Optimal Performance for DS-CDMA Systems with Hard Decision Parallel Interference Cancellation

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Abstract—**We study a multiuser detection system using code division multiple access (CDMA). We show that applying multistage hard decision parallel interference cancellation (HD-PIC) significantly improves performance compared to the matched filter system. In (multistage) HD-PIC, estimates of the interfering signals due to other users are used iteratively to improve knowledge of the desired signal. We use large deviation theory to show that the bit-error probability (BEP) is exponentially small and investigate the exponential rate of the BEP after several stages of HD-PIC. We propose to use the exponential rate of the BEP as a measure of performance, rather than taking the signal-to-noise ratio, which is not reliable in multiuser detection models.**

We show that the exponential rate of the BEP remains fixed after a finite number of stages, resulting in an optimal hard decision system. When the number of users becomes large, the exponential rate of the BEP converges to $(\log 2)/2-1/4$. We provide intuition concerning the number of stages necessary to obtain this asymptotic exponential rate. Finally, we give Chernoff bounds on the BEP's. These estimates show that the BEP's are quite small as long as $k = o(n/\log n)$ when the number of stages of HD-PIC is fixed, and even exponentially small when $k = \mathcal{O}(n)$ for the optimal HD-PIC system.

Keywords—**Code division multiple access, hard-decision parallel interference cancellation, large deviations, exponential rate, performance measures, optimal hard-decision system.**

I. Introduction

CURRENTLY, the third generation (3G) mobile communication system is being introduced. This system is based on *code division multiple access* (CDMA). The performance of CDMA is mainly limited by interference from other users: the multiple access interference. Particularly, the susceptibility to the near-far situation, which can significantly reduce capacity in the absence of good power control, is a problem. Therefore, there exists a great interest in techniques which improve the capacity of CDMA receivers (see [15], [2], [16] and the references therein). Initial research on multi-user receivers for CDMA has demonstrated the potential improvements in capacity and near-far resistance. The best known technique is a maximum likelihood sequence estimator [19], which obtains jointly optimum decisions for all users using maximum likelihood detection. Unfortunately, this technique is of high complexity. A straightforward technique is called *interference cancellation*, see [16], Chapter 4, [14], [9], [18] and the references therein. The idea is that we try to cancel the interference due to the other users. Interference cancellation, especially *hard-decision parallel interference cancellation* (HD-PIC), is the most promising and the most practical technique for uplink receivers (see [16]). This is because in WCDMA proposals all user signals are demodulated coherently, which makes implementation efficient. Apart from that, as we will see in this paper, HD-PIC significantly improves performance.

In the literature, not much attention has been given to obtain rigorous analytical results for (HD-)PIC systems. The papers that exist on this subject focus on approximating the *signal-to-noise ratio* (SNR), for example [1] and [20]. However, using the SNR as a measure of performance by substituting the SNR in the Gaussian error function implicitly relies on Gaussian assumptions. For PIC systems, this assumption is false and it leads to incorrect results as we will argue in Section II-E.

We use *large deviation theory* [3], [8] to obtain analytical results for the *bit-error probability* (BEP). More precisely, rather than calculating the SNR, we calculate the *exponential rate* of the BEP. We will first explain the essence of this rate. Suppose p_n is a sequence of probabilities and $p_n \to 0$ as $n \to \infty$. Often, this decay is exponentially in n, so that we investigate the exponential rate defined as

$$I = -\lim_{n\to\infty} \frac{1}{n} \log p_n.$$

For finite, but not too small n, we can read this as

$$p_n \sim e^{-nI}. \tag{1}$$

Thus, the probability p_n is mainly characterized by its exponential rate. We stress that the exponential rate of the BEP is *not* a probability.

It turns out that the BEP for *matched filter* (MF) and HD-PIC systems have the same structure as p_n. This makes it possible to use the exponential rate as a measure for the performance.

For the MF model, the SNR and the exponential rate are asymptotically equivalent, meaning that when the number of users is sufficiently large, both substituting the SNR into the Gaussian distribution and the use of the exponential rate imply that the BEP has the same asymptotic form. This is shown in Section II-E. In [17], a MF model is investigated using large deviation techniques. However, the emphasis lies on sampling techniques. For a one-stage soft-decision PIC model, results have been obtained in [6]. In that paper, large deviation techniques are used to prove properties of the exponential rate and the BEP. In [7] and

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[11], various results have been proven for the HD-PIC system.

In this paper, we start by repeating results of one-stage and multi-stage HD-PIC of [7] and [11]. We extend these results to values of the processing gain and the number of users that are comparable using Chernoff bounds.

The main novelty of this paper is that we obtain results for systems with infinite stages of HD-PIC, the so-called *optimal HD-PIC system*. One of the striking results is that after *finitely* many iterations, the exponential rate does not increase any further. In other words, applying more stages of interference cancellation does not improve performance anymore. Another striking result is that the obtained exponential rate is *strictly positive*, regardless of the number of users. We will give bounds on this exponential rate. Furthermore, we give intuition on the number of stages necessary to obtain this exponential rate.

II. System and background

In this section, we explain the system, and give previous results concerning HD-PIC.

A. System model

We first introduce a mathematical model for CDMA systems. We define the data signal $b_m(t)$ of the mth user as $b_m(t) = b_{m\,\lfloor t/T\rfloor}$, for $1 \le m \le k$, where $b_m = (\ldots, b_{m,-1}, b_{m0}, b_{m1}, \ldots) \in \{-1,+1\}^{\mathbb{Z}}$ and where for $x \in \mathbb{R}$, $\lfloor x\rfloor$ denotes the largest integer smaller than or equal to x. For each user m, $1 \le m \le k$, we have a coding sequence $a_m = (\ldots, a_{m,-1}, a_{m0}, a_{m1}, \ldots) \in \{-1,+1\}^{\mathbb{Z}}$ and we put $a_m(t)$, the rectangular spreading signal, generated by $(a_m)_{m=-\infty}^{\infty}$, i.e., $a_m(t) = a_{m\,\lfloor t/T_c\rfloor}$, where $T_c = T/n$, for some integer n. The variable n is often called the *processing gain*.

The transmitted coded signal of the mth user is then

$$s_m(t) = \sqrt{2P_m}\, b_m(t) a_m(t) \cos(\omega_c t), \quad 1 \le m \le k, \tag{2}$$

where P_m is the power of the mth user and ω_c the carrier frequency. The total transmitted signal is given by

$$r(t) = \sum_{j=1}^{k} s_j(t). \tag{3}$$

In practice, the signals do not need to be synchronized, i.e., it is not necessary that all users transmit using the same time grid. However, for technical reasons we do assume so. In [17], an asynchronous MF system is investigated, using large deviations techniques. One important result in that paper is that the asynchronous system has much similarities, compared to the synchronous system. In fact, the exponential rate for the synchronous and asynchronous system are the same. For the more advanced HD-PIC system, this idea carries over, so that the results investigated in this paper turn out not to differ from the results for an asynchronous system. Indeed, we expect that the asynchronous model has the *same* exponential rate of the BEP as the synchronized model. Therefore, the synchronicity assumption is without loss of generality.

We assume that there is no additive white Gaussian noise (AWGN). Thus, we are dealing with a *perfect channel*. However, we believe that many of our results remain true when there is little AWGN.

To retrieve the data bit b_{m0}, the signal $r(t)$ is multiplied by $a_m(t)\cos\omega_c t$ and then averaged over $[0,T]$. For simplicity, we pick $\omega_c T_c = \pi f_c$, where $f_c \in \mathbb{N}$ to get (cf. [6], Eqn. (3)-(4))

$$\begin{aligned}\frac{1}{T}\int_0^T r(t) a_m(t)\cos(\omega_c t)\,dt \\ = \sqrt{\frac{P_m}{2}} b_{m0} + \sum_{\substack{j=1\\ j\neq m}}^{k} \sqrt{\frac{P_j}{2}} b_{j0} \frac{1}{n}\sum_{i=0}^{n-1} a_{ji}a_{mi}.\end{aligned} \tag{4}$$

As is seen from (4), the decoded signal consists of the desired bit and interference due to the other users. In the ideal situation the vectors $(a_{m0},\ldots,a_{m,n-1})$ and $(a_{j0},\ldots,a_{j,n-1})$, $j \neq m$, would be orthogonal, so that $\sum_{i=1}^{n} a_{ji}a_{mi} = 0$. In practice, the a-sequences are generated by a random number generator. To model the pseudo-random sequence a, let A_{mi}, $m = 1,2,\ldots,k$, $i = 1,2,\ldots,n$, be an array of independent and identically distributed (i.i.d.) random variables with distribution

$$\mathbb{P}(A_{11} = +1) = \mathbb{P}(A_{11} = -1) = 1/2. \tag{5}$$

Assuming the coding sequences to be random, we model the signal of (4) as

$$\sqrt{\frac{P_m}{2}} b_{m0} + \sum_{\substack{j=1\\ j\neq m}}^{k} \sqrt{\frac{P_j}{2}} b_{j0} \frac{1}{n}\sum_{i=1}^{n} A_{ji}A_{mi},$$

where we have replaced $i = 0,\ldots,n-1$ by $i = 1,\ldots,n$, for notational convenience.

In practice, the a-sequences are not chosen as i.i.d. sequences. Rather, they are carefully chosen to have good correlation properties. Examples are Gold sequences [4] or Kasami sequences [10]. However, it is common in the literature to use random sequences, so that a detailed analysis is possible. Better performance can be achieved for well-chosen deterministic codes.

We let $\hat{b}_{m0}^{(1)}$ be the estimator for b_{m0} given by

$$\operatorname{sgnr}_m\left\{\sqrt{\frac{P_m}{2}} b_{m0} + \sum_{\substack{j=1\\ j\neq m}}^{k} \sqrt{\frac{P_j}{2}} b_{j0} \frac{1}{n}\sum_{i=1}^{n} A_{ji}A_{mi}\right\},$$

where, for $x \in \mathbb{R}$, the randomized sign-function is defined as

$$\operatorname{sgnr}_m(x) = \begin{cases} +1, & x > 0,\\ U_m, & x = 0,\\ -1, & x < 0.\end{cases} \tag{6}$$

with $\mathbb{P}(U_m = -1) = \mathbb{P}(U_m = +1) = 1/2$. The random variables U_m are independent of all other random variables in the system. Note that the mth user *always* makes the same decision when its signal produces a zero. There are other ways to define the sign-function, such as the choice where every time when a zero is detected, a new independent random U is chosen. Another option is to let the sign of 0 be equal to 0. We choose the definition in (6) for technical reasons only. We will comment more on other choices of sign functions in Section III below. We use the notation $^{(1)}$ in $\hat{b}^{(1)}_{m0}$ to indicate that this is a tentative decision only.

We now describe the hard-decision procedure. The powers P_m are assumed to be known. We estimate the data signal $s_j(t)$ for $t \in [0, T]$ by (recall (2))

$$\hat{s}^{(1)}_j(t) \quad = \quad \sqrt{2P_j}\, \hat{b}^{(1)}_{j0}\, a_j(t) \cos(\omega_c t).$$

Then we estimate the total interference for the mth user in $r(t)$ due to the other users by (recall (3))

$$\hat{r}^{(1)}_m(t) = \sum_{\substack{j=1\\ j\neq m}}^{k} \hat{s}^{(1)}_j(t)$$

We use the above to find a better estimate of the data bit b_{m0}, denoted by $\hat{b}^{(2)}_{m0}$, which is the sgnr_m of

$$\begin{aligned} &\frac{1}{T}\int_0^T (r(t) - \hat{r}^{(1)}_m(t)) a_m(t) \cos(\omega_c t) dt \qquad (7)\\ &= \sqrt{\frac{P_m}{2}} b_{m0} + \sum_{\substack{j=1\\ j\neq m}}^{k} \left(\frac{1}{n} \sum_{i=1}^{n} A_{ji} A_{mi} \right) \sqrt{\frac{P_j}{2}} \left(b_{j0} - \hat{b}^{(1)}_{j0} \right). \end{aligned}$$

We are now interested in $\mathbb{P}(\hat{b}^{(2)}_{m0} \neq b_{m0})$, which is the probability of a bit error after one stage of interference cancellation. We will see that this probability is indeed smaller than $\mathbb{P}(\hat{b}^{(1)}_{m0} \neq b_{m0})$, the probability of a bit error without cancellation. This motivates a repetition of the previous procedure. We obtain, similar to (7), the estimates $\hat{b}^{(s)}_{m0}$, which are the sgnr of

$$\sqrt{\frac{P_m}{2}} b_{m0} + \sum_{\substack{j=1\\ j\neq m}}^{k} \left(\frac{1}{n} \sum_{i=1}^{n} A_{ji} A_{mi} \right) \sqrt{\frac{P_j}{2}} \left(b_{j0} - \hat{b}^{(s-1)}_{j0} \right).$$

This is called *multistage HD-PIC*. When we have applied s steps of interference cancellation we speak of s-stage HD-PIC and the corresponding bit error probability is $\mathbb{P}(\hat{b}^{(s+1)}_{m0} \neq b_{m0})$.

B. Reformulation of the problem

We can write the probability of a bit error in a more convenient way. Namely, because $b^2_{mi} = 1$, we have

$$\begin{aligned} &\sqrt{\frac{P_m}{2}} b_{m0} + \sum_{\substack{j=1\\ j\neq m}}^{k} \sqrt{\frac{P_j}{2}} b_{j0} \frac{1}{n} \sum_{i=1}^{n} A_{ji} A_{mi}\\ &= b_{m0} \left(\sqrt{\frac{P_m}{2}} + \sum_{\substack{j=1\\ j\neq m}}^{k} \sqrt{\frac{P_j}{2}} \frac{1}{n} \sum_{i=1}^{n} b_{j0} A_{ji} b_{m0} A_{mi} \right). \end{aligned}$$

Since $A_{ji} \stackrel{d}{=} b_{j0} A_{ji}$ we have

$$\begin{aligned} \mathbb{P}(\hat{b}^{(1)}_{m0} \neq b_{m0}) &= \mathbb{P}\left(\frac{\hat{b}^{(1)}_{m0}}{b_{m0}} \neq 1 \right)\\ &= \mathbb{P}(\text{sgnr}_m(Z^{(1)}_m) \neq 1) = \mathbb{P}(Z^{(1)}_m < 0) + \frac{1}{2}\mathbb{P}(Z^{(1)}_m = 0), \end{aligned}$$

where $Z^{(1)}_m$, for $1 \leq m \leq k$, is defined as

$$Z^{(1)}_m = \sqrt{P_m} + \sum_{\substack{j=1\\ j\neq m}}^{k} \sqrt{P_j} \frac{1}{n} \sum_{i=1}^{n} A_{ji} A_{mi}.$$

We can easily bound

$$\mathbb{P}(Z^{(1)}_m < 0) \leq \mathbb{P}(\hat{b}^{(1)}_{m0} \neq b_{m0}) \leq \mathbb{P}(Z^{(1)}_m \leq 0). \qquad (8)$$

Note that the above bound is completely independent of the choice of the sign-function in (6). The upper and lower bound in (8) can be shown to be almost equal when $k \geq 3$ and n is large.

Similarly, we define for $s \geq 2$ and $1 \leq m \leq k$,

$$\begin{aligned} Z^{(s)}_m = \sqrt{P_m} + \sum_{\substack{j=1\\ j\neq m}}^{k} \sqrt{P_j} \left(\frac{1}{n} \sum_{i=1}^{n} A_{ji} A_{mi} \right) \qquad (9)\\ \times [1 - \text{sgnr}_j(Z^{(s-1)}_j)], \end{aligned}$$

to obtain as in (8)

$$\mathbb{P}(Z^{(s)}_m < 0) \leq \mathbb{P}(\hat{b}^{(s)}_{m0} \neq b_{m0}) \leq \mathbb{P}(Z^{(s)}_m \leq 0). \qquad (10)$$

We will now investigate the performance of HD-PIC. We will use as a measure of performance the exponential rate of a bit error, defined by

$$H^{(s)}_{k;m} = -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\hat{b}^{(s)}_m \neq b_m). \qquad (11)$$

For systems with equal powers, the users are exchangeable, so that the above quantity does not depend on m. In that case, we define $H^{(s)}_k = H^{(s)}_{k;m}$ for all m.

We will often use the system with equal powers as a reference model. We will denote this model by the *simple* system. Without loss of generality, we then take $P_m = 2$, so that the factors $\sqrt{P_m/2}$ disappear.

C. Previous results for HD-PIC

In [6], it is shown that the exponential rate without interference cancellation $H^{(1)}_k$, (denoted there by I_k), for the simple system for $k \geq 3$, is given by [1]

$$H^{(1)}_k = \frac{k-2}{2} \log\left(\frac{k-2}{k-1} \right) + \frac{k}{2} \log\left(\frac{k}{k-1} \right). \qquad (12)$$

In [7], we have obtained an analytic formula for the rate after one stage of HD-PIC.

[1] A different definition of the sign function is used there, but this does not influence the results.

Theorem II.1: When all powers are equal

$$H_k^{(2)} = \min_{1\le r\le k-1} H_{k,r}^{(2)}, \tag{13}$$

where

$$H_{k,r}^{(2)} = \sup_{s,t}\{-\log h(s,t)\}, \tag{14}$$

with $h(s,t) = h_{k,r}(s,t)$ *equal to*

$$2^{-r}\sum_{j=-r}^{r}\binom{r}{\frac{r+j}{2}} e^{s+2sj+tj+tj^2}(\cosh tj)^{k-r-1}.$$

In the statement of the theorem, one should not confuse $H_{k,r}^{(2)}$ with $H_{k;m}^{(2)}$ in (11).

We see that the problem is split into two optimization problems. First, r represents the number of bits that have been estimated wrongly in the first stage. For every r, we solve the large deviation minimization problem. Then we minimize over r to obtain the rate $H_k^{(2)}$.

