathbb{E}[n^{-1/3} Z_n(sn^{2/3})] \sim s\lambda - s^2\sigma^2/(2\mu),$$

while

$$n^{-1/3} Z_n(sn^{2/3}) - (s\lambda - s^2\sigma^2/(2\mu)) \xrightarrow{d} B_s.$$

Suggests that rescaled cluster sizes converge to successive excursions from past minima of process

$$B_s^{\lambda} = B_s + s\lambda - s^2\sigma^2/(2\mu).$$

Weak convergence of exploration process follows from

functional martingale central limit theorem.

Open Problems 5

Open Problem 5(a): Prove that that percolation on other random graphs such as configuration model obey identical scaling.

Open Problem 5(b): Identify mean-field conditions for spatial inhomogeneous random graphs:
Under what conditions does inhomogeneous percolation with spatial structure have same scaling limit as non-spatial model?

Literature 5

[1] Aldous, D. (1997) Brownian excursions, critical random graphs and the multiplicative coalescent. *AoP* **25**, 812–854.

[2] Aldous, D., and Limic, V. (1998) The entrance boundary of the multiplicative coalescent. *EJP* **3**, 1–59.

[3] Bollobás, B. and Janson, S. and Riordan, O. (2007) The phase transition in inhomogeneous random graphs, *RSA*, **31**, 3–122.

[4] Norros, I. and Reittu, H. (2006) On a conditionally Poissonian graph process, *AdAP*, **38**, 59–75.

Preprints 5

[P1] Bhamidi, S. and van der Hofstad, R. and van Leeuwaarden, J. Scaling limits for critical inhomogeneous random graphs with finite third moments.
`http://arxiv.org/abs/0907.4279`.

[P2] Bhamidi, S. and van der Hofstad, R. and van Leeuwaarden, J. Novel scaling limits for critical inhomogeneous random graphs.
`http://arxiv.org/abs/0909.1472`.

[P3] Van der Hofstad, R. Critical behavior in inhomogeneous random graphs.
`http://arxiv.org/abs/0902.0216`.

[P4] Turova, T. Diffusion approximation for the components in critical inhomogeneous random graphs of rank 1.
`http://arxiv.org/abs/0907.0897`.