Random Graphs and Complex Networks

Kaleidoscoopdag, Leiden

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Joint work with
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Six Degrees of Separation

"Everybody on this planet is separated only by six other people. Six degrees of separation. Between us and everybody else on this planet. The president of the United States. A gondolier in Venice... It's not just the big names. It's anyone. A native in the rain forest. (...) An Eskimo. I am bound to everyone on this planet by a trail of six people. It is a profound thought."

Small Worlds

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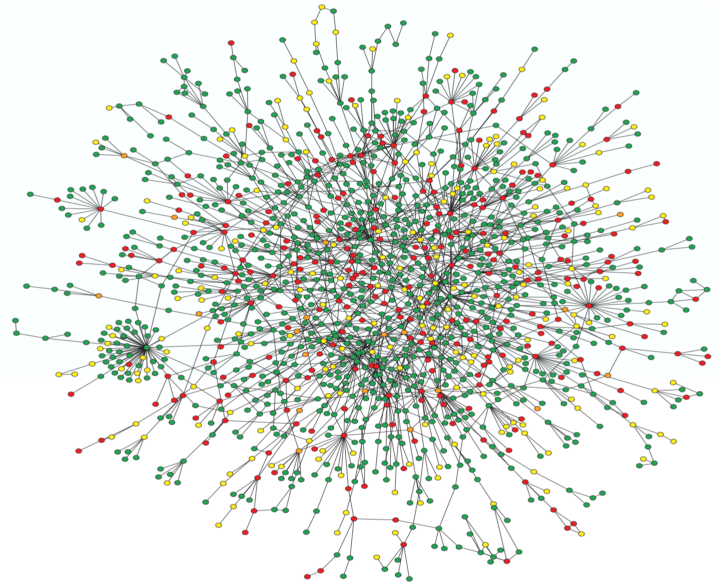
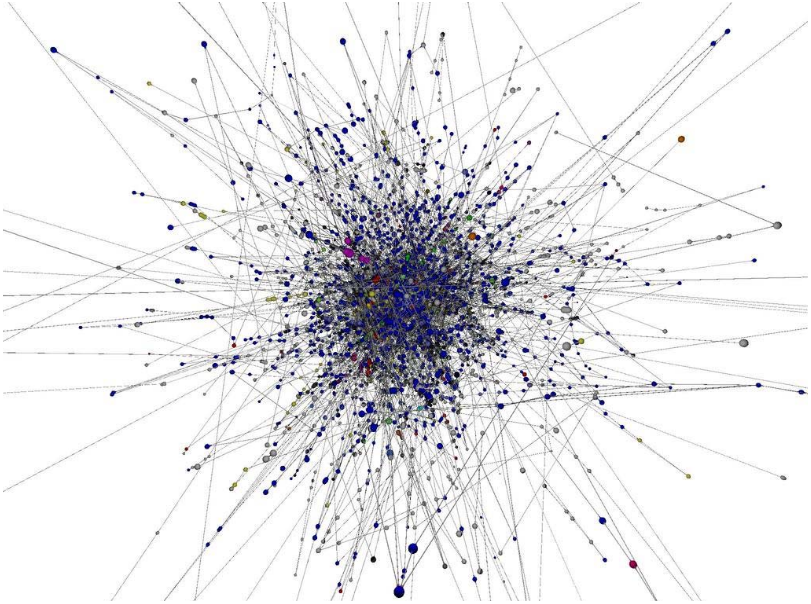
Scale-Free



Yeast protein interaction network
(Jeong, Mason, Barabási, and Oltvai (2001))



Internet topology in 2001
(http://www.fractalus.com/steve/stuff/ipmap/)

Real Networks

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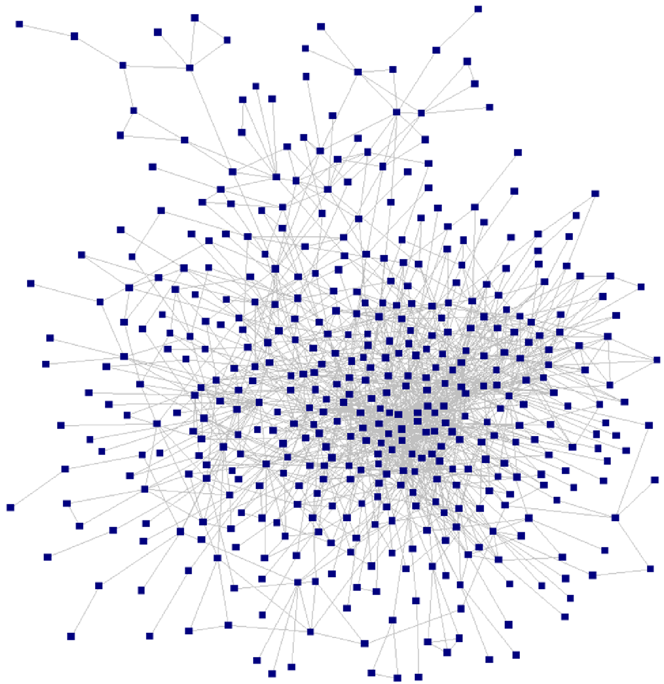
Focus as a primary example on Collaboration graph in mathematics.