We can give an analytical expression for $H_{k,1}^{(2)}$, similar to the rate for the MF system, that reads

$$\begin{aligned} H_{k,1}^{(2)} &= \frac{3}{4}\log 3 - \log 2 \\ &\quad + \frac{2k-5}{4}\log\left(\frac{2k-5}{2k-4}\right) + \frac{2k-3}{4}\log\left(\frac{2k-3}{2k-4}\right) \end{aligned} \tag{15}$$

For general r, we cannot obtain a closed form expression for $H_{k,r}^{(2)}$. However, standard numerical packages allow us to compute $H_{k,r}^{(2)}$ for all k and r. In Figure 1, $H_{k,r}^{(2)}$ is shown for $r = 1,\ldots,5$. Observe that the optimal r equals 1 for $k \le 9$. In probability language it means that the BEP, caused by 2 or more bit errors in the first stage is negligible compared to the BEP caused by 1 bit error in the first stage. For $k = 10,\ldots,25$, $r = 2$ will give the minimal rate, meaning that "typically" a bit error in the second stage (after HD-PIC) is caused by 2 bit errors in the first stage. This is further illustrated in Table I, where the optimal r_k is given for $k = 3,\ldots,250$.

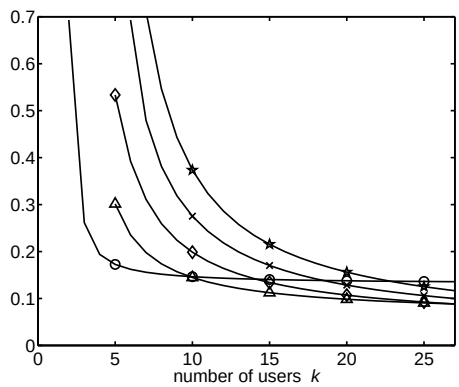

Fig. 1. Exponential rates $H_{k,r}^{(2)}$ for $r = 1,\ldots,5$ indicated with $\circ, \triangle, \diamond, \times, \star$ respectively.

For $1/k \to 0$ a Taylor expansion of (12) yields

$$H_k^{(1)} = \frac{1}{2k} + \mathcal{O}\left(\frac{1}{k^2}\right).$$

k	r_k
{3,...,9}	1
{10,...,26}	2
{27,...,51}	3
{52,...,84}	4
{85,...,125}	5
{126,...,174}	6
{175,...,231}	7
{232,...,250}	8

TABLE I
OPTIMAL VALUE r_k FOR $k = 3,\ldots,250$.

In fact,

$$\frac{1}{2k} \le H_k^{(1)} \le \frac{1}{2k} + \mathcal{O}\left(\frac{1}{k^2}\right). \tag{16}$$

In [7], it is shown that for the simple system, we have that the rate has an asymptotic scaling:

Theorem II.2: Fix $1 \le s < \infty$ and assume that the powers are all equal. Then, as $k \to \infty$,

$$H_k^{(s)} = \frac{s}{8}\sqrt[s]{\frac{4}{k}}\left(1 + \mathcal{O}\left(\frac{1}{\sqrt[s]{k}}\right)\right). \tag{17}$$

However, these results are asymptotic only. In Figure 2, we see that the approximation becomes worse when s increases.

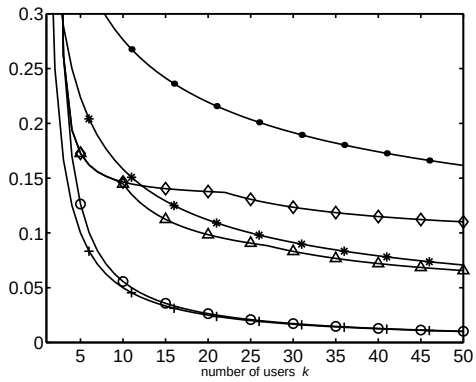

Fig. 2. Exponential rates $H_k^{(1)}$, $H_k^{(2)}$, respectively, $H_k^{(3)}$ and the asymptotic behaviour $\frac{1}{2k}$, $\frac{1}{2\sqrt{k}}$, respectively, $\frac{3}{8}\sqrt[3]{\frac{4}{k}}$. Here $\circ, \triangle, \diamond$ represent the exact values, and $+, *, \bullet$ the asymptotic values.

We next describe the extension of the above results for the simple system with equal powers to the case where the powers are unequal, shown in [11]. For the MF system, the following result is proven for the exponential rate $H_k^{(1)}$.

Theorem II.3: For $P_1/\sum_{m=1}^k P_m \to 0$,

$$H_{k;1}^{(1)} = \frac{P_1}{2\sum_{m=1}^k P_m} + \mathcal{O}\left(\left(\frac{P_1}{\sum_{m=1}^k P_m}\right)^2\right).$$

Theorem II.3 is based on a Taylor expansion of the rate. For the HD-PIC system, it is more involved to find an asymptotic expansion, but it can still be done.

Theorem II.4: For $P_1/\sum_{m=1}^k P_m \to 0$,

$$H_{k;1}^{(2)} \geq \frac{1}{2}\sqrt{\frac{P_1}{\sum_{m=1}^k P_m}} + \mathcal{O}\Big(\frac{P_1}{\sum_{m=1}^k P_m}\Big).$$

When there exists a $C > 0$, such that, uniformly in $\sum_{m=1}^k P_m/P_1$,

$$\min_R \left| \frac{\sum_{m\in R} P_m}{P_1} - \frac{1}{2}\sqrt{\frac{\sum_{m=1}^k P_m}{P_1}} \right| \leq C\Big(\frac{\sum_{m=1}^k P_m}{P_1}\Big)^{1/4},$$

where the minimum ranges over all $R \subset \{2,\ldots,k\}$, equality is attained:

$$H_{k;1}^{(2)} = \frac{1}{2}\sqrt{\frac{P_1}{\sum_{m=1}^k P_m}} + \mathcal{O}\Big(\frac{P_1}{\sum_{m=1}^k P_m}\Big).$$

A simple example where the above condition is satisfied is when $\max_{j,m} P_j/P_m \leq C$ for some C. We refer to [11] for more examples.

To see that the results above imply that HD-PIC really improves performance, it is useful to investigate the simple system. For this case, $H_k^{(1)}$ and $H_k^{(2)}$ reduce to

$$H_k^{(1)} \approx \frac{1}{2k} \qquad \text{and} \qquad H_k^{(2)} \approx \frac{1}{2\sqrt{k}}.$$

It is clear that HD-PIC gives a significant increase in performance compared to the MF receiver, since $1/(2\sqrt{k})$ is much large than $1/(2k)$, so that $\exp(-n/(2\sqrt{k}))$ is much smaller than $\exp(-n/(2k))$. Note the shift from $1/(2\cdot)$ to $1/(2\sqrt{\cdot})$. In order to preserve the desired quality level, it is thus possible to consider for example a decrease of the processing gain by a factor $\sqrt{k}$.

For the system with unequal powers, we see that

$$H_{k;1}^{(1)} \approx \frac{1}{2}\frac{P_1}{\sum_{m=1}^k P_m} \qquad \text{and} \qquad H_{k;1}^{(2)} \approx \frac{1}{2}\sqrt{\frac{P_1}{\sum_{m=1}^k P_m}}.$$

The same conclusion conclusion holds: interference cancellation significantly improves performance. The difference is that k is replaced by $\sum_{m=1}^k P_m/P_1$.

We will now give a heuristic explanation of Theorem II.2 We will start by explaining the result for $s = 2$, and then comment on $s > 2$. We will argue that

$$H_{k,r}^{(2)} \approx \frac{r}{2k} + \frac{1}{8r}. \tag{18}$$

Indeed, using (13), we then see that the minimum over r is obtained when $r \approx 2\sqrt{k}$, yielding $H_k^{(2)} \approx 1/(2\sqrt{k})$.

To see (18), note that $H_{k,r}^{(2)}$ is the rate of the event that $Z_2^{(1)} < 0, \ldots, Z_{r+1}^{(1)} < 0, Z_1^{(2)} < 0$, and all the other $Z_j^{(1)} > 0$ for $j = r+2,\ldots,k$. When we know all the signs of the $Z^{(1)}$'s, we can substitute these signs in the formula for $Z_1^{(2)}$. This yields that

$$Z_1^{(2)} \quad = \quad 1 + \frac{2}{n}\sum_{j=2}^{r+1}\sum_{l=1}^{n} A_{1l}A_{jl}. \tag{19}$$

Note that $\mathbb{E}\,(Z_j^{(1)}) = 1$ for all j. Therefore, the event $\{Z_j^{(1)} \geq 0\}$ is not a large deviation. This explains that we can show that the events $Z_j^{(1)} > 0$ for $j = r+2,\ldots,k$ do not contribute to the rate and we can leave these events out in the definition of $H_{k,r}^{(2)}$. Finally, independence of the sets $\{Z_2^{(1)} < 0\},\ldots,\{Z_{r+1}^{(1)} < 0\}$, and $\{Z_1^{(2)} < 0\}$ is clearly false for all finite k. However, it turns out to be asymptotically true for k large, as shown in [7]. This yields that

$$H_{k,r}^{(2)} \approx \lim_{n\to\infty} \sum_{m=2}^{r+1} -\frac{1}{n}\log \mathbb{P}(Z_m^{(1)} < 0)$$
$$-\frac{1}{n}\log \mathbb{P}\Big(1 + \frac{2}{n}\sum_{j=2}^{r+1}\sum_{l=1}^{n} A_{1l}A_{jl} < 0\Big).$$

The first term produces $rH_k^{(1)} \approx r/(2k)$, the second term can be computed to give $1/(8r)$ when r is large. This gives an informal explanation of the asymptotic result in (19), and explains that to have one bit error at the first stage of HD-PIC, we need roughly $\sqrt{k}/2$ bit errors at level one. This explains that one-stage of HD-PIC significantly decreases the BEP.

We finally give an informal explanation of Theorem II.2 for $s > 2$. For $R = (R_\sigma^{(s)})_{\sigma=1}^s$, let $H_{k,R}^{(s)}$ denote the rate of the event that $\{Z_m^{(\sigma)} < 0\}$ for all $m \in R_\sigma^{(s)}$, and $\{Z_m^{(\sigma)} > 0\}$ for all $m \notin R_\sigma^{(s)}$. Then we have that

$$H_k^{(s)} = \min_R H_{k,R}^{(s)}. \tag{20}$$

Let $r_\sigma^{(s)} = |R_\sigma^{(s)}|$. In [7], we show that we have

$$H_{k,R}^{(s)} \approx \frac{r_1^{(s)}}{2k} + \frac{r_2^{(s)}}{8r_1^{(s)}} + \ldots + \frac{r_{s-2}^{(s)}}{8r_{s-1}^{(s)}} + \frac{1}{8r_{s-1}^{(s)}}. \tag{21}$$

Note that the right hand side does not depend on the precise structure of $(R_\sigma^{(s)})_{\sigma=1}^s$, but only on the sizes. The reason is that *all* dependence between $Z^{(\sigma)}$ is different levels is dictated by the signs of these quantities. After these signs are substituted, the random variables are asymptotically independent for large k, like in the case for $s = 2$.

Minimization of the right hand side of (21) over $r_1^{(s)},\ldots,r_{s-1}^{(s)}$ yields

$$r_\sigma^{(s)} \approx \big(\frac{k}{4}\big)^{(s-\sigma)/s}. \tag{22}$$

Substitution yields (17).

D. Chernoff bounds and large k depending on n

The Chernoff bound can be used to bound probabilities from above by an exponential, and is true regardless of k, n. For the MF system, the Chernoff bound is straightforward to calculate. The Chernoff bound is given by

$$\mathbb{P}(\hat{b}_{10}^{(1)} \neq b_{10}) \leq e^{-nH_k^{(1)}} \tag{23}$$

and this holds for any n and k. Recall from (16) that $H_k^{(1)} \geq 1/(2k)$, so that $\mathbb{P}(\hat{b}_{10}^{(1)} \neq b_{10}) \leq e^{-n/(2k)}$ for any n

and k. For the HD-PIC system, a similar statement can be derived. When we investigate for example the case of equal powers, we are able to prove

$$\mathbb{P}(\hat{b}_{10}^{(2)} \neq b_{10}) \leq \sum_{r=1}^{k-1} \binom{k-1}{r} e^{-nH_{k,r}^{(2)}}. \tag{24}$$

The Chernoff bound is not tight in the sense that it approximates the BEP with any desired precision. Instead it gives an upper bound in a very simple form. Note that if n is large, the Chernoff bound is dominated by the term which has the smallest rate. Take for example $k = 12$ and $n = 63$. We can show that (see Table I). $H_{k,2}^{(2)}$ minimizes $H_{k,r}^{(2)}$. Therefore we expect that most of the sum is contributed by the second term. Substitution leads to

$$\begin{aligned}\mathbb{P}(\hat{b}_{10}^{(2)} \neq b_{10}) \quad &\leq \quad 1.32 \cdot 10^{-3} + 1.76 \cdot 10^{-2} + 5.28 \cdot 10^{-3} \\ &\quad +3.44 \cdot 10^{-4} + \ldots = 2.45 \cdot 10^{-2}.\end{aligned}$$

The main contribution clearly comes from $r = 2$ (72%), the contributions from $r = 3$ and $r = 1$ are also relevant (22% and 5%), but the remainder of the sum is negligible (1%).

In Figure 3, simulated BEP's are compared with the Chernoff bound in (24) as a function of n. It is seen that the Chernoff bound indeed gives upper bounds. The exponential rate appears in this figure as the slope of the BEP and the Chernoff bound. The difference in slopes is very small, indicating that the large deviation approach is indeed promising for performance evaluation. These exact BEP's are obtained by an importance sampling procedure (see [12] and [13] for the details).

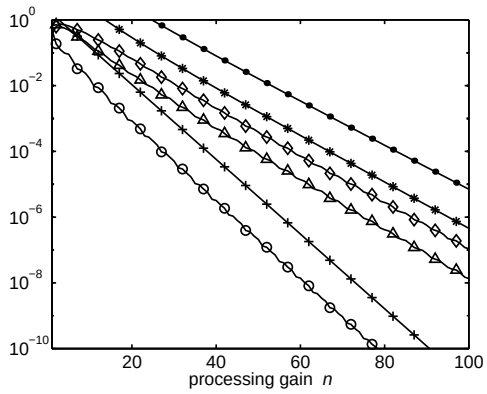

Fig. 3. BEP and Chernoff bound. The BEP for $k = 3, 6, 9$ is marked with $\circ, \triangle, \diamond$, respectively. The Chernoff bound for $k = 3, 6, 9$ is marked with $+, *, \bullet$, respectively.

We next use the Chernoff bound in order to take k large with n. From a practical point of view, one always wishes to have as many users as possible, so that the situation where k is fixed and $n \to \infty$ may be violated. Instead, we now take $k = k_n \to \infty$. We prove the following result:

Theorem II.5: When $k_n \to \infty$ such that $k_n = o(\frac{n}{\log n})$,

$$\lim_{n\to\infty} -\frac{\sqrt[s]{k_n}}{n} \log \mathbb{P}(\hat{b}_{10}^{(s)} \neq b_{10}) \geq \frac{s}{8}\sqrt[s]{4}. \tag{25}$$

The proof is given in Section V-B.

The above result implies that when $k_n \to \infty$ such that $k_n = o(\frac{n}{\log n})$, we have that $\mathbb{P}(\hat{b}_{10}^{(2)} \neq b_{10})$ is to leading asymptotics equal to $\exp(-\frac{n}{2\sqrt{k_n}})$. This should be contrasted with the behaviour that is expected when k is of the order n. This is the central limit regime where we expect that the bit error probability converges as $n \to \infty$ to a non-zero constant. It pays to take k slightly smaller than n instead of a multiple of n, at least when considering a finite number of HD-PIC stages! In Section III-B we will give a more precise explanation of the central limit behaviour and its relation to the HD-PIC results.

We remark that the Chernoff bound for $s = 2$ can easily be extended to certain cases with different powers. Indeed, when we assume that $\max_{j,m} \frac{P_j}{P_m} \leq C$, we can easily show that

$$\lim_{n\to\infty} -\frac{\sqrt{\sum_m P_m}}{n} \log \mathbb{P}(\hat{b}_{10}^{(s)} \neq b_{10}) \geq \frac{1}{2}\sqrt{P_1}. \tag{26}$$

E. Measures of performance: signal-to-noise ratio vs. exponential rate

We next investigate the relation of the above result to the signal-to-noise ratio (SNR), which is defined as

$$\text{SNR} = \frac{\mathbb{E}\,(Z)}{\sqrt{\text{Var}(Z)}},$$

where Z denotes the decision statistic in the model under consideration. We consider the case $n \gg k \gg 1$ and $P_m = 2$. We will show that in this case the SNR is not a good measure of performance, and therefore, we propose to use the exponential rate instead.

The asymptotic result for $H_k^{(1)}$ in (16) yields that the exponential rate of the BEP is approximately $1/(2k)$. Hence, the BEP is approximately

$$\text{BEP} \sim e^{-\frac{n}{2k}}. \tag{27}$$

For the MF system, we know that SNR$= \sqrt{n/(k-1)}$, since

$$\mathbb{E}\,(Z_m^{(1)}) = 1 \quad \text{and} \quad \text{Var}(Z_m^{(1)}) = (k-1)/n.$$

To get an approximation for the BEP, one often substitutes the SNR in the well-known Q-function. Since (see [5])

$$\frac{e^{-x^2/2}}{\sqrt{2\pi x^2}}(1 - \frac{1}{x^2}) < Q(x) < \frac{e^{-x^2/2}}{\sqrt{2\pi x^2}},$$

substituting the SNR yields the approximation

$$\text{BEP} \sim e^{-\text{SNR}^2/2}$$

This approach gives the approximation

$$\text{BEP} \sim e^{-\frac{n}{2(k-1)}} \sim e^{-\frac{n}{2k}},$$

which agrees with the large deviation result in (27).

