Critical behavior in inhomogeneous random graphs

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EURANDOM

European Meeting of Statisticians, Toulouse, July 20-24, 2009

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Critical behavior Erdős-Rényi random graph

Erdős-Rényi random graph is random subgraph of complete graph on n vertices where each of $\binom{n}{2}$ edges is occupied with probab. p.

Phase transition: (Erdős and Rényi (60))
For $p = (1+\varepsilon)/n$, largest component is
(a) $\Theta_{\mathbb{P}}(\log n)$ for $\varepsilon < 0$;
(b) $\Theta_{\mathbb{P}}(n)$ for $\varepsilon > 0$;

Scaling window: (Bollobás (84) and Łuczak (90))
For $p = (1/n)(1+\lambda n^{-1/3})$, largest component is $\Theta_{\mathbb{P}}(n^{2/3})$.

Extension: Aldous (97): Weak convergence of ordered clusters.

Key question:
How much remains valid when we let go of homogeneity vertices?

Rank-1 inhomogeneous random graphs

Vertex set $[n] := \{1, 2, \ldots, n\}$.
Attach edge with probability p_{ij} between vertices i and j, where

$$p_{ij} = \frac{w_i w_j}{l_n},$$

and

$$l_n = \sum_{i=1}^{n} w_i,$$

and different edges are independent.

Note that w_i is expected degree vertex i.
When $w_i = \lambda$, we retrieve Erdős-Rényi random graph with $p = \lambda/n$.

Assume throughout talk $w_i^2/l_n \leq 1$ for all $i \in [n]$.

Choice of weights

Take $\boldsymbol{w} = (w_1, \ldots, w_n)$ as

$$w_i = [1 - F]^{-1}(i/n),$$

where $F(x)$ is distribution function.
Interpretation: number of vertices i with $w_i \leq x$ is close to $F(x)$.

Simple example:

$$F(x) = \begin{cases} 0 & \text{for } x < a, \\ 1 - (a/x)^{\tau-1} & \text{for } x \geq a, \end{cases}$$

in which case

$$[1 - F]^{-1}(u) = a(1/u)^{-1/(\tau-1)}, \qquad \text{so that} \qquad w_j = a(n/j)^{1/(\tau-1)}.$$

Critical value

Bollobás-Janson-Riordan (2007): Let $W \sim F$, then

- largest component $\sim \rho n$ for $\nu = \mathbb{E}[W^2]/\mathbb{E}[W] > 1$;
- largest component $o(n)$ for $\nu = \mathbb{E}[W^2]/\mathbb{E}[W] \leq 1$.

Thus, critical value IRG is

$$\nu = 1.$$

In simple example $F(x) = 1 - (a/x)^{\tau-1}$ for $x \geq a$

$$\mathbb{E}[W] = \frac{a(\tau-1)}{\tau-2}, \qquad \mathbb{E}[W^2] = \frac{a^2(\tau-1)}{\tau-3},$$

so that critical case arises when $a = (\tau-3)/(\tau-2)$:

$$\nu = \frac{\mathbb{E}[W^2]}{\mathbb{E}[W]} = \frac{a(\tau-2)}{\tau-3} = 1.$$

Power-law degrees

Let

$$1 - F(x) \sim cx^{-(\tau-1)} \qquad \text{for } x \text{ sufficiently large.}$$

Further, let $|\mathcal{C}_{\max}|$ denote largest connected component.

Theorem 1. (vdH 09) Assume that $\nu = 1$.

(a) Let $\tau > 4$. Then, there exists $b > 0$ such that for all $\omega \geq 1$

$$\mathbb{P}\Big(\frac{1}{\omega}n^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega n^{2/3}\Big) \geq 1 - \frac{b}{\omega} \qquad \text{as } n \to \infty.$$

(b) Let $\tau \in (3, 4)$. Then, there exists $b > 0$ such that for all $\omega \geq 1$

$$\mathbb{P}\Big(\frac{1}{\omega}n^{(\tau-2)/(\tau-1)} \leq |\mathcal{C}_{\max}| \leq \omega n^{(\tau-2)/(\tau-1)}\Big) \geq 1 - \frac{b}{\omega} \qquad \text{as } n \to \infty.$$

Scaling limit for $\tau > 4$

Let $\mu = \mathbb{E}[W], \sigma^2 = \mathbb{E}[W^3]/\mathbb{E}[W]$. Consider

$$B_s^\lambda = \sigma B_s + s\lambda - s^2\sigma^2/(2\mu),$$

where B is standard Brownian motion. Let

$$R_s^\lambda = B_s^\lambda - \min_{0 \le u \le s} B_s^\lambda.$$

Aldous (1997): Excursions of R^λ can be ranked in increasing order as $\gamma_1(\lambda) > \gamma_2(\lambda) > \dots$.