http://www.ams.org/mathscinet/otherTools.html



Collaboration graph in mathematics
(`http://www.orgnet.com/Erdos.html`)

Power-law degree sequences

Degree sequence $(f(1), f(2), f(3), \ldots)$ of graph:

$f(1)$ is number of elements with degree equal to 1,
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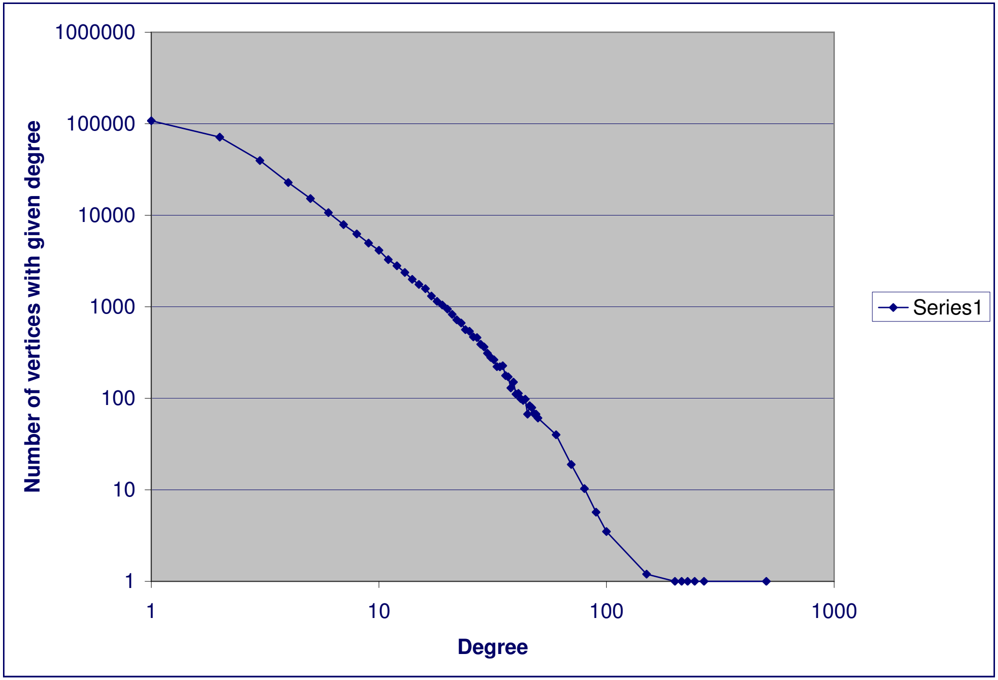
Is dubbed scale-free phenomenon.



Degree sequence of collaboration graph among mathematicians
(`http://www.oakland.edu/enp`)

Power law random graph: Configuration Model

Let N be number of vertices. Consider an i.i.d. sequence of degrees $D_1, D_2, \ldots, D_N$, with

$$\mathbb{P}(D_1 = k) = c_\tau k^{-\tau}$$

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Power law degree sequence built in in model!

Graph construction

How to construct graph with above degree sequence?
Assign to vertex j degree D_j.

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Now have N separate vertices and incident to vertex i have D_i 'stubs' or half edges. Connect stubs to create edges as follows:

Number stubs from 1 to L_N in any order.
First connect first stub at random with one of *other* $L_N - 1$ stubs.
Continue with second stub (when not connected to first) and so on, until all stubs are connected...

Scaling in power law random graph

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- When $\tau \in (1,2)$, (HHZ04)