For the simple system with one-stage of HD-PIC, Theorem II.2 that $H_k^{(2)} \approx 1/(2\sqrt{k})$, yielding

$$\text{BEP} \sim e^{-\frac{n}{2\sqrt{k}}}. \tag{28}$$

However, for a system with one stage of HD-PIC, using the SNR results in

$$\text{BEP} \leq \exp\left\{-\frac{e^{\frac{n}{2k}}}{2(k-1)^2}\right\}, \tag{29}$$

which is shown below. The latter value is far too small compared to the true asymptotics of the BEP in (28), which clearly indicates that the Gaussian approximation using the SNR is no good. The reasoning above can be adapted for other models, such as multistage HD-PIC or the optimal HD-PIC system described below. Therefore, using the SNR together with a Gaussian approximation for such multiuser detection systems leads to faulty approximations for the BEP.

To prove the upper bound (29), observe that

$$\begin{aligned} \mathbb{E}\left(\hat{b}_1^{(2)}\right) &= 1 + 2(k-1)\mathbb{P}(\hat{b}_2^{(1)} \neq b_2)\mathbb{E}\left(A_{21}A_{11}|\hat{b}_2^{(1)} \neq b_2\right) \\ &\approx 1, \end{aligned} \tag{30}$$

when n is large compared to k since $\mathbb{P}(\hat{b}_2^{(1)} \neq b_2) \approx 0$. Moreover,

$$\text{Var}(Z_1^{(2)}) \leq \mathbb{E}\left((Z_1^{(2)} - 1)^2\right) \leq (k-1)^2\mathbb{P}(\hat{b}_2^{(1)} \neq b_2),$$

where for the latter bound we use that $|A_{ij}| = 1$. From the Chernoff bound (see (23)), and the fact that $H_k^{(1)} \geq \frac{1}{2k}$ (see (16)), we end up with

$$\text{Var}(Z_1^{(2)}) \leq (k-1)^2 e^{-\frac{n}{2k}}.$$

This yields that

$$\text{SNR} \geq \frac{e^{\frac{n}{4k}}}{(k-1)},$$

so that

$$e^{-\text{SNR}^2/2} \leq \exp\left\{-\frac{e^{\frac{n}{2k}}}{2(k-1)^2}\right\}.$$

III. The optimal system

In this section, we show that after a finite number of stages of HD-PIC, the exponential rate of the BEP remains fixed. Then we study properties of this optimal hard decision system, such as its exponential rate, and Chernoff bounds for the BEP's. The results show that the BEP is exponentially small whenever $k \leq \delta n$ for some $\delta > 0$ sufficiently small, which is quite remarkable.

A. Optimal hard decisions

We first list some easy consequences for the system with unequal powers. First of all, we define the worst case rate of a bit error to be

$$H_k^{(s)} = \min_{m=1}^{k} H_{k;m}^{(s)}. \tag{31}$$

Hence, $H_k^{(s)}$ is the exponential rate of the bit error probability of that user that has *largest* bit error probability.

Theorem III.1: (a) $H_k^{(s)}$ is monotone in s.
(b) There exists a $s_k \leq 2^k + 1$, such that $H_k^{(s)} = H_k^{(s_k)}$ for all $s \geq s_k$.

The above shows that we can speak of the system for $s = s_k$ as the *optimal system*. For equal powers, we know that the rate for *all* users does not improve for $s \geq s_k$. For unequal powers, we can think of this system as having the optimal worse case rate, in the sense that the rate of the bit error minimized over all users is optimal.

We will use the definition of the sign-function in (6). However, in the proof, we will comment on other definitions of the sign-function where a similar result can be shown.

We will now prove the above theorem.

Proof: First of all, $s \mapsto H_k^{(s)}$ is non-decreasing. Indeed, to have a bit error, one of the interfering users has to have a bit error in the previous stage.

For the simple system, we have exchangeability of users, so that the probability of a bit error at stage s is smaller than that of stage $s-1$ and the desired statement follows. The above statement even proves that the probability of a bit error is non-decreasing.

This simple argument fails to hold when the powers are unequal. However, in this case, we can show in precisely the same way that the maximal bit error is non-decreasing. Then we take m such that the bit error in stage s is maximal for that user m. We can just repeat the argument given above, and see that we need to have a bit error from another user in the previous stage. However, the latter probability is at most the maximal probability of a bit error.

By the monotonicity above $H_k^{(\infty)} = \lim_{s\to\infty} H_k^{(s)}$ exists. To see why $H_k^{(\infty)}$ with k fixed is reached in a finite number of stages s_k and $s_k \leq 2^k + 1$, define

$$R_\sigma^{(s)} = \{m : \text{sgnr}_m(Z_m^{(\sigma)}) \neq 1\}. \tag{32}$$

Then $R_\sigma^{(s)}$ is the set of bit errors at stage s and level σ, i.e., the set of indices j for which $\text{sgnr}_j(Z_j^{(\sigma)}) < 0$ when we enforce that $\text{sgnr}_m(Z_m^{(s)}) < 0$. We observe that necessarily $R_{\sigma'}^{(s)} = R_{\sigma''}^{(s)}$, for some $\sigma' < \sigma'' \leq 2^k + 1$. Then $Z_m^{(\sigma''+1)} \equiv Z_m^{(\sigma'+1)}$ for all $1 \leq m \leq k$ and thus $R_{\sigma'+1}^{(s)} = R_{\sigma''+1}^{(s)}$. It follows that $R_{\sigma'+1}^{(s)} = R_{\sigma''+1}^{(s)}$ and thus $R_{\sigma+i(\sigma''-\sigma)}^{(s)} = R_\sigma^{(s)}$ for all $\sigma' \leq \sigma < \sigma''$, for all $i \in \mathbb{N} \cup \{0\}$. By (6), all stages beyond σ'' are determined by the stages $1, \ldots, \sigma''$ and do not contribute to the rate. Clearly some users have bit errors for a stage $s' > \sigma''$. Thus, the scenario described above is one scenario to force a bit error for a user at stage s'. However, $H_k^{(s')}$ is the worst-case rate over all scenarios and all users. Therefore, $H_k^{(s')} \leq H_k^{(s)}$. Since $H_k^{(s)}$ is non-decreasing, necessarily $H_k^{(s')} = H_k^{(s)}$ for all $s' \geq \sigma''$. Since $s \geq \sigma''$, the desired statement follows. □

The statement of Theorem III.1 can be extended to other definitions of the sign-function.

For example, when we let the sign of 0 to be 0, then we can copy the original proof, apart from the fact that we let $R^{(s)}_{\sigma;0}$ be the set of values m where $Z^{(\sigma)}_m = 0$, and $R^{(s)}_{\sigma;1}$ the set where $Z^{(\sigma)}_m < 0$. Then, when $s \geq 3^k + 1$, we must $R^{(s)}_{\sigma';i} = R^{(s)}_{\sigma'';i}$, for $i = 0, 1$ and some $\sigma' < \sigma'' \leq 3^k + 1$. This again implies a periodic scenario.

For the system with equal powers, we are also allowed to use a sign-function that assumes values ± 1 independently every time it is used, rather than fixing it per user. Indeed, the limit determining $H^{(\infty)}_k$ exists, so we can reach the limit just by using the odd n. In this case, we cannot have $Z^{(\sigma)}_m = 0$, except when $\sigma = 1$. Therefore, every user draws at most one time a random sign-function, which makes the statement equivalent to the statement with our original definition of sgnr. For unequal powers this argument unfortunately does not hold. However, $\mathbb{Q}$ is dense in $\mathbb{R}$, so that for *almost all* powers $(P_1, \ldots, P_k)$ and for all integers n, $Z^{(\sigma)}_m \neq 0$. This explains that the problems arising from the definition of the sign of zero are somewhat academic.

We will next investigate this optimal system. Recall that because of the hard decisions, the decisions at stage σ only depend on the decisions at stage $\sigma - 1$. For illustration purposes, we will first investigate the system with 2 users. We will use that when user 1 has no bit error at stage σ, user 2 necessarily has no bit error at stage $\sigma + 1$. We denote a bit error by $\square$ and no bit error by $\bullet$ in Figure 4. All possible five scenarios (except the trivial scenario where no errors at all are made) are shown.

stage \ user	1	2	1	2	1	2	1	2	1	2
1	$\square$	$\bullet$	$\bullet$	$\square$	$\square$	$\square$	$\square$	$\square$	$\square$	$\square$
2	$\bullet$	$\square$	$\square$	$\bullet$	$\square$	$\bullet$	$\bullet$	$\square$	$\square$	$\square$
3	$\square$	$\bullet$	$\bullet$	$\square$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\square$	$\square$
4	$\bullet$	$\square$	$\square$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\square$	$\square$

Fig. 4. Possible scenarios for the HD-PIC model with 2 users.

It is clear that within 3 stages, the set where bit errors are made becomes periodic. We will call this a *periodic scenario*. This is somewhat smaller than $2^2 + 1$. A periodic scenario has a big "advantage" over non-periodic scenarios. As long as the scenario is not yet periodic, specifying which users have bit errors at a certain stage results in a decrease of the BEP. Indeed, users do not tend to have a bit error; it is more likely to estimate a bit correctly. However, in a periodic scenario, this is not true anymore. For example, to get a bit error for user 1 at stage $s = 1001$, it is sufficient to specify the positions of bit errors in stages 1,2 and 3 (the first and last scenario in Figure 4 will do). From that point onwards, the bit errors are determined by the bit errors in the first three stages.

Two essentially different scenarios are characterizing the behaviour of the optimal system for $k = 2$. The first one is the so-called *disjoint* scenario, where at every stage user 1 has a bit error and user 2 not, or vice versa. The other scenario, which we will call the *overlapping* scenario is the scenario where at every stage both user 1 and user 2 have a bit error. Note that for both the disjoint as the overlapping scenario, the periodic behaviour kicks in at stage 1. For both scenarios, we can calculate the exponential rate. The minimum of the two exponential rates indicates which scenario typically is observed.

When $k \geq 3$, we extend those scenarios in the following way. For every r, at stage $1, 3, 5, \ldots$ bit errors are made for users in some set R_1, with $|R_1| = r$. At stage $2, 4, 6, \ldots$, bit errors are made for users in the set R_2 with $|R_2| = r$. When $R_1 \cap R_2 = \varnothing$, we speak of the disjoint scenario. Whenever $R_1 = R_2$, we will call it the overlapping scenario. All other scenarios are called *partly overlapping*.

The statement of Theorem III.1(b) is that after at most $2^k + 1$ stages, the set where bit-errors are made is periodic after a certain stage. We expect that the scenarios that we typically observe are periodic already at stage 1. Indeed, specifying the bit errors at the initial non-periodic stages make the BEP smaller, while for the periodic stages we do not need to specify the positions of bit errors, as these are determined by the first few stages. The longer the number of stages where the behaviour is not yet periodic, the smaller the BEP and thus this behaviour is less typical. This suggests that the scenarios that we typically observe are either the disjoint, the partly overlapping, or the overlapping scenario. The overlapping scenario has period 1, while the other scenarios have period 2, so stretching the above heuristic even further, it is tempting to assume that the overlapping scenario gives the smallest exponential rate, and is therefore typical. However, this is not true when k is small (see also *2)* below).

We observe the following phenomena.

1) The partly overlapping scenario is never optimal. It seems that both extremes (the overlapping and disjoint scenario) do a better job.
2) For small k, the disjoint scenario is optimal. The reason is quite simple. For r fixed, in the disjoint scenario, a user at stage 2 has contribution from r noise terms, whereas for the overlapping scenario the user has contribution from only $r - 1$ terms (the user does not interfere with its own signal). For higher k, however, the overlapping scenario is optimal. For the case $k \to \infty$, it is implicitly proven in the proof of Theorem III.2(a) that the overlapping scenario is indeed optimal, and the partly overlapping scenario is never optimal.
3) For $k \to \infty$, also $r \to \infty$, but much slower than k. The numerical results indicate that $r \approx \sqrt{k}/2$.

To illustrate *2)* and *3)*, the optimal r is shown for $k = 1 - 1000$ in Table II. Also, it is indicated whether the disjoint or the overlapping scenario is optimal.

In Figure 5, the exponential rates $H^{(s_k)}_k$ are given, together $H^{(s)}_k$ for $s = 1, 2, 3$. The results for $s = 3$ are obtained using similar techniques as in Theorem II.1. However, we have not stated the result for $s = 3$ in this paper. The

k	r	k	r
1-30	1(d)[1]	337-414	10(o)
31-73	2(d)	415-499	11(o)
74-83	3(d)	500-592	12(o)
84-107	5(o)[2]	593-694	13(o)
108-153	6(o)	695-803	14(o)
154-206	7(o)	804-920	15(o)
207-267	8(o)	920-1000	16(o)
268-336	9(o)		

[1](d) means disjoint scenario is optimal,
[2](o) means overlapping scenario is optimal.

TABLE II
EXPECTED OPTIMAL SCENARIO FOR OPTIMAL HD-PIC MODEL.

rate $H_k^{(s_k)}$ is in fact the rate corresponding to the disjoint scenario for $r = 1$ or $r = 2$. For $k = 3$, it is seen that $H_3^{(1)} = H_3^{(2)} = H_3^{(3)}$, so that $s_3 = 1$. For $4 \leq k \leq 9$, we see $H_k^{(1)} < H_k^{(2)} = H_k^{(3)}$, so that $s_k = 2$. We see that one stage of HD-PIC gives an improvement in exponential rate. However, adding one more stage does not result in any improvement. For $10 \leq k \leq 22$, 2-stage HD-PIC gives an improvement over one stage HD-PIC. The numerical result show that there is a block scenario (in fact the disjoint scenario with $r = 1$) with the same rate. Therefore $s_k = 3$ in this case. For $23 \leq k \leq 50$, we expect $s_k = 4$, even though we cannot calculate $H_k^{(4)}$.

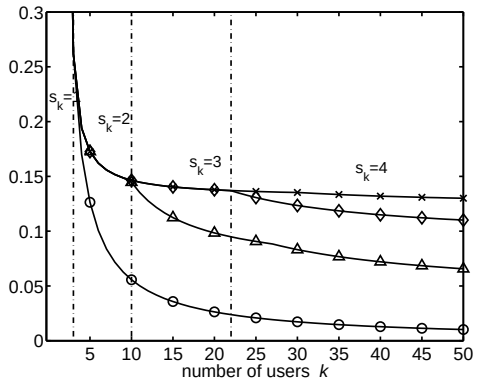

Fig. 5. The exponential rates $H_k^{(s_k)}$ and $H_k^{(s)}$ for $s = 1, 2, 3$.

We now turn to a lower bound of the exponential rate for the optimal system. Clearly, (17) has no meaning for $s \to \infty$, but assuming (17) and the monotonicity of $s \mapsto H_k^{(s)}$ it follows for the simple system that for all $\epsilon > 0$, $k^\epsilon H_k^{(\infty)} \to \infty$ when $k \to \infty$. Thus, if $H_k^{(\infty)}$ converges to 0 as $k \to \infty$, it does so slower than any power of $1/k$. The theorem below states that the exponential rate of the optimal system remains strictly positive as $k \to \infty$, and we identify the limiting rate under a certain assumption on the powers $P_1, \ldots, P_m$ that we will define now. We will assume that $(\mathcal{P}_1 - \mathcal{P}_4)$ hold, where the conditions $(\mathcal{P}_1 - \mathcal{P}_4)$ are defined by

$(\mathcal{P}_1)$ There exists a $\delta > 0$ s. t. $\sharp\{j : P_j \in [\delta, 1/\delta]\} \to \infty$,
$(\mathcal{P}_2)$ $\lim_{k\to\infty} kP^{-1} < \infty$,
$(\mathcal{P}_3)$ $k^{-1}P^{\max} \to 0$,
$(\mathcal{P}_4)$ $kP^{\min} \to \infty$.

Here $P^{\max} = \max_m P_m, P^{\min} = \min_m P_m$. The main result in this section in the following theorem.

Theorem III.2: (a) For the general system, for all $s > 2^k$,

$$H_k^{(s)} \geq \frac{1}{2}\log 2 - \frac{1}{4} \approx 0.09657. \tag{33}$$

For the system with equal powers the bound is obtained for $s > k$.
(b) If the powers are such that $(\mathcal{P}_1 - \mathcal{P}_4)$ hold, then

$$\lim_{k\to\infty} H_k^{(s_k)} = \frac{1}{2}\log 2 - \frac{1}{4}. \tag{34}$$

We stress that the case of equal powers is covered in Theorem III.2(b).

In Figure 6, an upper and lower bound is given of the exponential rate $H_k^{(s_k)}$ for equal powers. We expect that the upper bound is in fact equal to $H_k^{(s_k)}$, but we lack a proof.

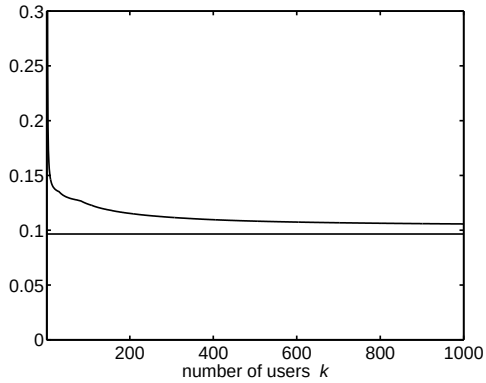

Fig. 6. Upper and lower bounds for the exponential rates $H_k^{(s_k)}$.

The proof of the above result will be given in the next sections. We will start with bounds on moment generating functions in Section IV. These bounds will be used in the proof of Theorem III.2 in Section V.

B. Chernoff bounds and large k depending on n

In Section V-B, we prove the following Chernoff bound for the optimal system:

Theorem III.3: When $k_n \to \infty$ and for $s > 2^{k_n}$

$$\mathbb{P}(\hat{b}_{10}^{(s)} \neq b_{10}) \leq 8^{k_n} e^{-In}, \tag{35}$$

where $I = \frac{1}{2}\log 2 - \frac{1}{4}$.
When the powers are equal, the same result holds when $s > k_n$.