Let $|\mathcal{C}_{(1)}(\lambda)| \ge |\mathcal{C}_{(2)}(\lambda)| \ge |\mathcal{C}_{(3)}(\lambda)| \dots$ denote sizes of components with weights $\tilde{w}_i = (1 + \lambda n^{-1/3}) w_i$ arranged in increasing order.

Theorem 2. (BvdHvL 09a) Assume that $\nu = 1$, and $\mathbb{E}[W^3] < \infty$. Then

$$\left(n^{-2/3}|\mathcal{C}_{(i)}(\lambda)|\right)_{i \ge 1} \xrightarrow{d} \left(\gamma_i(\lambda)\right)_{i \ge 1}.$$

Scaling limit for $\tau \in (3,4)$

Let $|\mathcal{C}_{(1)}(\lambda)| \geq |\mathcal{C}_{(2)}(\lambda)| \geq |\mathcal{C}_{(3)}(\lambda)| \ldots$ denote sizes of components with weights $\tilde{w}_i = (1 + \lambda n^{-(\tau-3)/(\tau-1)})w_i$ arranged in increasing order.

Theorem 3. (BvdHvL 09b) Assume that $\nu = 1$, and $\tau \in (3,4)$. Then,

$$\left(n^{-(\tau-2)/(\tau-1)}|\mathcal{C}_{(i)}(\lambda)|\right)_{i\geq 1} \xrightarrow{d} \left(H_i(\lambda)\right)_{i\geq 1}.$$

Moreover, for every i, j fixed

$$\mathbb{P}(i \longleftrightarrow j) \to q_{ij}(\lambda) \in (0,1).$$

Limits $H_i(\lambda)$ correspond to hitting times of 0 of a certain fascinating 'thinned' Lévy process.

Cluster exploration for $\tau > 4$

Take $\lambda = 0$.

- For all ordered pairs of vertices (i, j), let $U(i, j)$ be i.i.d. $U(0, 1)$.
- Choose vertex $v(1)$ with probability proportional to $\boldsymbol{w}$, so that

$$\mathbb{P}(v(1) = i) = w_i / l_n.$$

Children of $v(1)$ are those vertices j for which

$$U(v(1), j) \leq \frac{w_{v(1)} w_j}{l_n}.$$

Label children of $v(1)$ as $v(2), v(3), \ldots v(c(1) + 1)$ in increasing order of their $U(v(1), \cdot)$ values.

- Move to $v(2)$, explore all of its children, and label them as before.

Once we finish exploring one component, move onto next component by choosing starting vertex in size-biased manner amongst remaining vertices.

Size-biased reordering

Size-biased order $v^*(1), v^*(2), \ldots, v^*(n)$ is random reordering of vertex set $[n]$ where

- $v^*(1) = i$ with prob. w_i/l_n;
- given $v^*(1), \ldots, v^*(i-1)$, $v^*(i) = j \in [n] \setminus \{v^*(1)\}$ with prob. proportional to w_j.

Key ingredient proof:

$$(v(i))_{i\in[n]} \qquad \text{is size-biased reordering.}$$

Number of new neighbors $c(i)$ of $v(i)$ is close to

$$c(i) = \mathrm{Poi}\Big(w_{v(i)} \sum_{j\in[n]\setminus\{v(1),\ldots,v(i)\}} w_j/l_n\Big).$$

Connected components

Recall number of new neighbors of $v(i)$ is close to

$$c(i) = \mathrm{Poi}\Big(w_{v(i)} \sum_{j \in [n] \setminus \{v(1),\ldots,v(i)\}} w_j / l_n\Big).$$

Denote cluster exploration process Z_n by $Z_n(0) = 0$ and

$$Z_n(i) = Z_n(i-1) + c(i) - 1.$$

Denote first hitting time of $-j$ by

$$\eta(j) = \min\{i : Z_n(i) = -j\}.$$

Then, all connected component sizes are given by successive excursions from past minima

$$\mathcal{C}^*(j) = \eta(j) - \eta(j-1).$$

Scaling limit of cluster exploration

Process $t \mapsto n^{-1/3} Z_n(sn^{2/3})$ is close to Brownian motion with changing drift given by

$$\mathbb{E}[n^{-1/3} Z_n(sn^{2/3})] \sim s\lambda - s^2\sigma^2/(2\mu),$$

while

$$n^{-1/3} Z_n(sn^{2/3}) - (s\lambda - s^2\sigma^2/(2\mu)) \xrightarrow{d} B_s.$$

Suggests that rescaled cluster sizes converge to successive excursions from past minima of process

$$B_s^\lambda = B_s + s\lambda - s^2\sigma^2/(2\mu).$$

Weak convergence of exploration process follows from

functional martingale central limit theorem.

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