$$H_N \quad \text{uniformly bounded}.$$

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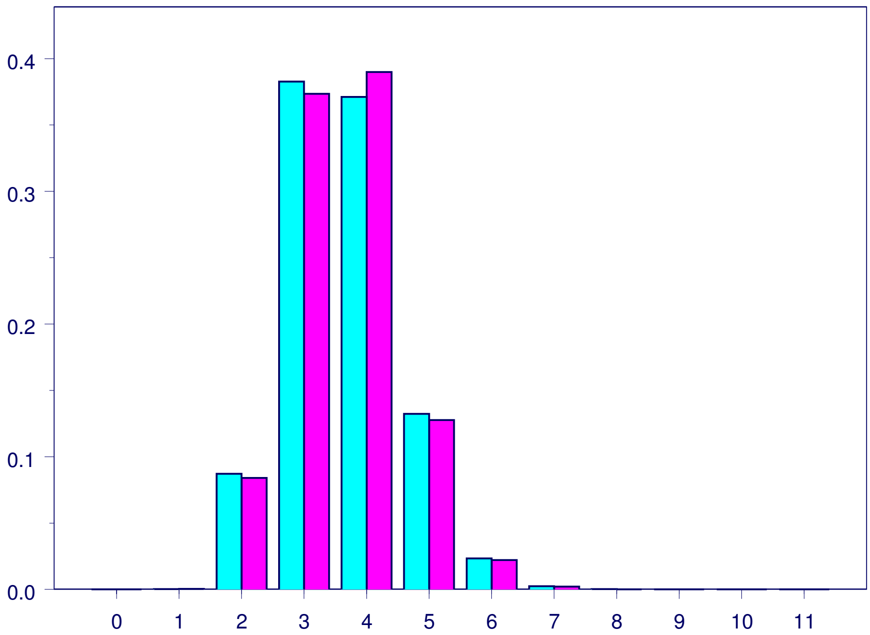
Extension: Fluctuations around leading order are uniformly bounded, and we compute its distribution.

Allows us to compare distances to measurements.



Number of AS traversed in hopcount data (blue) compared to the model (purple) with $\tau = 2.25, N = 10,940.$

Preferential Attachment

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Each of m edges of t^{th} vertex connects independently. Probability it connects to the i^{th} vertex equals

$$\frac{d_i(t-1)+\delta}{(2m+\delta)(t-1)},$$

where $\delta \geq m$ is parameter model.

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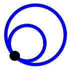

$(m = 2, \delta = 0)$

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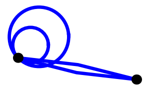

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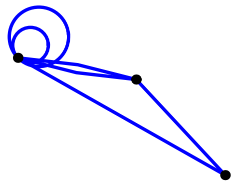

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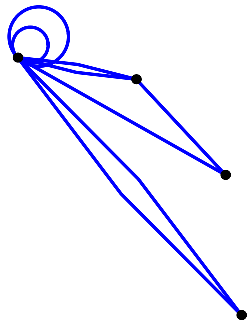

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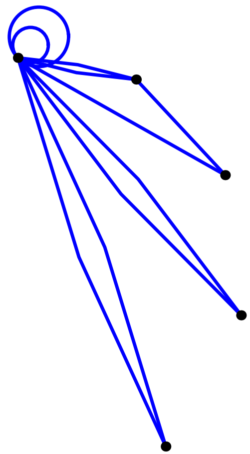

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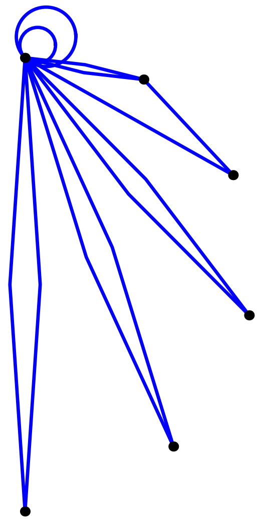

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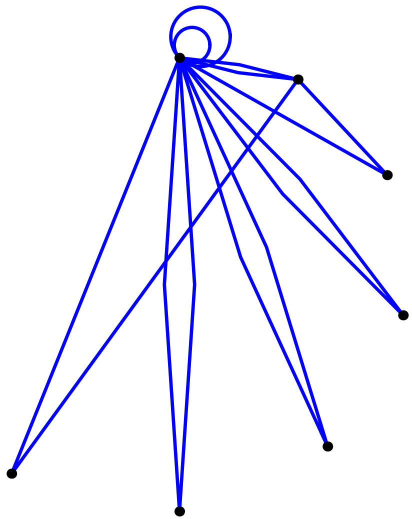

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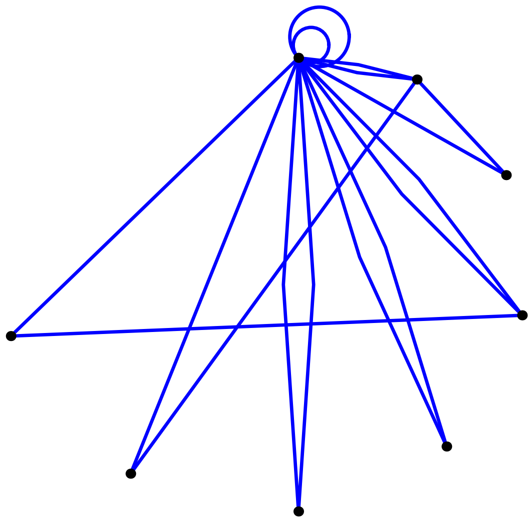