The above result shows that the limit of k and n can be taken simultaneously, instead of the order common in large deviation theory (first $n \to \infty$, and subsequently $k \to \infty$). Moreover, Theorem III.3 shows that when $k = \delta n$ with $\delta < I/(3\log 2)$, that then the BEP is exponentially small.

Since for $k = \mathcal{O}(n)$, we have central limit behaviour, the above Chernoff bound implies that applying HD-PIC sufficiently often enables us to reduce the BEP from $\mathcal{O}(1)$ for the MF system, to exponentially small for the optimal HD system. We now give a more extensive heuristic explanation for this effect.

We take $k = \delta n$ for some constant $\delta > 0$. We assume that the powers are equal, and for simplicity take $P_m = 2$. In this case, we have that by the central limit theorem, $Z_m^{(1)} \approx \mathcal{N}(1, \delta)$. Hence, the BEP is approximately $\epsilon^{(1)} = Q(1/\sqrt{\delta})$. Therefore, we expect $\epsilon^{(1)} n$ bit errors in the first stage ($s = 1$). Now, given that there are $\epsilon^{(1)} n$ bit errors in the first stage, we have that $Z_m^{(2)} \approx \mathcal{N}(1, 4\epsilon^{(1)})$. Therefore, the BEP is approximately $\epsilon^{(2)} = Q(1/(2\epsilon^{(1)}))$, and we expect $\epsilon^{(2)} n$ bit errors in the second stage ($s = 2$). Repeating the above argument, we find that, with $\epsilon^{(s)} = Q(1/(2\epsilon^{(s-1)}))$, we expect approximately $\epsilon^{(s)} n$ bit errors in the sth stage. The above is probably quite accurate when s is finite and $n \to \infty$, but certainly not when s tends to infinity with n. However, from Theorem II.2 combined with Theorem III.2, we do know that $s_k \to \infty$ when $k \to \infty$. Therefore, the above argument fails to hold for the optimal system, and we see that the optimal HD-PIC system has a much smaller BEP than the BEP after any finite number of stages.

C. On the number of stages to optimality

In this section, we will investigate the number of stages necessary to reach the asymptotic optimal rate $\frac{1}{2}\log 2 - \frac{1}{4}$. We will prove the following theorem. In its statement, we will recall that $P = \sum_m P_m$, and we will use the notation $p = \min_m P_m$.

Theorem III.4: For the general system, for all $0 < \epsilon < \frac{1}{2}\log 2 - \frac{1}{4}$ and for all s such that $s \geq \epsilon^{-1} \log \frac{P}{p}$,

$$H_k^{(s)} \geq \frac{1}{2}\log 2 - \frac{1}{4} - \epsilon. \tag{36}$$

We know that $I = \frac{1}{2}\log 2 - \frac{1}{4}$ is the asymptotically optimal rate for the system where all the powers are equal. Note that when all powers are comparable (e.g., when $\max_{j,m} \frac{P_j}{P_m} < C$ for some constant $C < \infty$ uniformly in k), that then $\epsilon^{-1} \log \frac{P}{p} \approx \epsilon^{-1} \log k$. Theorem III.4 states that when we apply at least that many stages of HD-PIC, that then the limiting rate will be asymptotically bounded from below by I. This suggests that s_k grows roughly like $\log k$. It is an interesting problem to determine how s_k grows more precisely. It follows from Theorem III.2 together with Theorem II.2 that $s_k \to \infty$. However, it would be of practical importance to know the precise rate, or even an upper bound on s_k. We have the following conjecture:

Conjecture III.5: For the general system,

$$\limsup_{k\to\infty} \frac{\log s_k}{\log \log k} = 1. \tag{37}$$

Conjecture III.5 says that $\frac{s_k}{\log k}$ cannot grow or decrease faster than any small power of $\log k$.

Theorem III.4, together with Theorem III.2, suggests that $s_k \leq \epsilon_k^{-1} \log k$ for every $\epsilon_k \downarrow 0$. We believe that in fact a logarithmic number of stages is required to obtain the optimal HD-PIC system. However, we have no proof for this belief. This belief stems from the fact that we expect the proof of Theorem II.2 to hold for some s that tend to infinity with k sufficiently slowly. More precisely, we expect that the strategy described in Section II-C remain true for as long as $s_k \leq (\log k)^{1-\epsilon}$ for any $\epsilon > 0$. The reason is that in (22), there is an essential change when $s \ll \log k$ compared to $s_k = \mathcal{O}(\log k)$, in the sense that for the former

$$\frac{r_\sigma^{(s)}}{r_{\sigma+1}^{(s)}} \to \infty,$$

whereas for the latter, the above converges to a constant. When this convergence is towards a constant, we cannot expect (21) to be a good approximation. Moreover, note that when $s = (\log k)^{1-\epsilon}$, then substitution of s into the right hand-side of (17) gives

$$\frac{s}{8} e^{-\log k/s} = \frac{(\log k)^{1-\epsilon}}{8} e^{-(\log k)^{\epsilon}} \to 0$$

for all positive ϵ. This is clearly far away from the rate of the optimal HD-PIC system in Theorem III.2. Therefore, we believe $s_k \geq (\log k)^{1-\epsilon}$ for all $\epsilon > 0$, which explains Conjecture III.5.

IV. Bounds on moment generating functions

In this section, we will give sharp bounds on certain moment generating functions that will prove to be essential in the analysis of the optimal HD-PIC system.

We define

$$S_n = \sum_{i=1}^{n} \sqrt{P_i} X_i, \tag{38}$$

where

$$\mathbb{P}(X_i = \pm 1) = \frac{1}{2}.$$

We will also use the notation $P = \sum_{i=1}^{n} P_i$. The main result of this section is

Proposition IV.1:
(a) For all $n \in \mathbb{N}$, $0 \leq t < \frac{1}{P}$ and all $s \in \mathbb{R}$,

$$\mathbb{E}\left[e^{sS_n + \frac{t}{2}S_n^2}\right] \leq \frac{e^{\frac{Ps^2}{2(1-Pt)}}}{\sqrt{1-Pt}}, \tag{39}$$

(b) For all $-\frac{1}{P} \leq t \leq \frac{1}{P}$

$$\mathbb{E}\left[e^{\frac{t}{2}S_n^2}\right] \leq \frac{1}{\sqrt{1-Pt}}. \tag{40}$$

Note that if n is large, then $S_n \approx \sqrt{P} Z$, where Z has a standard normal distribution. The bounds in (39-40) show that the moment generating function of S_n and S_n^2, for the appropriate ranges of the variables t and s are at least bounded from above by the moment generating functions of $\sqrt{P}Z$ and PZ^2, $e^{Ps^2/2}$ and $1/\sqrt{1-Pt}$, respectively.

Proof of Proposition IV.1(a). This bound is easiest. Let Z have a standard normal distribution, then we know that $\mathbb{E}\left(e^{tZ}\right) = e^{t^2/2}$. Hence, we get that

$$\begin{aligned}\mathbb{E}\left[e^{sS_n+\frac{t}{2}S_n^2}\right] &= \mathbb{E}\left[e^{(s+\sqrt{t}Z)S_n}\right] \\ &= \mathbb{E}\left[\prod_{i=1}^n \cosh(\sqrt{P_i}(s+\sqrt{t}Z))\right].\end{aligned}$$

We use that $\cosh(t) \leq e^{t^2/2}$, to arrive at

$$\mathbb{E}\left[e^{sS_n+\frac{t}{2}S_n^2}\right] \leq \mathbb{E}\left[\prod_{i=1}^n e^{\frac{P_i}{2}(s+\sqrt{t}Z)^2}\right] = e^{\frac{Ps^2}{2}}\mathbb{E}\left[e^{Ps\sqrt{t}Z+tPZ^2}\right].$$

We complete the proof by noting that, for $t \leq \frac{1}{P}$,

$$\mathbb{E}\left[e^{Ps\sqrt{t}Z+tPZ^2}\right] = \frac{e^{\frac{P^2s^2t}{2(1-Pt)}}}{\sqrt{1-Pt}},$$

and by rearranging terms. □

Proof of Proposition IV.1(b). The claim for $0 \leq t \leq \frac{1}{P}$ follows from Proposition IV.1(a) proved above. The claim for $t < 0$ is more difficult, and we will use induction on n. Define

$$f_n(t) = \mathbb{E}\left[e^{\frac{t}{2}S_n^2}\right]. \tag{41}$$

The induction hypothesis is that

$$f_n(t) \leq \frac{1}{\sqrt{1-Pt}} \qquad \text{for all} \quad -\frac{1}{P} \leq t \leq 0. \tag{42}$$

Clearly, for $n = 0$, the above is trivial, as both the left and right hand side are equal to 1. We next advance the induction. We write

$$\begin{aligned}f_n(t) &= \mathbb{E}\left[e^{\frac{t}{2}S_n^2}\right] = e^{\frac{tP_n}{2}}\mathbb{E}\left[e^{\frac{t}{2}S_{n-1}^2+\sqrt{P_n}tS_{n-1}X_n}\right] \\ &= e^{\frac{tP_n}{2}}\mathbb{E}\left[e^{\frac{t}{2}S_{n-1}^2}\cosh(\sqrt{P_n}tS_{n-1})\right].\end{aligned}$$

We again use that $\cosh(t) \leq e^{t^2/2}$ to arrive at

$$f_n(t) = \mathbb{E}\left[e^{\frac{t}{2}S_n^2}\right] \leq e^{\frac{tP_n}{2}}\mathbb{E}\left[e^{\frac{t+P_nt^2}{2}S_{n-1}^2}\right] = e^{\frac{tP_n}{2}}f_{n-1}(t+P_nt^2).$$

To prove the claim, we first show that for $-\frac{1}{P} \leq t \leq 0$,

$$-\frac{1}{P-P_n} \leq t + P_nt^2 \leq 0$$

Indeed, since $-\frac{1}{P} \leq t \leq 0$, we have $0 \leq 1 + P_nt \leq 1$, so that

$$\begin{aligned}&-1 \leq (-1-P_nt)(1+P_nt) \leq (Pt-P_nt)(1+P_nt) \\ &= (P-P_n)(t+P_nt^2) = (P-P_n)t(1+P_nt) \leq 0,\end{aligned}$$

where the last inequality follows from $P-P_n \geq 0$ and $t \leq 0$. We therefore can substitute the induction hypothesis (42) for $n-1$, so that it remains to show that

$$\frac{e^{tP_n/2}}{\sqrt{1-(P-P_n)(t+P_nt^2)}} \leq \frac{1}{\sqrt{1-Pt}}.$$

Since $e^x \geq 1 + x + x^2/2$ for all $x \geq 0$,

$$e^{tP_n/2} = \frac{1}{\sqrt{e^{-tP_n}}} \leq \frac{1}{\sqrt{1-P_nt+P_n^2t^2/2}}.$$

Multiplying and rearranging terms gives

$$\begin{aligned}&(1-P_nt+P_n^2t^2/2)(1-(P-P_n)(t+P_nt^2)) \\ &= 1+t[-P_n-(P-P_n)] \\ &\quad +t^2[-P_n(P-P_n)+P_n(P-P_n)+P_n^2/2] \\ &\quad +t^3[-(P-P_n)P_n^2/2+P_n^2(P-P_n)] \\ &\quad +t^4[-P_n^3(P-P_n)/2] \\ &= 1-Pt+P_n^2t^2/2+P_n^2(P-4P_n/3)t^3/2 \\ &\quad -P_n^3(P-P_n)t^4/3 \\ &= 1-Pt+\frac{P_n^2t^2}{2}\Big(1+Pt-P_nt(1+(P-P_n)t)\Big) \\ &\geq 1-Pt,\end{aligned}$$

since $1 + Pt \geq 0$, $t \leq 0$ and $1 + (P - P_n)t \geq 0$. This completes the proof. □

V. Proofs

In this section we will first prove Theorem III.2(a) and III.4, where we will use Proposition IV.1. Then, we will prove the Chernoff bounds in Theorem II.5 and III.3 in Section V-B. This proof makes use of the proof of Theoremn Theorem III.2(a). Finally, we prove Theorem III.2(b) for equal powers. The proof of Theorem III.2(b) for unequal powers satisfying the power conditions ($\mathcal{P}_1 - \mathcal{P}_4$) will be given in the appendix.

A. Proof of Theorem III.2(a) and Theorem III.4

In this section, we will prove Theorem III.2(a) and Theorem III.4 simultaneously. For completeness, we will repeat the statements. For $s = 2^k + 1$,

$$H_k^{(s)} \geq \frac{1}{2}\log 2 - \frac{1}{4}. \tag{43}$$

For every $0 < \epsilon < I = \frac{1}{2}\log 2 - \frac{1}{4}$. For $s = \epsilon^{-1}\log(P/p)$,

$$H_k^{(s)} \geq \frac{1}{2}\log 2 - \frac{1}{4} - \epsilon. \tag{44}$$

In Section III we have shown that there is an optimal HD-PIC system, and that for $s \geq 2^k + 1$, the set of bit errors is periodic. For any set $A \subset \{1, \ldots, k\}$, we let $P_A = \sum_{i\in A} P_i$. When $s = 2^k + 1$, there must be a $\sigma \leq 2^k + 1$ such that $P_{R_\sigma^{(s)}} \geq P_{R_{\sigma-1}^{(s)}}$. In fact, when the powers are equal, we have that $P_{R_\sigma^{(s)}} = P|R_\sigma^{(s)}|$, and the above must happen already when $s \geq k$. We focus on that level σ and are only interested in the event $\{P_{R_\sigma^{(s)}} \geq P_{R_{\sigma-1}^{(s)}}\}$.

Furthermore, when $s = \epsilon^{-1}\log(P/p) + 1$, there must be a $\sigma \leq \epsilon^{-1}\log(P/p) + 1$ such that $P_{R_\sigma^{(s)}} \geq (1-\epsilon)P_{R_{\sigma-1}^{(s)}}$.

Indeed, when this should not be the case after $\sigma - 1 = \epsilon^{-1}\log(P/p)$ stages, we have

$$P_{R^{(s)}_{\sigma-1}} \le (1-\epsilon)^{\sigma-1} P_{R^{(s)}_0} = \exp\Big(\frac{\log(1-\epsilon)}{\epsilon}\log\frac{P}{p}\Big)P.$$

Since $1-\epsilon \le e^{-\epsilon}$, also $\log(1-\epsilon)/\epsilon \le -1$, so that

$$P_{R^{(s)}_{\sigma-1}} \le \exp\Big(-\log\frac{P}{p}\Big)P = p.$$

However, always $P_{R^{(s)}_\sigma} \ge p$, so that

$$P_{R^{(s)}_\sigma} \ge p \ge (1-\epsilon)P_{R^{(s)}_{\sigma-1}},$$

so certainly after at most $s = \log(P/p)/\epsilon + 1$ stages the desired event has occurred. We focus on the level σ at which the desired event occurs and are only interested in that occurrence.

At this point we remark that once we obtain for $P_{R^{(s)}_\sigma} \ge (1-\epsilon)P_{R^{(s)}_{\sigma-1}}$ that

$$-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(P_{R^{(s)}_\sigma} \ge (1-\epsilon)P_{R^{(s)}_{\sigma-1}}\Big) \ge \frac{1}{2}\log 2 - \frac{1}{4} - \epsilon,$$

we immediately obtain the result for $\epsilon = 0$, which is the statement (43).

We use that when $A = R^{(s)}_{\sigma-1}$ and $B = R^{(s)}_\sigma$, that then for all $m \in B$, we have that

$$\frac{1}{2}\sqrt{2P_m}Z^{(\sigma)}_m = P_m + \frac{2}{n}\sum_{l=1}^{n}\sum_{j\in A\setminus\{m\}}\sqrt{P_jP_m}A_{jl}A_{ml}.$$

As the number of configuration $(R^{(s)}_\sigma)^s_{\sigma=1}$ is just finite, we obtain that $H^{(s)}_k$ is bounded from below by the minimum over A and B such that $P_B \ge (1-\epsilon)P_A$ of

$$\begin{aligned}
&-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\sum_{j\in A}\sum_{l=1}^{n}\sqrt{P_jP_m}A_{jl}A_{ml} + \frac{nP_m}{2} \le 0 \forall m\in B\Big)\\
&\ge -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\sum_{\substack{j\in A\\ m\in B\setminus\{j\}}}\sum_{l=1}^{n}\sqrt{P_jP_m}A_{jl}A_{ml} + \frac{nP_B}{2} \le 0\Big)\\
&\ge -\log\Big(e^{\frac{P_B-2P_{A\cap B}}{4}t}\mathbb{E}\big(e^{\frac{t}{2}\sum_{(j,m)\in A\times B}\sqrt{P_jP_m}A_{j1}A_{m1}}\big)\Big),
\end{aligned}$$

where the last inequality is the exponential Chebycheff's inequality for $t \le 0$. We write $S_A = \sum_{j\in A}\sqrt{P_j}A_{j1}$ to end up with

$$H^{(s)}_k \ge \min_{\substack{A,B\\ P_B\ge(1-\epsilon)P_A}} -\log\Big(\mathbb{E}\big(e^{\frac{t}{2}S_AS_B}\big)\Big) + \frac{P_B-2P_{A\cap B}}{4}t. \quad (45)$$

We will now bound the moment generating function from below using Proposition IV.1.

We first write $S_A = S_{A\cap B} + S_{A\setminus B}$, and we use the fact that $S_{A\setminus B}$ is independent from $(S_B, S_{B\cap A})$ to get that $\mathbb{E}\big(e^{\frac{t}{2}S_BS_A}\big)$ equals

$$\begin{aligned}
&\mathbb{E}\big(e^{\frac{t}{2}S_BS_{B\cap A}}e^{\frac{t}{2}S_BS_{A\setminus B}}\big) \quad (46)\\
&= \mathbb{E}\big(e^{\frac{t}{2}S_BS_{B\cap A}}\prod_{j\in A\setminus B}\cosh(\frac{t}{2}\sqrt{P_j}S_B)\big)\\
&\le \mathbb{E}\big(e^{\frac{t}{2}S_BS_{B\cap A}}e^{\frac{t^2}{8}P_{A\setminus B}S_B^2}\big)
\end{aligned}$$

We write the right hand side of (46) as

$$\mathbb{E}\left(e^{(\frac{t}{2}+\frac{t^2}{8}P_{A\setminus B})S^2_{A\cap B}}e^{(\frac{t}{2}+\frac{t^2}{4}P_{A\setminus B})S_{B\cap A}S_{B\setminus A}+\frac{t^2}{8}P_{A\setminus B}S^2_{B\setminus A}}\right)$$

and use Proposition IV.1(a) with $s = (\frac{t}{2}+\frac{t^2}{4}P_{A\setminus B})S_{B\cap A}$ and $t = \frac{t^2}{4}P_{A\setminus B} \ge 0$ to bound the expectation over $S_{A\setminus B}$ as

$$\begin{aligned}
&\frac{1}{\sqrt{1-P_{A\setminus B}P_{B\setminus A}t^2/4}}\\
&\quad\times\mathbb{E}\left(e^{(\frac{t}{2}+\frac{t^2}{8}P_{A\setminus B})S^2_{A\cap B}}e^{P_{A\setminus B}\frac{(t/2+P_{B\setminus A}t^2/4)^2}{2(1-P_{A\setminus B}P_{B\setminus A}2t^2/4)}S^2_{B\cap A}}\right)\\
&= \frac{1}{\sqrt{1-P_{A\setminus B}P_{B\setminus A}t^2/4}}\,\mathbb{E}\left(e^{\tilde{t}S^2_{B\cap A}/2}\right)\\
&\le \frac{1}{\sqrt{1-P_{A\setminus B}P_{B\setminus A}t^2/4}}\,\frac{1}{\sqrt{1-P_{B\cap A}\tilde{t}}},
\end{aligned}$$

where the last inequality is valid as long as $|\tilde{t}| \le \frac{1}{P_{B\cap A}}$ and where

$$\begin{aligned}
\tilde{t} &= t + \frac{t^2P_{A\setminus B}}{4} + P_{B\setminus A}\frac{(t/2+P_{A\setminus B}t^2/4)^2}{1-P_{A\setminus B}P_{B\setminus A}t^2/4}\\
&= \frac{t+(P_{B\setminus A}+P_{A\setminus B})t^2/4}{1-P_{A\setminus B}P_{B\setminus A}t^2/4}.
\end{aligned}$$

From the first expression for $\tilde{t}$ it is clear that $\tilde{t} \ge t$, so that we restrict $-1/P_{A\cap B} \le t \le 0$. From the second expression for $\tilde{t}$ above it is straightforward to prove that $\tilde{t} \le 0$. We proceed by multiplying the two square roots to obtain

$$\mathbb{E}\big(e^{\frac{t}{2}S_AS_B}\big) \le \frac{1}{\sqrt{1-tP_{B\cap A}-\frac{t^2}{4}(P_AP_B-P^2_{B\cap A})}}, \quad (47)$$

since

$$P_{A\setminus B}P_{B\setminus A} + P_{A\cap B}(P_{A\setminus B}+P_{B\setminus A}) = P_AP_B - P^2_{B\cap A}.$$

Substituting (47) into (45) yields that for all $t \ge -\frac{1}{P_{A\cap B}}$ we have that $H^{(s)}_k$ is bounded from below by

$$\begin{aligned}
&\min_{\substack{A,B\\ P_B\ge(1-\epsilon)P_A}}\frac{1}{2}\log\Big(1-tP_{A\cap B}-\frac{t^2}{4}(P_AP_B-P^2_{B\cap A})\Big)\\
&\qquad -\frac{P_B-2P_{B\cap A}}{4}t. \quad (48)
\end{aligned}$$

Since $-\frac{1}{P_{B\cap A}} \le -\frac{1}{P_B} \le 0$, substituting $t = -\frac{1}{P_B}$ results in

$$\begin{aligned}
H^{(s)}_k &\ge \min_{\substack{A,B\\ P_B\ge(1-\epsilon)P_A}}\frac{1}{2}\log\Big(1-\frac{P_A}{4P_B}+\frac{P_{B\cap A}}{P_B}+\frac{P^2_{B\cap A}}{4P^2_B}\Big)\\
&\quad -\frac{1}{4}\Big(2\frac{P_{B\cap A}}{P_B}-1\Big). \quad (49)
\end{aligned}$$

It is clear that this lower bound of $H_k^{(s)}$ is decreasing in P_A and that $P_A \le \frac{1}{1-\epsilon}P_B$. Therefore substituting $P_A = \frac{1}{1-\epsilon}P_B$ still gives a lower bound. Clearly, $\frac{P_{A\cap B}}{P_B} \in [0,1]$. The above lower bound is therefore attained at

$$\min_{0\le\alpha\le1} \frac{1}{2}\log\left(1-\frac{1}{4(1-\epsilon)}+\alpha+\frac{\alpha^2}{4}\right)-\frac{1}{4}(2\alpha-1) = \min_{0\le\alpha\le1} f(\alpha). \tag{50}$$

Differentiating $f(\alpha)$ w.r.t. α gives that $f'(\alpha)$ equals

$$-\frac{1}{2}+\frac{1}{2}\frac{1+\frac{\alpha}{2}}{1-\frac{1}{4(1-\epsilon)}+\alpha+\frac{\alpha^2}{4}} = \frac{\frac{1}{4(1-\epsilon)}-\frac{\alpha}{2}-\frac{\alpha^2}{4}}{1-\frac{1}{4(1-\epsilon)}+\alpha+\frac{\alpha^2}{4}}.$$

Hence, $f'(\alpha)>0$ for $\alpha<\sqrt{1+1/(1-\epsilon)}-1$ and $f'(\alpha)<0$ for $\alpha>\sqrt{1+1/(1-\epsilon)}-1$. Therefore, the minimum of f is attained at either $\alpha=0$ or at $\alpha=1$. Substitution yields that

$$\begin{aligned} f(0) &= \frac{1}{2}\log\left(1-\frac{1}{4(1-\epsilon)}\right)+\frac{1}{4} \\ &= \frac{1}{2}\log\frac{3}{4}+\frac{1}{4}+\frac{1}{2}\log\left(1-\frac{1}{3}\left(\frac{1}{1-\epsilon}-1\right)\right) \end{aligned}$$

and

$$\begin{aligned} f(1) &= \frac{1}{2}\log\left(\frac{9}{4}-\frac{1}{4(1-\epsilon)}\right)-\frac{1}{4} \\ &= \frac{1}{2}\log 2-\frac{1}{4}+\frac{1}{2}\log\left(1-\frac{1}{8}\left(\frac{1}{1-\epsilon}-1\right)\right). \end{aligned}$$

Finally observe that for $0\le\epsilon\le 2/3$,

$$e^{-2\epsilon} \le 1-2\epsilon+2\epsilon^2 \le 1-2\epsilon+4\epsilon/3 = 1-2\epsilon/3,$$

so that $-2\epsilon \ge \log(1-2\epsilon/3)$. Furthermore, for $0\le\epsilon\le 1/2$, $1/(1-\epsilon)\le 1/(2\epsilon)$. Substituting this yields

$$\log\left(1-\frac{1}{3}\left(1-\frac{1}{1-\epsilon}\right)\right) \ge -2\epsilon,$$
$$\log\left(1-\frac{1}{8}\left(1-\frac{1}{1-\epsilon}\right)\right) \ge -\frac{3}{4}\epsilon \ge -2\epsilon.$$

Therefore, since $\frac{1}{2}\log\frac{3}{4}+1/4 > \frac{1}{2}\log 2+1/4$,

$$H_k^{(s)} \ge \min\{f(0),f(1)\} = \frac{1}{2}\log 2-\frac{1}{4}-\epsilon.$$

This completes the proof of Theorem III.4. Substituting $\epsilon=0$ gives Theorem III.2(a). □

B. Proof of the Chernoff bounds

We will start by proving Theorem II.5 for $s=2$. The extension to $s>2$ will follow later, and is a small adaptation of the proof for $s=2$.

We have that

$$\begin{aligned} &\mathbb{P}(\mathrm{sgnr}_1(Z_1^{(2)})<0) \le \mathbb{P}(Z_1^{(2)}\le 0) \\ &\le \sum_{r=1}^{k_n-1}\binom{k_n-1}{r}\mathbb{P}\Big(Z_1^{(2)}\le 0, \max_{m=2}^{r+1} Z_m^{(1)}\le 0, \min_{m=r+2}^{k_n} Z_m^{(1)}\ge 0\Big). \end{aligned}$$

We split the sum over r in two parts: $1\le r\le 4\sqrt{k_n}$ and $r>4\sqrt{k_n}$. We start with the first term. The Chernoff bound gives

$$\mathbb{P}\Big(Z_1^{(2)}\le 0, \max_{m=2}^{r+1} Z_m^{(1)}\le 0, \min_{m=r+2}^{k_n} Z_m^{(1)}\ge 0\Big) \le e^{-nH_{k_n,r}^{(2)}}.$$

We bound, using that $\binom{k_n-1}{r}\le k_n^r$ and $e^{-nH_{k_n,r}^{(2)}} \le e^{-nH_{k_n}^{(2)}}$,

$$\begin{aligned} \sum_{r=1}^{4\sqrt{k_n}}\binom{k_n-1}{r}e^{-nH_{k_n,r}^{(2)}} &\le k_n^{4\sqrt{k_n}+1}e^{-nH_{k_n}^{(2)}} \\ &= e^{4\frac{k_n\log k_n}{\sqrt{k_n}}+\frac{\sqrt{k_n}\log k_n}{\sqrt{k_n}}}e^{-nH_{k_n}^{(2)}}. \end{aligned}$$

The first term on the right hand-side is $e^{o(\frac{n}{\sqrt{k_n}})}$, since $k_n = o(\frac{n}{\log n})$ implies $k_n\log k_n = o(n)$. Therefore, this term is bounded from above by

$$e^{o(\frac{n}{\sqrt{k_n}})}e^{-nH_{k_n}^{(2)}} = e^{o(\frac{n}{\sqrt{k_n}})}e^{-\frac{n}{2\sqrt{k_n}}(1+o(1))}.$$

This proves that the first term has the right order, and it remains to show that the other term is an error term. In order to do this, we will first prove the following lemma.

Lemma V.1: For every k, n

$$\mathbb{P}\Big(Z_1^{(2)}\le 0, \max_{m=2}^{r+1} Z_m^{(1)}\le 0, \min_{m=r+2}^{k} Z_m^{(1)}\ge 0\Big) \le e^{-\frac{r}{4k}n}.$$

Proof: We bound, using $\mathbb{P}(A\cap B)\le\mathbb{P}(A)$,

$$\begin{aligned} &\mathbb{P}\Big(Z_1^{(2)}\le 0, \max_{m=2}^{r+1} Z_m^{(1)}\le 0, \min_{m=r+2}^{k} Z_m^{(1)}\ge 0\Big) \\ &\le \mathbb{P}\Big(\max_{m=1}^{r} Z_m^{(1)}\le 0\Big) \le \mathbb{P}\Big(\sum_{m=1}^{r} Z_m^{(1)}\le 0\Big). \end{aligned}$$

We can compute that

$$\sum_{m=1}^{r} Z_m^{(1)} = \frac{1}{n}\sum_{l=1}^{n}\sum_{m=1}^{r} A_{ml}\sum_{i=1}^{k} A_{il}, \tag{51}$$

Therefore, by the exponential Chebycheff inequality, for every $t\le 0$, the probability of interest is bounded by

$$\Big(\min_{t\le 0}\mathbb{E}\left(e^{\frac{t}{2}S_rS_k}\right)\Big)^n \le \Big(\min_{t\le 0}\mathbb{E}\left(e^{\frac{t}{2}S_r^2}e^{\frac{t}{2}S_r(S_k-S_r)}\right)\Big)^n,$$

where we write $S_m=\sum_{l=1}^m A_{1l}$. We next bound the moment generating function from above. We first use the independence of S_r and S_k-S_r to obtain

$$\mathbb{E}\left(e^{\frac{t}{2}S_r^2}e^{\frac{t}{2}S_r(S_k-S_r)}\right) = \mathbb{E}\left(e^{\frac{t}{2}S_r^2}\cosh\Big(\frac{t}{2}S_r\Big)^{k-r}\right). \tag{52}$$

We next use the bound $1\le\cosh(t)\le e^{t^2/2}$ to bound the expression above as

$$\mathbb{E}\left(e^{\frac{t}{2}S_r^2}\cosh\Big(\frac{t}{2}S_r\Big)^{k}\right) \le \mathbb{E}\left(e^{(t+\frac{kt^2}{4})S_r^2/2}\right). \tag{53}$$

As long as $t + \frac{kt^2}{4} \geq -\frac{1}{r}$, we can use Proposition IV.1(b) to obtain

$$\mathbb{E}\, e^{\frac{t}{2}S_r^2} e^{\frac{t}{2}S_r(S_k - S_r)} \leq \frac{1}{\sqrt{1 - rt - \frac{rkt^2}{4}}}. \tag{54}$$

Therefore, we arrive at

$$\mathbb{P}\Big(Z_1^{(2)} \leq 0, \max_{m=2}^{r+1} Z_m^{(1)} \leq 0, \min_{m=r+2}^{k} Z_m^{(1)} \geq 0\Big)^{1/n} \leq \min_{t \leq 0} \exp\Big(-\frac{1}{2}\log\Big(1 - rt - \frac{rkt^2}{4}\Big)\Big).$$

The optimal t is attained at $t = -2/k$. For this choice, we have that $t + kt^2/4 = -1/k \geq -1/r$. This justifies (54). We further observe that for $t = -2/k$, we have that $\log\big(1 - rt - \frac{rkt^2}{4}\big) = \log\big(1 + \frac{r}{k}\big)$. Finally, observe that $\frac{1}{2}\log(1+x) \geq x/4$ for all $0 \leq x \leq 1$, which completes the proof of the lemma. □

We now complete the proof of Theorem II.5 for $s = 2$ using Lemma V.1. Since $k_n e^{-\frac{n}{4k_n}} = o(1)$, the sum over r satisfying $r > 4\sqrt{k_n}$ is bounded by

$$\sum_{r > 4\sqrt{k_n}} k_n^r e^{-\frac{nr}{4k_n}} = \Big(e^{\log k_n - \frac{n}{4k_n}}\Big)^{4\sqrt{k_n}} \sum_{r>0} \Big(e^{\log k_n - \frac{n}{4k_n}}\Big)^r \leq 2\exp\Big(4\sqrt{k_n}\log k_n - \frac{n}{\sqrt{k_n}}\Big).$$

This satisfies the required bound since $\sqrt{k_n}\log k_n = o(\frac{n}{\sqrt{k_n}})$, so that we have

$$\mathbb{P}(\mathrm{sgnr}_1(Z_1^{(2)}) < 0) \leq e^{-\frac{n}{2\sqrt{k_n}}(1+o(1))} + 2e^{-\frac{n}{\sqrt{k_n}}(1+o(1))} = e^{-\frac{n}{2\sqrt{k_n}}(1+o(1))}.$$

The proof for $s > 2$ is similar, and we point out the differences only. We can use the proof of Lemma V.1 to show that the probability that there are at least $c_1 k_n^{(s-1)/s}$ bit errors in the first stage is an error term if $c_1 > 0$ is large enough. Therefore, we only have to deal with the case where there are at most $c_1 k_n^{(s-1)/s}$ bit errors in the first stage. In this case, an easy extension of the proof of Lemma V.1 shows that the probability that there are at least $c_2 k_n^{(s-2)/s}$ bit errors in the second stage is an error term if $c_2 > 0$ is large enough. Therefore, we may also assume that there are at most $c_2 k_n^{(s-2)/s}$ bit errors in the first stage. We can repeat this argument, so that we only have to deal with the probability that user 0 has a bit error in stage s, intersected by the events $|R_\sigma^{(s)}| \leq c_\sigma k_n^{(s-\sigma)/s}$. We now can use the Chernoff bound and show that the binomial factors are of lower order.