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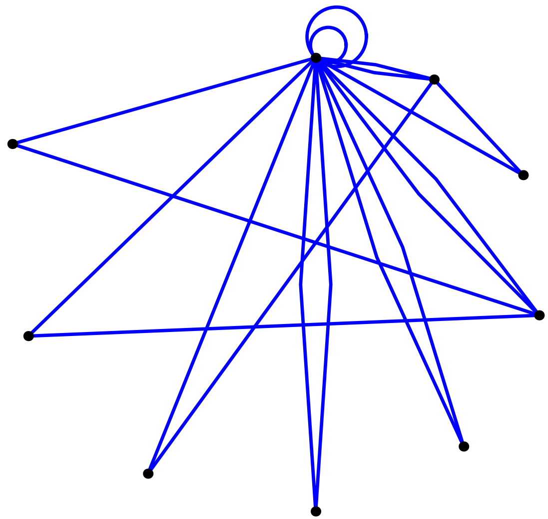

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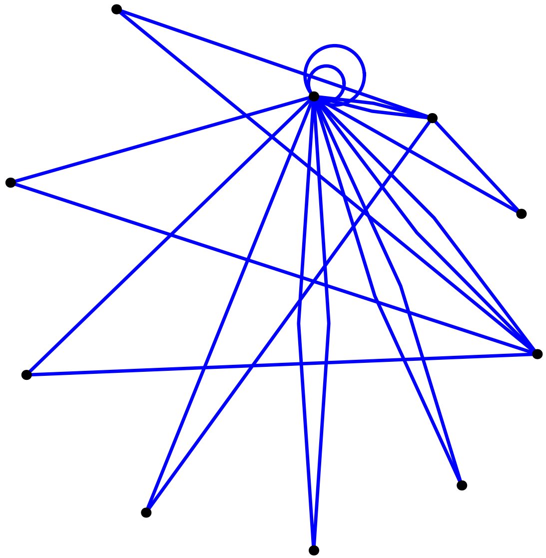

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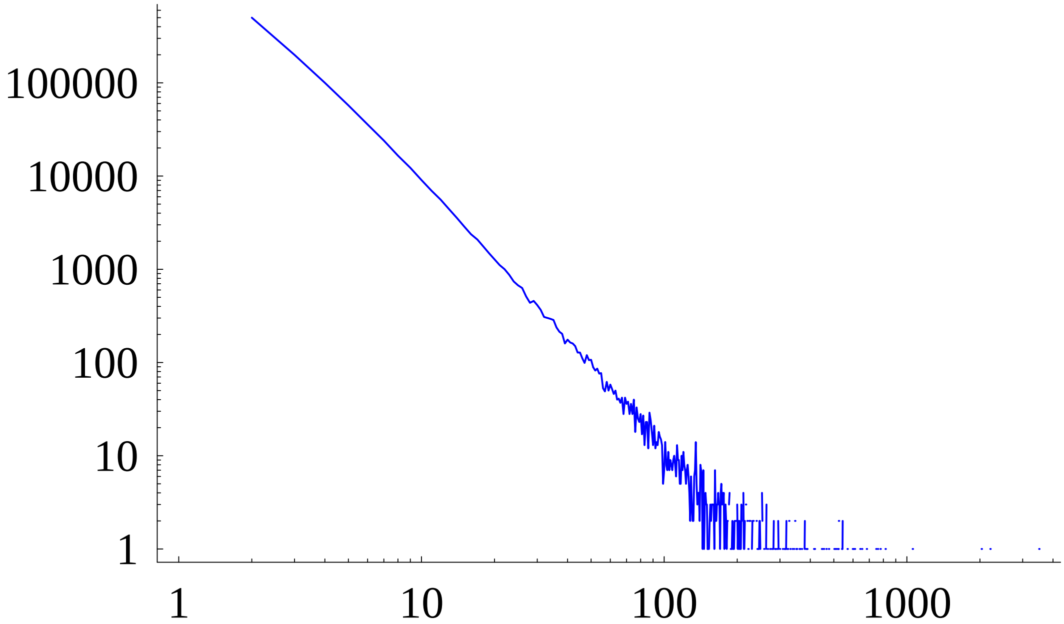
Scale-Free Nature



$$(m = 2, \delta = 0, \tau = 3 + \frac{\delta}{m} = 3)$$

"...the scale-free topology is evidence of organizing principles acting at each stage of the network formation. (...) No matter how large and complex a network becomes, as long as preferential attachment and growth are present it will maintain its hub-dominated scale-free topology."

Challenges PA models

Non-rigorous literature predicts that scaling distances in preferential attachment models similar to the one in configuration model with equal power-law exponent degrees.

No rigorous foundation for this prediction!
Strong form of universality.