□

We next turn to the proof of Theorem III.3, which is quite easy when we use results of the proof of Theorem III.2 in Section V-A. In the proof, we have used the fact that for $s \geq 2^{k_n} + 1$, there must be a stage σ such that $P_{R_\sigma^{(s)}} \geq P_{R_{\sigma-1}^{(s)}}$. The rate of this event is proven to be bounded from below by $I = \frac{1}{2}\log 2 - \frac{1}{4}$. The number of possible stages at which this can happen is $2^{k_n} + 1$, and the number of possibilities for $R_\sigma^{(s)}$ and $R_{\sigma-1}^{(s)}$ are $2^{k_n} - 1$ each, since $R_\sigma^{(s)}$ and $R_{\sigma-1}^{(s)}$ cannot be empty. The above argument leads to an overall factor of $(2^{k_n}+1)(2^{k_n}-1)^2 \leq 8^{k_n}$. This completes the proof, since

$$\begin{aligned}\mathbb{P}(\hat{b}_{10}^{(s)} \neq b_{10}) &\leq \mathbb{P}\Big(\bigcup_{\sigma \leq 2^{k_n}+1} \{P_{R_\sigma^{(s)}} \geq P_{R_{\sigma-1}^{(s)}}\}\Big) \\ &\leq 8^{k_n} \min_{A,B: P_B \geq P_A} \mathbb{P}(R_\sigma^{(s)} = A, R_{\sigma-1}^{(s)} = B) \\ &\leq 8^{k_n} e^{-In}.\end{aligned}$$

We note that the above proof also holds if we choose the sign(0) to be equal to $\pm$ independently every time. □

C. Proof of Theorem III.2(b)

In order to prove Theorem III.2(b), we have to find a strategy that has asymptotic rate $\frac{1}{2}\log 2 - \frac{1}{4}$. For simplicity, we will assume that all powers are equal. The proof for unequal powers is more technical and is therefore deferred to Appendix A. For simplicity, we assume that the powers are all equal to 2. We note that when

$$R_2 = R_1,$$

that necessarily for all $\sigma \geq 1$

$$R_\sigma = R_1.$$

Hence, we have now found a strategy that implies bit errors at all stages, so that the rate of this event is an upper bound for the rate of the optimal system. We fix $r = |R_1|$ and for technical reasons we assume r to be odd. We will first investigate the rate

$$-\lim_{n\to\infty} \frac{1}{n}\log \mathbb{P}(R_2 = R_1, |R_1| = r).$$

Due to the fact that all users are exchangeable, and the rate function of the vector $((Z_m^{(1)}, Z_m^{(2)}))_{m=1}^k$ is convex, we have that

$$\begin{aligned}&-\lim_{n\to\infty} \frac{1}{n}\log \mathbb{P}(R_2 = R_1, |R_1| = r) \\ &= -\lim_{n\to\infty} \frac{1}{n}\log \mathbb{P}\Big(\sum_{m=1}^r Z_m^{(1)} \leq 0, \sum_{m=r+1}^k Z_m^{(1)} \geq 0, \\ &\qquad \sum_{m=1}^r \bar{Z}_m^{(2)} \leq 0, \sum_{m=r+1}^k \bar{Z}_m^{(2)} \geq 0\Big),\end{aligned}$$

where $\bar{Z}_m^{(2)}$ denotes $Z_m^{(2)}$ where the signs of the $Z_m^{(1)}$ are substituted. This statement will be proven in more detail in Appendix A.

We next note that

$$\begin{aligned}\sum_{m=1}^{k} Z_m^{(1)} &= \frac{1}{n}\sum_{i=1}^{n}\Big(\sum_{m=1}^{k}\Big(1+\sum_{\substack{j=1\\ j\neq m}}^{k} A_{ji}A_{mi}\Big)\Big)\\ &= \frac{1}{n}\sum_{i=1}^{n}\Big(\sum_{m,j=1}^{k} A_{ji}A_{mi}\Big)\\ &= \frac{1}{n}\sum_{i=1}^{n}\Big(\sum_{m=1}^{k} A_{mi}\Big)^2 \geq 0.\end{aligned}$$

Moreover, we have that $\sum_{m=1}^{r} Z_m^{(1)} \leq 0$. Hence, $\sum_{m=r+1}^{k} Z_m^{(1)} \geq 0$, so that we can remove the event $\{\sum_{m=r+1}^{k} Z_m^{(1)} \geq 0\}$. Since $\mathbb{P}(X \leq 0, Y \leq 0) \geq \mathbb{P}(X - Y/2 \leq 0, Y \leq 0)$, we can bound the exponential rate from above by

$$-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\sum_{m=1}^{r} Z_m^{(1)} - \frac{1}{2}\bar{Z}_m^{(2)} \leq 0,\ \sum_{m=1}^{r}\bar{Z}_m^{(2)} \leq 0,\ \sum_{m=r+1}^{k}\bar{Z}_m^{(2)} \geq 0\Big),$$

where for $1 \leq m \leq r$,

$$\begin{aligned}Z_m^{(1)} - \frac{1}{2}\bar{Z}_m^{(2)} &= \frac{1}{n}\sum_{i=1}^{n}\Big(\frac{1}{2}+\sum_{j=r+1}^{k} A_{ji}A_{mi}\Big),\\ \bar{Z}_m^{(2)} &= \frac{1}{n}\sum_{i=1}^{n}\Big(1+2\sum_{\substack{j=1\\ j\neq m}}^{r} A_{ji}A_{mi}\Big)\\ &= \frac{1}{n}\sum_{i=1}^{n}\Big(-1+2\sum_{j=1}^{r} A_{ji}A_{mi}\Big),\end{aligned}$$

while for $r+1 \leq m \leq k$,

$$\bar{Z}_m^{(2)} = \frac{1}{n}\sum_{i=1}^{n}\Big(1+2\sum_{j=1}^{r} A_{mi}A_{ji}\Big).$$

We abbreviate

$$\begin{aligned}E_1 &= \Big\{-r+2\sum_{m,j=1}^{r}\frac{1}{n}\sum_{i=1}^{n} A_{mi}A_{ji} \leq 0\Big\},\\ E_2 &= \Big\{\frac{r}{2}+\sum_{m=r+1}^{k}\sum_{j=1}^{r}\frac{1}{n}\sum_{i=1}^{n} A_{mi}A_{ji} \leq 0\Big\},\\ E_3 &= \Big\{\sum_{m=r+1}^{k}\sum_{j=1}^{r}\Big[1+\frac{1}{n}\sum_{i=1}^{n} A_{mi}A_{ji}\Big] \geq 0\Big\}.\end{aligned}$$

Clearly,

$$\mathbb{P}(E_1\cap E_2) = \mathbb{P}(E_1\cap E_2\cap E_3) + \mathbb{P}(E_1\cap E_2\cap E_3^c),$$

so that, according to the largest-exponent-wins principle,

$$-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2) = \min\Big\{-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3),\ -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3^c)\Big\}.$$

We wish to show that for $k\to\infty$ and r large,

$$-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2) = -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3).$$

In order to do so, we show that for all r

$$\liminf_{k\to\infty} -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3^c) \geq \frac{1}{2}\log\frac{3}{4}+\frac{1}{4}. \quad (55)$$

Furthermore, we will show that

$$-\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2) \leq -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1) \quad (56)$$

and as $r\to\infty$,

$$-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1) = \frac{1}{2}\log 2 - \frac{1}{4} + o(1). \quad (57)$$

This implies directly when r is large

$$\begin{aligned}&-\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3)\\ &\quad= -\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2)\\ &\quad\leq -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1)\\ &\quad= \frac{1}{2}\log 2 - \frac{1}{4} + o(1) = 0.096573\ldots + o(1).\end{aligned}$$

Indeed, when $x \leq \min\{y,z\}$ and $x < z$, then necessarily $x = y$. Equations (55), (56) and (57) also imply the statement in Theorem III.2(b) as we will show now. Taking $r\to\infty$ gives

$$-\lim_{r\to\infty}\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3) \leq \frac{1}{2}\log 2 - \frac{1}{4}.$$

The remainder of this proof therefore focuses on proving (55), (56) and (57). We prove (55) in the following lemma, (56) in Lemma V.3 and (57) in Lemma V.4.

Lemma V.2: As $k\to\infty$,

$$\liminf_{k\to\infty} -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3^c) \geq \frac{1}{2}\log\frac{3}{4}+\frac{1}{4}. \quad (58)$$

Proof: We bound the rate of interest from below by

$$\begin{aligned}-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_3^c) &= -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\sum_{m=r+1}^{k}\bar{Z}_m^{(2)} \leq 0\Big)\\ &= -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\frac{1}{n}\sum_{i=1}^{n}\sum_{m=r+1}^{k}\Big[1+2\sum_{j=0}^{r-1}A_{ji}A_{mi}\Big] < 0\Big).\end{aligned}$$

We can follow the proof of Section V-A with $A = \{0,\ldots,r-1\}$ and $B = \{r,\ldots,k-1\}$ (see e.g. (49) with $A\cap B = \varnothing$). This results in

$$-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_3^c) \geq \frac{1}{2}\log\big(1-\frac{r}{4(k-r)}\big)+\frac{1}{4} \geq \frac{1}{2}\log\frac{3}{4}+\frac{1}{4},$$

when $r \leq k/2$. □

The strategy of the proof is first to characterize the behaviour for $k \to \infty$ and then showing that $r \to \infty$ gives the desired result.

Lemma V.3: For r fixed,

$$\lim_{k\to\infty} - \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_2) \leq - \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(E_1). \quad (59)$$

Proof: We focus on

$$- \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Bigg(\frac{r}{2} + \sum_{m=r}^{k-1} \sum_{j=0}^{r-1} \frac{1}{n} \sum_{i=1}^{n} A_{mi} A_{ji} \leq 0, \\ -r + 2 \sum_{m,j=0}^{r} \frac{1}{n} \sum_{i=1}^{n} A_{mi} A_{ji} \leq 0 \Bigg).$$

Using Cramér's Theorem and invoking the notation $R = \{0, \ldots, r-1\}$ and $R_0^+ = \{r, \ldots, k-1\}$ gives that the rate above is given by

$$- \inf_{t_1, t_2 \leq 0} \log \mathbb{E} \left(e^{t_1 (r/2 + S_R S_{R_0^+}) t_2 (2S_R^2 - r)} \right).$$

Since S_R and $S_{R_0^+}$ are independent, we can bound the rate of interest from above by

$$- \inf_{t_1, t_2 \leq 0} \log \left(e^{t_1 r/2} \cosh(t_1 S_R)^{k-r} \mathbb{E}\, e^{t_2 (2S_R^2 - r)} \right). \quad (60)$$

It is sufficient to prove that for all $\delta > 0$, $|t_1| \leq \delta$ as $k \to \infty$. Indeed, since $\cosh x \geq 1$, the rate on the right hand-side is bounded of (60) from above by

$$\begin{aligned} &- \inf_{|t_1| \leq \delta, t_2 \leq 0} \log \left(e^{t_1 r/2} (\cosh(t_1 S_R))^{k-r} \mathbb{E}\, e^{t(2S_R^2 - r)} \right) \\ &\leq - \inf_{t_2 \leq 0} \log \left(e^{-\delta r/2} \mathbb{E}\, e^{t(2S_R^2 - r)} \right) \\ &= \frac{\delta r}{2} - \inf_{t_2} \log \mathbb{E}\, e^{t_2 (2S_R^2 - r)} = \frac{\delta r}{2} - \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(E_1). \end{aligned}$$

Since δ is arbitrary small and r is fixed, the desired statement then follows. It is hence sufficient to prove that under $|t_1| \geq \delta$, the infimum can not be attained for k sufficiently large. We prove this statement by contradiction. Assume that the infimum is attained for a $|t_1| \geq \delta$. Since r is odd, $\cosh(t_1 j) \geq \cosh(t_1) \geq e^{|t_1|}/2$, so that

$$e^{t_1 r/2} \cosh(t_1 j)^r \geq 2^{-r} e^{-|t_1| r/2} e^{r|t_1|} \geq 2^{-r}.$$

Substituting the result above gives

$$\begin{aligned} &- \inf_{|t_1| \geq \delta, t_2 \leq 0} \log \left(e^{t_1 r/2} (\cosh(t_1 j))^{k-r} \mathbb{E}\, e^{t(2S_R^2 - r)} \right) \\ &\leq - \inf_{|t_1| \geq \delta, t_2 \leq 0} \log \left(e^{-r \log 2} (\cosh(t_1 j))^{k-2r} \mathbb{E}\, e^{t(2S_R^2 - r)} \right) \\ &\leq - \inf_{t_2 \leq 0} \log \left(e^{-r \log 2} (\cosh \delta)^{k-2r} \mathbb{E}\, e^{t(2S_R^2 - r)} \right) \\ &= - \inf_{t_2 \leq 0} \log \mathbb{E}\, e^{t_2 (2S_R^2 - r)} + r \log 2 - (k - 2r) \log \cosh \delta. \end{aligned}$$

Since $\cosh(\delta) > 1$, the last term tends to $-\infty$ for $k \to \infty$. Thus, assuming $t_1 \geq \delta$ leads to $-\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(E_1) < 0$, which is a contradiction. □

To complete the proof, we let $r \to \infty$.

Lemma V.4: For $r \to \infty$,

$$- \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(E_1) \leq \frac{1}{2} \log 2 - \frac{1}{4} + o(1).$$

Proof: We show

$$\lim_{r\to\infty} - \inf_{t_2 \leq 0} \log \mathbb{E}\, e^{t_2 (2S_r^2 - r)} = \frac{1}{2} \log 2 - \frac{1}{4}.$$

The first step is to show that whenever $t_2 \leq -1/r$, $h(t_2) = \mathbb{E}\, e^{t_2 (2S_r^2 - r)} > 1$. We then can conclude that the infimum is not attained, since $h(0) = 1 < h(t_2)$ for $t_2 \leq -1/r$. Since $h(t_2)$ is a moment-generating function, it is log-convex, so that it suffices to show that $h(-1/r) > 1$. Indeed, for all $t_2 \leq -1/r$, there exists an $\alpha \in (0, 1]$, such that $\alpha t_2 = -1/r$. It now follows that

$$\begin{aligned} 1 < h(\alpha t_2) &= h\big(\alpha t_2 + (1-\alpha) \cdot 0\big) \\ &\leq \alpha h(t_2) + (1-\alpha) h(0) = \alpha h(t_2) + (1-\alpha), \end{aligned}$$

so that $h(t_2) > 1$. In order to prove that $h(-1/r) > 1$, we observe that

$$e^{(-\frac{1}{r})(2S_r^2 - r)} = e^{1 - 2(\frac{S_r}{\sqrt{r}})^2} \xrightarrow{\mathcal{D}} e^{1-2Z^2},$$

where $Z \stackrel{d}{=} \mathcal{N}(0,1)$. Furthermore, since $\mathbb{E}\, e^{(-\frac{\alpha}{r})(2S_r^2 - r)} \leq e^{\alpha} < \infty$ for all $\alpha > 1$, it then follows from [5] Example 7.10.15, that as $r \to \infty$,

$$\mathbb{E}\, e^{1 - 2(\frac{S_r}{\sqrt{r}})^2} \to \mathbb{E}\, e^{1-2Z^2} = \frac{e}{\sqrt{5}} \approx 1.21565 > 1.$$

Indeed, the infimum is not attained. Finally, using that $t_2 = -\beta/r$ for some $\beta \in [0, 1)$, we can again use the argument above to conclude that

$$\mathbb{E}\, e^{\beta - 2\beta(\frac{S_r}{\sqrt{r}})^2} \to \mathbb{E}\, e^{\beta - 2\beta Z^2} = \frac{e^{\beta}}{\sqrt{1 + 4\beta}}.$$

Minimizing over β gives $\beta^* = 1/4$, resulting in

$$\lim_{r\to\infty} - \inf_{t_2 \leq 0} \log \mathbb{E}\, e^{t_2 (2S_r^2 - r)} = -\log \frac{e^{1/4}}{\sqrt{2}} = \frac{1}{2} \log 2 - \frac{1}{4},$$

which is the desired result. □

This completes the proof of Theorem III.2(b) in the case of equal powers. □

VI. Conclusions

Investigated is a DS-CDMA system with HD-PIC. The model incorporates interfering users with unequal powers. The processing gain is denoted by n and the number of users by k. Large deviation theory is used to obtain qualitative statements concerning the performance of

the system. The exponential rate has been proven to be a good measure of quality. For the MF system, the exponential rate is asymptotically equivalent with the well-known signal-to-noise ratio. We note that for the HD-PIC system the signal-to-noise ratio is *not* a good measure, since there is no Gaussian behaviour for large k. However, the rate is not based on (Gaussian) assumptions and this allows us to draw qualitative conclusions concerning the behaviour of HD-PIC systems by using the rate of the BEP.

We have also investigated the asymptotic exponential rate of the BEP for multistage HD-PIC as the number of users tends to infinity. The asymptotic exponential rate of the BEP shows that HD-PIC keeps increasing performance as the number of users increases!

We have shown that the exponential rate of the BEP remains unchanged after a finite number of HD-PIC stages. The number of stages of HD-PIC after which this happens is 1 for 3 up to 9 users, and 2 for 10 up to 22 users when all powers are equal. This is the first analytical proof of this effect often observed in practice. We have shown that the exponential rate of the BEP remains uniformly positive as the number of users increases. Under mild conditions on the powers, we have identified the limiting exponential rate of the BEP after applying sufficiently stages of HD-PIC as $\frac{1}{2}\log 2 - \frac{1}{4}$ when the number of users increases.

The exponential rate have been used to give Chernoff bounds. The bound is not tight in the sense that it approximates the BEP with any desired precision. Instead, it gives an upper bound that is valid for all n and k, making it a robust upper bound. For s fixed, the Chernoff bound is valid as long as $k = o(\frac{\log n}{n})$, whereas for the optimal HD-PIC system, it is valid as long as $k = o(n)$.

The analysis in this paper answers many questions. Nevertheless, many other questions remain open. We try to summarize the most important ones:
1. What happens to the results when AWGN and unequal powers are considered in the case where we apply more than one stage of HD-PIC? We expect that the result will be generalized as in Theorem II.4. The proof of Theorem II.4 has been extended to include AWGN in [11].
2. Can we include AWGN in the proof for the optimal HD-PIC system? We believe that we can prove upper and lower bounds on the asymptotic exponential rate of the BEP using the methods in this paper, but that these bounds are not sharp. In order to prove asymptotics, new ideas will need to be developed.
3. It is well known that e^{-nI} is not a good approximation to p_n when $-\frac{1}{n}\log p_n \to I$. Can we compute the second order asymptotics, e.g. in the case when we apply one-stage on HD-PIC?
4. Can we say more about how many iterations of HD-PIC are necessary to obtain the optimal system? For instance, can we prove Conjecture III.5?

Appendix

A. Proof of Theorem III.2(b) for unequal powers

We recall that we have the following condition on the powers $(P_1, \ldots, P_k)$:

$(\mathcal{P}_1)$ There exists a $\delta > 0$ s. t. $\sharp\{j : P_j \in [\delta, 1/\delta]\} \to \infty$,
$(\mathcal{P}_2)$ $\lim_{k\to\infty} kP^{-1} < \infty$,
$(\mathcal{P}_3)$ $k^{-1}P^{\max} \to 0$,
$(\mathcal{P}_4)$ $kP^{\min} \to \infty$,

where $P^{\max} = \max_m P_m, P^{\min} = \min_m P_m$.

The result of the statement is that under these power conditions, we have that

$$\lim_{k\to\infty} H_k^{(s_k)} = \frac{1}{2}\log 2 - \frac{1}{4}.$$

Theorem III.2(a) states that $H_k^{(s_k)} \geq \frac{1}{2}\log 2 - \frac{1}{4}$, so that it is sufficient to prove

$$\lim_{k\to\infty} H_k^{(s_k)} \leq \frac{1}{2}\log 2 - \frac{1}{4}.$$

We will prove the theorem similar to the way we have proven the result for equal powers. However, the derivations will be more technical.

Similarly to the case in which all powers are equal, we will focus on the event $\{R_2 = R_1 = R\}$. This specifies R_σ for all σ. We will show the theorem, using a symmetrizing argument. More precisely, we replace different powers by the same value. In this case, we can use exchangeability, together with convexity arguments in order to simplify the analysis. We intend to follow the strategy in Section V-C as closely as possible.

We take $R \subset \{j : P_j \in [\delta, 1/\delta]\}$ and we assume that $|R|$ is fixed and odd and k is large.

Our starting point is the probability

$$\begin{aligned}\mathbb{P}(R_2 = R_1 = R) &= \mathbb{P}\Big(\max_{m\in R} Z_m^{(1)} \leq 0,\ \min_{m\in R^c} Z_m^{(1)} \geq 0,\\ &\qquad \max_{m\in R} \bar{Z}_m^{(2)} \leq 0,\ \min_{m\in R^c} \bar{Z}_m^{(2)} \geq 0\Big),\end{aligned}$$

where

$$Z_m^{(1)} = \frac{1}{n}\sum_{i=1}^{n}\Big(P_m^{1/2} + \sum_{\substack{j=1\\ j\neq m}}^{k} P_j^{1/2} A_{ji}A_{mi}\Big), \tag{61}$$

$$\bar{Z}_m^{(2)} = \frac{1}{n}\sum_{i=1}^{n}\Big(P_m^{1/2} + 2\sum_{\substack{j\in R\\ j\neq m}} P_j^{1/2} A_{ji}A_{mi}\Big). \tag{62}$$

Since $\mathbb{P}(A \leq 0, B \leq 0) \geq \mathbb{P}(A - B/2 \leq 0, B \leq 0)$, we can bound the probability from below by

$$\begin{aligned}\mathbb{P}\Big(&\max_{m\in R} Z_m^{(1)} - \frac{1}{2}\bar{Z}_m^{(2)} \leq 0,\ \min_{m\in R^c} Z_m^{(1)} - \frac{1}{2}\bar{Z}_m^{(2)} \geq 0,\\ &\max_{m\in R} \bar{Z}_m^{(2)} \leq 0,\ \min_{m\in R^c} \bar{Z}_m^{(2)} \geq 0\Big),\end{aligned}$$

where for $m \in R$,

$$\begin{aligned} Z_m^{(1)} - \frac{1}{2}\bar{Z}_m^{(2)} &= \frac{1}{n}\sum_{i=1}^n \Big(\frac{1}{2}P_m^{1/2} + \sum_{j\in R^c} P_j^{1/2} A_{ji}A_{mi}\Big), \\ \bar{Z}_m^{(2)} &= \frac{1}{n}\sum_{i=1}^n \Big(P_m^{1/2} + 2\sum_{\substack{j\in R\\ j\neq m}} P_j^{1/2} A_{ji}A_{mi}\Big), \end{aligned}$$

while for $m \in R^c$

$$\begin{aligned} Z_m^{(1)} - \frac{1}{2}\bar{Z}_m^{(2)} &= \frac{1}{n}\sum_{i=1}^n \Big(\frac{1}{2}P_m^{1/2} + \sum_{\substack{j\in R^c\\ j\neq m}} P_j^{1/2} A_{ji}A_{mi}\Big) \\ \bar{Z}_m^{(2)} &= \frac{1}{n}\sum_{i=1}^n \Big(P_m^{1/2} + 2\sum_{j\in R} P_j^{1/2} A_{ji}A_{mi}\Big). \end{aligned}$$

We can further bound the probability of interest from below by the probability of the intersection of the events

$$\begin{aligned} &\max_{m\in R, j\in R^c} \frac{1}{n}\sum_{i=1}^n A_{mi}A_{ji} \le -\frac{(P_{R^c}^{\max}P_R^{\max})^{1/2}}{2P_{R^c}}, \\ &\max_{m,j\in R,\, j\neq m} \frac{1}{n}\sum_{i=1}^n A_{mi}A_{ji} \le -\frac{(P_R^{\max})^{1/2}}{2(|R|-1)(P_R^{\min})^{1/2}}, \quad (63) \\ &\min_{m,j\in R^c, j\neq m} \frac{1}{n}\sum_{i=1}^n A_{mi}A_{ji} \ge 0, \\ &\min_{m\in R, j\in R^c} \frac{1}{n}\sum_{i=1}^n A_{mi}A_{ji} \ge -\frac{(P_{R^c}^{\min})^{1/2}}{2|R|(P_R^{\max})^{1/2}}, \end{aligned}$$

where $P_R^{\max} = \max_{m\in R} P_m$, $P_R^{\min} = \min_{m\in R} P_m$.

Indeed, using $\sum_{m\in R^c} P_m^{1/2} \ge \frac{P_{R^c}}{(P_{R^c}^{\max})^{1/2}}$, we have for $m \in R$

$$\begin{aligned} Z_m^{(1)} - \frac{\bar{Z}_m^{(2)}}{2} &= \frac{1}{n}\sum_{i=1}^n \Big(\frac{1}{2}P_m^{1/2} + \sum_{j\in R^c} P_j^{1/2} A_{ji}A_{mi}\Big) \\ &\le \frac{1}{2}(P_R^{\max})^{1/2} - \sum_{j\in R^c} P_j^{1/2}\frac{(P_{R^c}^{\max}P_R^{\max})^{1/2}}{2P_{R^c}} \\ &\le \frac{1}{2}(P_R^{\max})^{1/2} - \frac{(P_R^{\max})^{1/2}}{2} = 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} \bar{Z}_m^{(2)} &= \frac{1}{n}\sum_{i=1}^n \Big(P_m^{1/2} + 2\sum_{\substack{j\in R\\ j\neq m}} P_j^{1/2} A_{ji}A_{mi}\Big) \\ &\le (P_R^{\max})^{1/2} - 2\sum_{\substack{j\in R\\ j\neq m}} (P_R^{\min})^{1/2}\frac{(P_R^{\max})^{1/2}}{2(|R|-1)(P_R^{\min})^{1/2}} \\ &\le (P_R^{\max})^{1/2} - \frac{(|R|-1)(P_R^{\max}P_R^{\min})^{1/2}}{(|R|-1)(P_R^{\min})^{1/2}} = 0. \end{aligned}$$

For $m \in R^c$, similar arguments give the desired result:

$$\begin{aligned} Z_m^{(1)} - \frac{\bar{Z}_m^{(2)}}{2} &= \frac{1}{n}\sum_{i=1}^n \Big(\frac{1}{2}P_m^{1/2} + \sum_{j\in R^c} P_j^{1/2} A_{ji}A_{mi}\Big) \\ &\ge \frac{1}{2}P_m^{1/2} \ge 0, \\ \bar{Z}_m^{(2)} &= \frac{1}{n}\sum_{i=1}^n \Big(P_m^{1/2} + 2\sum_{j\in R} P_j^{1/2} A_{ji}A_{mi}\Big) \\ &\ge (P_{R^c}^{\min})^{1/2} - 2\sum_{j\in R}(P_R^{\max})^{1/2}\frac{(P_{R^c}^{\min})^{1/2}}{2|R|(P_R^{\max})^{1/2}} = 0. \end{aligned}$$

In order to obtain a positive probability, we must assure that

$$-\frac{(P_{R^c}^{\max}P_R^{\max})^{1/2}}{2P_{R^c}} \ge -\frac{(P_{R^c}^{\min})^{1/2}}{2|R|(P_R^{\max})^{1/2}}, \quad (64)$$

or equivalently,

$$\frac{(P_R^{\max})^2 P_{R^c}^{\max}|R|^2}{P_{R^c}^{\min}P_{R^c}^2} \le 1.$$

We use $P_{R^c}^{\max} \le P^{\max}$, together with $P_{R^c}^{\min} \ge P^{\min}$ and $P_R^{\max} \le 1/\delta$, to obtain from $(\mathcal{P}_2), (\mathcal{P}_3)$ and $(\mathcal{P}_4)$ that

$$\begin{aligned} \frac{(P_R^{\max})^2 P_{R^c}^{\max}|R|^2}{P_{R^c}^{\min}P_{R^c}^2} &\le \delta^{-2}|R|^2\frac{P^{\max}}{P^2P^{\min}} \\ &= \delta^{-2}|R|^2\frac{k^2}{P^2}\frac{P^{\max}}{k}\frac{1}{kP^{\min}}, \end{aligned}$$

which tends to zero for $k \to \infty$. Hence, the inequality holds, since $|R|$ is fixed and k is sufficiently large.

We have now symmetrized the problem. The next step is to observe that the events are exchangeable for m, j. Together with convexity of the rate function, we can replace the max and min by $\sum$, as we will now show. This is a standard large deviation argument.

We write $\vec{a} \mapsto J_k(\vec{a})$ to be the rate function of the cross correlations $\vec{W} = \big(W_{jm}\big)_{1\le j<m\le k}$, where

$$W_{jm} = \frac{1}{n}\sum_{i=1}^n A_{ij}A_{im}. \quad (65)$$

This means that, for any set A, we let

$$J_k(A) = -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(\vec{W}\in A), \quad (66)$$

and we can compute $J_k(A)$ using

$$J_k(A) = \min_{\vec{a}\in A} J_k(\vec{a}). \quad (67)$$

This rate function $\vec{a} \mapsto J_k(\vec{a})$ exists by Cramér's Theorem, is convex by [8], Thm. III.27, and is minimal for $\vec{a} = \vec{0}$, where it takes the value $J_k(\vec{0}) = 0$. We note that

$$H_k^{(s_k)} \le \min_{\vec{a}\in A} J_k(\vec{a}), \quad (68)$$

where A is defined to be the set of vectors such that the coordinates satisfy the inequalities in (63). We now let

$$E_1 = \Big\{\sum_{\substack{m,j\in R\\ m\neq j}}\frac{1}{n}\sum_{i=1}^n A_{mi}A_{ji} \le -|R|\frac{(P_R^{\max})^{1/2}}{2(P_R^{\min})^{1/2}}\Big\},$$

$$E_2 = \Big\{ \sum_{\substack{m\in R\\ j\in R^c}} \frac{1}{n}\sum_{i=1}^n A_{mi}A_{ji} \le -|R||R^c| \frac{(P_{R^c}^{\max}P_R^{\max})^{1/2}}{2P_{R^c}} \Big\},$$

$$E_3 = \Big\{ \sum_{\substack{m,j\in R^c\\ m\neq j}} \frac{1}{n}\sum_{i=1}^n A_{mi}A_{ji} \ge 0 \Big\}, \tag{69}$$

$$E_4 = \Big\{ \sum_{\substack{m\in R\\ j\in R^c}} \frac{1}{n}\sum_{i=1}^n A_{mi}A_{ji} \ge -|R||R^c| \frac{(P_{R^c}^{\min})^{1/2}}{2|R|(P_R^{\max})^{1/2}} \Big\}.$$

Then we see that

$$-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3\cap E_4) = \min_{\vec{b}\in B} J_k(\vec{b}), \tag{70}$$

where B is the set such that the inequalities in (69) hold. Note that $A\subset B$, and we wish to show that the minimum in (70) is attained for a $\vec{b}^*\in A$. That proves that the rate on the r.h.s. of (68) equals the rate of $\mathbb{P}(E_1\cap E_2\cap E_3\cap E_4)$.

Let $\vec{b}^*$ be the minimizer of (70). Note that by exchangeability, if we interchange the elements in R and in R^c, we obtain the same rate. We write $\mathcal{P}_R$ for the set of permutations of the elements in R and $\mathcal{P}_{R^c}$ the set of permutations of the elements in R^c, and we define $\mathcal{P}=\mathcal{P}_R\times\mathcal{P}_{R^c}$. Hence, for a $p\in\mathcal{P}$, we have $p=(p_1,p_2)$ where $p_1\in\mathcal{P}_R, p_2\in\mathcal{P}_{R^c}$.

Then we obtain that

$$\min_{\vec{b}\in B} J_k(\vec{b}) = \frac{1}{|\mathcal{P}|}\sum_{p\in\mathcal{P}} J_k(p(\vec{b}^*)). \tag{71}$$

By convexity, we obtain that

$$\min_{\vec{b}\in B} J_k(\vec{b}) \ge J_k\Big(\frac{1}{|\mathcal{P}|}\sum_{p\in\mathcal{P}} p(\vec{b}^*)\Big). \tag{72}$$

We have ended up with the vector $\vec{a}^* = \frac{1}{|\mathcal{P}|}\sum_{p\in\mathcal{P}} p(\vec{b}^*)$. This vector takes only three different values, as we will explain now. Indeed, let $j\neq j'$ be any two indices in R, and $m\neq m'$ any two indices in R^c. Then a^*_{nl} takes the value $a^*_{jj'}$ for $n,l\in R$, the value a^*_{jm} for $n\in R, l\in R^c$ and $a^*_{mm'}$ for any $n,l\in R^c$. Hence, we must have that each of the elements satisfies the inequalities in (63), so that $\vec{a}^*\in A$. This completes the proof.

We continue to determine the rate of $\mathbb{P}(E_1\cap E_2\cap E_3\cap E_4)$. We write

$$\mathbb{P}(E_1\cap E_2\cap E_3) = \mathbb{P}(E_1\cap E_2\cap E_3\cap E_4) + \mathbb{P}(E_1\cap E_2\cap E_3\cap E_4^c).$$

By the largest-exponent-wins principle, we have

$$\begin{aligned} &-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3)\\ &\quad= \min\Big\{ -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3\cap E_4),\\ &\qquad -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3\cap E_4^c)\Big\}. \end{aligned}$$

We wish to show that for $k\to\infty$ and $|R|$ large,

$$\begin{aligned} &-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3) \\ &\quad= -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3\cap E_4). \end{aligned} \tag{73}$$

In order to do so, we show that for all R

$$\liminf_{k\to\infty} -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3\cap E_4^c) \ge \frac{1}{2}\log\frac{3}{4}+\frac{1}{4}. \tag{74}$$

Furthermore, we will show that

$$-\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3) = -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1) \tag{75}$$

and, when $|R|\to\infty$

$$-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1) \le \frac{1}{2}\log 2 - \frac{1}{4} + o(1). \tag{76}$$

This implies directly that when $|R|$ is large

$$\begin{aligned} &-\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3\cap E_4)\\ &= -\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3)\\ &\quad\le -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1) \le \frac{1}{2}\log 2 - \frac{1}{4} + o(1). \end{aligned}$$

Indeed, we note that $\frac{1}{2}\log\frac{3}{4}+\frac{1}{4} > \frac{1}{2}\log 2 - \frac{1}{4}$. When $x\le\min\{y,z\}$ and $x<z$, then necessarily $x=y$. We will apply this principle with

$$\begin{aligned} x &= -\lim_{|R|\to\infty}\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3),\\ y &= -\lim_{|R|\to\infty}\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3\cap E_4),\\ z &= -\lim_{|R|\to\infty}\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3\cap E_4^c). \end{aligned}$$

Indeed, (75) and (76) show that

$$x = -\lim_{|R|\to\infty}\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3) \le \frac{1}{2}\log 2 - \frac{1}{4}.$$

Equation (73) shows that $z>x$, so that necessarily $y=x$. Moreover, (76) shows that $x\le\frac{1}{2}\log 2-\frac{1}{4}$. This proves the claim, since it gives that

$$y = -\lim_{|R|\to\infty}\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3) \le \frac{1}{2}\log 2 - \frac{1}{4}. \tag{77}$$

The remainder of this proof therefore focuses on proving (74), (75) and (76). We prove (74) in the following lemma, (75) in Lemma A.2 and (76) in Lemma A.3.

Lemma A.1: For k sufficiently large, under power conditions $(\mathcal{P}_1)$ and $(\mathcal{P}_4)$,

$$-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3\cap E_4^c) \ge \frac{1}{2}\log\frac{3}{4}+\frac{1}{4}.$$

Proof: We have

$$
\begin{aligned}
&-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3\cap E_4^c)\\
&\geq -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_4^c)\\
&=\sup_{t_4\leq 0}-\log\left\{\mathbb{E}\, e^{t_4|R^c|\frac{(P_{R^c}^{\min})^{1/2}}{2(P_R^{\max})^{1/2}}+t_4\sum_{m\in R,j\in R^c}A_{m1}A_{j1}}\right\}.
\end{aligned}
$$

Recall the definition of S_R, which is $S_R=\sum_{m\in R}A_{m1}$. Since R^c and R are by definition disjoint, and $\cosh(t)\leq e^{t^2/2}$, it follows that

$$
\mathbb{E}\, e^{t_4 S_{R^c}S_R}=\mathbb{E}\,(\cosh(t_4S_R))^{|R^c|}\leq \mathbb{E}\, e^{|R^c|t_4^2S_R^2/2}.
$$

Furthermore, by (40), this is further bounded from above by $(1-t_4^2|R||R^c|)^{-1/2}$, as long as $t_4^2|R||R^c|\leq 1$. Note that conditions $(\mathcal{P}_1)$ and $(\mathcal{P}_4)$ imply that $P_{R^c}^{\min}|R^c|\geq P_R^{\max}|R|$ when k is sufficiently large, because $P_R^{\max}|R|$ is bounded and $|R^c|=k(1-o(1))$. Substituting this yields

$$
\begin{aligned}
&-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3\cap E_4^c)\\
&\geq \sup_{t_4\leq 0}-t_4\frac{|R^c|(P_{R^c}^{\min})^{1/2}}{2(P_R^{\max})^{1/2}}+\frac{1}{2}\log(1-t_4^2|R||R^c|)\\
&\geq \sup_{t_4\leq 0}-t_4\frac{|R|^{1/2}|R^c|^{1/2}}{2}+\frac{1}{2}\log(1-t_4^2|R||R^c|).
\end{aligned}
$$

We substitute $t_4=-\frac{1}{2|R|^{1/2}|R^c|^{1/2}}$ to obtain a lower bound. Observe that the substituted t_4 indeed fulfills $t_4^2|R||R^c|\leq 1$. Substitution of the above result yields

$$
-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3\cap E_4^c)\geq\frac{1}{2}\log\frac{3}{4}+\frac{1}{4}.
$$

□

The second lemma is more involved than the one above.

Lemma A.2: For $|R|$ fixed and odd, under condition $(\mathcal{P}_1-\mathcal{P}_4)$,

$$
-\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3)\leq-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1).
$$

Proof: Cramér's Theorem gives that

$$
-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1\cap E_2\cap E_3)=\sup_{\underline{t}\in D}-\log\mathbb{E}\, e^{Y_1+Y_2+Y_3},
$$

where $\underline{t}=(t_1,t_2,t_3)$, $D=\{\underline{t}:t_1,t_3\leq 0,\ t_2\geq 0\}$ and

$$
\begin{aligned}
Y_1 &= t_1\left(\frac{|R|(P_R^{\max})^{1/2}}{2(P_R^{\min})^{1/2}}+\sum_{\substack{m,j\in R\\ m\neq j}}A_{m1}A_{j1}\right),\\
Y_2 &= t_2\left(\sum_{\substack{m,j\in R^c\\ m\neq j}}A_{m1}A_{j1}\right),\\
Y_3 &= t_3\left(-|R||R^c|\frac{(P_{R^c}^{\max}P_R^{\max})^{1/2}}{2P_{R^c}}+\sum_{m\in R,j\in R^c}A_{m1}A_{j1}\right).
\end{aligned}
$$

We rewrite this, invoking $S_A=\sum_{j\in A}A_{j1}$ as

$$
\begin{aligned}
Y_1 &= t_1\left(|R|\left(\frac{(P_R^{\max})^{1/2}}{2(P_R^{\min})^{1/2}}-1\right)+S_R^2\right),\\
Y_2 &= t_2\left(-|R^c|+S_{R^c}^2\right),\\
Y_3 &= t_3\left(-|R||R^c|\frac{(P_{R^c}^{\max}P_R^{\max})^{1/2}}{2P_{R^c}}+S_RS_{R^c}\right).
\end{aligned}
\tag{78}
$$

It is sufficient to prove that for $k\to\infty$, for all t_1,t_2

$$
\mathbb{E}\, e^{Y_1+Y_2+Y_3}\geq\mathbb{E}\, e^{Y_1}(1-o(1)).
$$

Indeed, then

$$
\begin{aligned}
-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1) &= \sup_{t_1\leq 0}-\log\mathbb{E}\, e^{Y_1}\\
&\leq \sup_{t_1\leq 0}\{-\log\mathbb{E}\, e^{Y_1+Y_2+Y_3}\}+o(1)\\
&\leq \sup_{\underline{t}\in D}\{-\log\mathbb{E}\, e^{Y_1+Y_2+Y_3}\}+o(1),
\end{aligned}
$$

so that for $k\to\infty$, the statement of the lemma follows. We abbreviate

$$
h(t_1,t_2,t_3)=\mathbb{E}\, e^{Y_1+Y_2+Y_3}
$$

We will prove that

$$
h(t_1,t_2,t_3)\geq\mathbb{E}\, e^{Y_1}(1-o(1)),\qquad k\to\infty.
\tag{79}
$$

Instead of maximizing $-\log h(t_1,t_2,t_3)$, we will minimize $h(t_1,t_2,t_3)$. To do this, we will define an appropriate ellipse $\mathcal{E}$ with elements t_2,t_3 with $(0,0)\in\mathcal{E}^0$, the interior of $\mathcal{E}$. In order to show that the supremum of $-\log h(t_1,t_2,t_3)$ is attained in $\mathcal{E}^0$, it is sufficient to show that on the boundary of the ellipse $h(t_1,t_2,t_3)>\mathbb{E}\, e^{Y_1}$. Since $h(t_1,0,0)=\mathbb{E}\, e^{Y_1}$ and h is log-convex (since it is a moment-generating function), we can then conclude that $h(t_1,t_2,t_3)>\mathbb{E}\, e^{Y_1}$ outside the ellipse. Indeed, whenever $(t_2,t_3)\notin\mathcal{E}$, there exists a unique $0<\alpha<1$ such that $\alpha(t_2,t_3)\in\partial\mathcal{E}$. From convexity of h and $h\big(t_1,\alpha(t_2,t_3)\big)>\mathbb{E}\, e^{Y_1}$ it follows that

$$
\begin{aligned}
\mathbb{E}\, e^{Y_1} &< h(\alpha(t_1,t_2),t_3)=h(\alpha(t_1,t_2)+(1-\alpha)(0,0),t_3)\\
&\leq \alpha h(t_1,t_2,t_3)+(1-\alpha)h(0,0,t_3)\\
&= \alpha h(t_1,t_2,t_3)+(1-\alpha)\mathbb{E}\, e^{Y_1}.
\end{aligned}
$$

It immediately follows that $h(t_1,t_2,t_3)>\mathbb{E}\, e^{Y_1}$, so that (t_1,t_2,t_3) cannot be the minimizer of $h(t_1,t_2,t_3)$ over D. When $(t_2,t_3)\in\mathcal{E}^0$, we can prove $h(t_1,t_2,t_3)\geq\mathbb{E}\, e^{Y_1}(1-o(1))$ for $k\to\infty$.

Since $e^x\geq 1+x+x^2/2+x^3/6$, we have

$$
\mathbb{E}\, e^{Y_1+Y_2+Y_3}\geq\mathbb{E}\left(e^{Y_1}(1+Y_2+Y_3+\frac{1}{2}(Y_2+Y_3)^2+\frac{1}{6}(Y_2+Y_3)^3)\right).
$$

We will first calculate the required moments. Since $|R|$ is odd, we have that $S_R^2\geq 1$, so that

$$
\mathbb{E}\, e^{Y_1}Y_2 = 0,\qquad \mathbb{E}\, e^{Y_1}Y_2Y_3=0,
$$

$$\begin{aligned}
\mathbb{E}\, e^{Y_1} Y_3 &= \mathbb{E}\, e^{Y_1}\Big(-t_3|R||R^c|\frac{(P^{\max}_{R^c}P^{\max}_R)^{1/2}}{2P_{R^c}}\Big),\\
\mathbb{E}\, e^{Y_1} Y_2^2 &= \mathbb{E}\, e^{Y_1} t_2^2(3|R^c|^2+2|R^c|) \geq \mathbb{E}\, e^{Y_3} 3t_2^2|R^c|^2,\\
\mathbb{E}\, e^{Y_1} Y_3^2 &= \mathbb{E}\, e^{Y_1}\Big(t_3^2\frac{|R|^2|R^c|^2P^{\max}_{R^c}P^{\max}_R}{4P^2_{R^c}} + t_3^2 S_R^2|R^c|\Big)\\
&\geq \mathbb{E}\, e^{Y_1} t_3^2|R^c|. \qquad (80)
\end{aligned}$$

Concerning $\mathbb{E}\, e^{Y_1}(Y_2+Y_3)^3$,

$$\mathbb{E}\, e^{Y_1}(Y_2+Y_3)^3 \geq -4\mathbb{E}\, e^{Y_1}(|Y_2^3|+|Y_3^3|).$$

Using (78) and $|S_R| \leq |R|$ results in

$$|Y_3^3| \leq 4|t_3|^3\Big(|R|^3|R^c|^3\frac{(P^{\max}_{R^c})^{3/2}(P^{\max}_R)^{3/2}}{8P^3_{R^c}} + |R|^3|S^3_{R^c}|\Big),$$

and

$$|Y_2^3| \leq 4t_2^3\Big(|R^c|^3 + |S^6_{R^c}|\Big),$$

so that (using $\mathbb{E}\,|S^m_{R^c}| \leq C|R^c|^{m/2}$ for some $C<\infty$)

$$\begin{aligned}
&-\mathbb{E}\, e^{Y_1}(|Y_2^3|+|Y_3^3|)\\
&\quad\geq -\mathbb{E}\, e^{Y_1}\Big(C|t_3|^3|R|^3|R^c|^3\frac{(P^{\max}_{R^c})^{3/2}(P^{\max}_R)^{3/2}}{P^3_{R^c}}\\
&\qquad + C|t_3|^3|R|^3|R^c|^{3/2} + Ct_2^3|R^c|^3\Big).
\end{aligned}$$

We next use the moments of (80) to obtain

$$\begin{aligned}
&h(t_1,t_2,t_3)\\
&\quad\geq \mathbb{E}\, e^{Y_1}\Big(1+2|R^c|^2t_2^2 - t_3|R||R^c|\frac{(P^{\max}_{R^c}P^{\max}_R)^{1/2}}{2P_{R^c}}\\
&\qquad + \frac{t_3^2|R^c|}{2} + e(t_2,t_3)\Big) \qquad (81)\\
&\quad= \mathbb{E}\, e^{Y_1}\Big(1-\mathcal{H}+\frac{3|R^c|^2}{2}t_2^2 + \frac{|R^c|}{2}(t_3-t_3^*)^2 + e(t_2,t_3)\Big),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{H} &= \frac{|R|^2|R^c|P^{\max}_{R^c}P^{\max}_R}{8P^2_{R^c}},\\
t_3^* &= -\frac{|R|(P^{\max}_{R^c}P^{\max}_R)^{1/2}}{2P_{R^c}},\\
e(t_1,t_2) &= -Ct_2^3|R^c|^3 - C|t_1|^3|R|^3|R^c|^{3/2}\\
&\quad -C|t_1|^3|R|^3|R^c|^3\frac{(P^{\max}_{R^c})^{3/2}(P^{\max}_R)^{3/2}}{P^3_{R^c}}.
\end{aligned}$$

According to $(\mathcal{P}_1)$, we have that $P_R \leq P^{\max}_R|R| \leq \delta^{-1}|R|$ is bounded, so that $P = P_{R^c}+P_R = P_{R^c}(1-o(1))$. This implies, using $(\mathcal{P}_1-\mathcal{P}_4)$, that as $k\to\infty$,

$$\mathcal{H} \leq \frac{|R|^2kP^{\max}\delta^{-1}}{8P^2(1-o(1))} = \frac{|R|^2\delta^{-1}}{8}\frac{P^{\max}}{k}\frac{k^2}{P^2}(1+o(1)) \to 0. \qquad (82)$$

We define the ellipse $\mathcal{E}$ as

$$\mathcal{E} = \Big\{(t_2,t_3) : \frac{3|R^c|^2}{2}t_2^2 + \frac{|R^c|}{2}(t_3-t_3^*)^2 \leq 2\mathcal{H}\Big\}.$$

To show that on $\partial\mathcal{E}$, $h(t_1,t_2,t_3) > \mathbb{E}\, e^{Y_1}$, it is sufficient to show that for every $|R|$ fixed

$$|e(t_2,t_3)| = o(\mathcal{H}). \qquad (83)$$

Indeed, on $\partial\mathcal{E}$, according to (82)

$$h(t_1,t_2,t_3) = \mathbb{E}\, e^{Y_1}\Big(1-\mathcal{H}+2\mathcal{H}+e(t_2,t_3)\Big) > \mathbb{E}\, e^{Y_1}.$$

Therefore, we focus on $e(t_2,t_3)$ for $(t_2,t_3)\in\partial\mathcal{E}$. Note that for $(t_2,t_3)\in\mathcal{E}$ by the triangle inequality,

$$\begin{aligned}
|t_2| &\leq \frac{1}{\sqrt{6}}\frac{|R|(P^{\max}_{R^c}P^{\max}_R)^{1/2}}{|R^c|^{1/2}P_{R^c}}, \qquad (84)\\
|t_3| &\leq \frac{|R|(P^{\max}_{R^c}P^{\max}_R)^{1/2}}{P_{R^c}}. \qquad (85)
\end{aligned}$$

Substituting this into the formula for $e(t_2,t_3)$ gives

$$\begin{aligned}
|e(t_2,t_3)| &\leq C\frac{|R|^3(P^{\max}_{R^c}P^{\max}_R)^{3/2}}{P^3_{R^c}}\\
&\quad\times\Big(|R|^3|R^c|^3\frac{(P^{\max}_{R^c}P^{\max}_R)^{3/2}}{P^3_{R^c}} + |R|^3|R^c|^{3/2}\Big)\\
&\quad + C\frac{|R|^3(P^{\max}_{R^c}P^{\max}_R)^{3/2}}{|R^c|^{3/2}P_{R^c}}|R^c|^3\\
&\leq C(\mathcal{H}^3+|R|^3\mathcal{H}^{3/2}) + C\mathcal{H}^{3/2}.
\end{aligned}$$

Since $\mathcal{H}=o(1)$ according to (82) and $|R|$ is fixed, indeed $|e(t_2,t_3)| = o(\mathcal{H})$. This proves that

$$h(t_1,t_2,t_3) > \mathbb{E}\, e^{Y_1}, \qquad (t_2,t_3)\notin\mathcal{E}. \qquad (86)$$

Finally, for $(t_2,t_3)\in\mathcal{E}^0$, we use the results above to obtain

$$h(t_1,t_2,t_3) \geq \mathbb{E}\, e^{Y_1}\Big(1-\mathcal{H}+\frac{3|R^c|^2}{2}t_2^2+\frac{|R^c|}{2}(t_3-t_3^*)^2-o(\mathcal{H})\Big).$$

Minimization over t_2 and t_3 of the right hand side clearly gives $t_2=0$ and $t_3=t_3^*$, so that for all $(t_2,t_3)\in\mathcal{E}$

$$h(t_1,t_2,t_3) \geq \mathbb{E}\, e^{Y_3}\Big(1-\mathcal{H}-o(\mathcal{H})\Big).$$

Finally, since $\mathcal{H}=o(1)$, $1-\mathcal{H}-o(\mathcal{H}) = 1-o(1)$. Together with (86), this proves the claim in (79). □

Lemma A.3: For $|R|\to\infty$, under condition $(\mathcal{P}_1)$,

$$-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1) \leq \frac{1}{2}\log 2 - \frac{1}{4} + o(1).$$

Proof: Clearly,

$$-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(E_1) = \sup_{t_1\leq 0}\Big\{-\log\mathbb{E}\, e^{Y_1}\Big\},$$

where

$$Y_1 = t_1\Big(|R|\Big(\frac{(P^{\max}_R)^{1/2}}{2(P^{\min}_R)^{1/2}}-1\Big) + S_R^2\Big).$$

When we prove

$$\lim_{|R|\to\infty} \sup_{t_1\le 0}\left\{-\log \mathbb{E}\, e^{Y_1}\right\} \le \frac{1}{2}\log 2 - \frac{1}{4}, \qquad (87)$$

we conclude that for $|R|$ sufficiently large,

$$-\lim_{n\to\infty}\frac{1}{n}\log \mathbb{P}(E_1) \le \frac{1}{2}\log 2 - \frac{1}{4} + o(1).$$

Thus, it remains to show (87). We denote

$$j_R = \frac{1}{2}\frac{(P_R^{\max})^{1/2}}{(P_R^{\min})^{1/2}} - 1/2.$$

The crucial step in this part of the proof is that we are still allowed to choose R. So far, we have only assumed that $R \subset \{j : P_j \in [\delta, 1/\delta]\}$ and that $|R|$ is odd. By condition $(\mathcal{P}_1)$, we can choose R such that whenever $|R| = r$, there exists a $\bar{P}$ with

$$\max_{m\in R}|P_m - \bar{P}| \le \frac{\bar{P}}{r}.$$

Thus, we know

$$P_R^{\max} \le \bar{P}(1 + 1/|R|)\bar{P} \qquad \text{and} \qquad P_R^{\min} \ge \bar{P}(1 - 1/|R|)\bar{P},$$

so that for $|R| \to \infty$,

$$0 \le j_R \le \frac{1}{2}\sqrt{\frac{1 + 1/|R|}{(1 - 1/|R|)}} - \frac{1}{2} = \sqrt{1 + \frac{2}{|R| - 1}} - 1 \to 0.$$

We next write

$$\begin{aligned} &-\inf_{t_1\le 0}\log \mathbb{E}\,\exp\left(t_1\left(\frac{P_R^{\max}}{2P_R^{\min}} - \frac{1}{2}\right) + t_1 S_R^2\right) \\ &\quad = -\inf_{t_1\le 0}\log \mathbb{E}\,\exp\left(t_1 j_R + t_1(S_R^2 - |R|/2)\right). \end{aligned}$$

We can now follow the argument in the proof of Lemma V.4, together with the observation that whenever $X_n \xrightarrow{\mathcal{D}} X$ and $c_n \to c$ that also $c_n X_n \xrightarrow{\mathcal{D}} cX$. Indeed, in all weak convergence arguments $-1/|R| \le t_1 \le 0$, so that we are allowed to replace $e^{t_1 j_R}$ by its limit 1. This proves that indeed for $|R|$ large

$$-\inf_{t_1\le 0}\log \mathbb{E}\, e^{Y_1} \le \frac{1}{2}\log 2 - \frac{1}{4} + o(1).$$

□

This completes the proof of Theorem III.2(b) in the case of unequal powers. □

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