Critical percolation in high dimensions:

critical exponents, finite size scaling and random walks

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EURANDOM

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Percolation

Plan of the lectures:

Hour 1 (RvdH):
Critical percolation in high dimensions;

Hour 2 (AN):
Random walks on random fractals: the Alexander-Orbach conjecture;

Hour 3 (RvdH):
Finite size scaling in high-dimensional percolation;

Hour 4 (AN):
Critical percolation on expanders of high girth.

Lecture 1:

Critical percolation in high dimensions.

Percolation on a tree

Let T_r be tree where root has degree $r-1$ and every other vertex has degree r.

Make each edge independently occupied with probability p, vacant otherwise.
Let $\mathcal{C}(0)$ be cluster of root, i.e.,

$$\mathcal{C}(0) = \{x \in T_r \colon 0 \longleftrightarrow x\},$$

where $0 \longleftrightarrow x$ means that there is a path of occupied bonds connecting 0 and x.

Let

$$\theta(p) = \mathbb{P}_p(|\mathcal{C}(0)| = \infty), \qquad \chi(p) = \mathbb{E}_p[|\mathcal{C}(0)|].$$

Phase transition percolation on a tree

By connection to branching process, we see that

$$\theta(p) = \begin{cases} 0 & p \le 1/(r-1), \\ > 0 & p > 1/(r-1). \end{cases}$$

Also,

$$\chi(p) = \begin{cases} \frac{1}{1-(r-1)p} & p < 1/(r-1), \\ +\infty & p \ge 1/(r-1). \end{cases}$$

Thus, percolation on a tree has phase transition, and $p_c(T_r) = 1/(r-1)$, where

$$p_c(T_r) = \inf\{p \colon \theta(p) > 0\} = \sup\{p \colon \chi(p) < \infty\}.$$

Critical exponents percolation tree

Again using branching process methodology, we see that

$$\theta(p) \sim (p - p_c), \qquad \chi(p) \sim (p_c - p)^{-1}.$$

Moreover,

$$\mathbb{P}_{p_c}(|\mathcal{C}(0)| \geq n) \sim n^{-1/2}, \qquad \mathbb{P}_{p_c}\big(\mathrm{diameter}(|\mathcal{C}(0)|) \geq n\big) \sim n^{-1}.$$

Critical exponents:

$$\beta = 1, \qquad \gamma = 1, \qquad \delta = 2, \qquad \rho_{\mathrm{int}} = 1.$$

High-dimensional percolation

Physics prediction:
Percolation in high-dimensions $(d > d_c = 6)$ behaves as it does on regular infinite tree:

- No infinite critical cluster;
- Critical exponents are same as ones on tree.

Informal reason:
When dimension is high, space is so vast, that faraway pieces of percolation cluster no longer interact.
Thus, geometry "trivializes", and for most questions answer is same as for percolation on an infinite regular tree.

Goal lectures:
Show how part of these claims can be made rigorous.

Percolation on $\mathbb{Z}^d$

Bonds join x to y for $x, y \in \mathbb{Z}^d$. Make bonds (x, y) independently

occupied	with probability p,
vacant	with probability $1 - p$,

where $p \in [0, 1]$ is percolation parameter.

Key examples:
- nearest-neighbor percolation;
- spread-out percolation, where range of bonds grows proportionally with parameter L, and L is often taken to be large.

Phase transition

Percolation has a phase transition, i.e, there is a critical probability $p_c = p_c(d, L) \in (0, \infty)$, such that

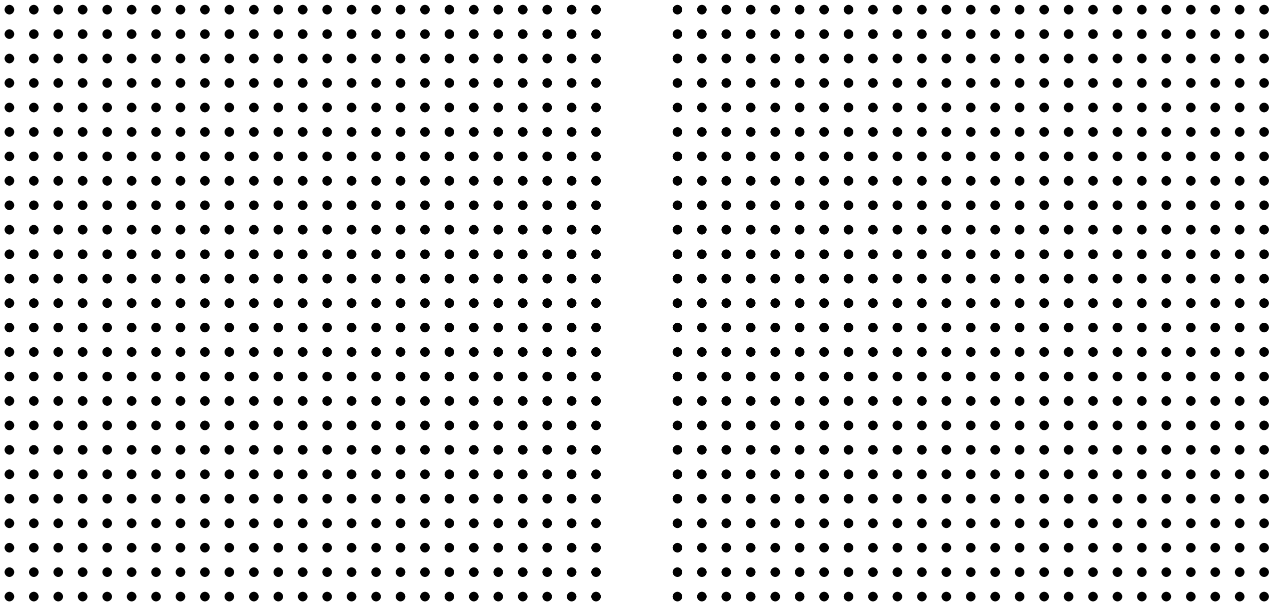
- For $p < p_c$, a.s. no infinite cluster exists.
- For $p > p_c$, a.s. a (unique) infinite cluster.
- For $p = p_c$, behavior not understood and dimension dependent.

No percolation at criticality for $d = 2$,
and for nn $d \geq 19$ and spread-out model with $d > 6$ (Hara-Slade 90).

Proving continuity percolation function one of main challenges probability and statistical physics.

At criticality, large clusters are abundantly present.

Percolation



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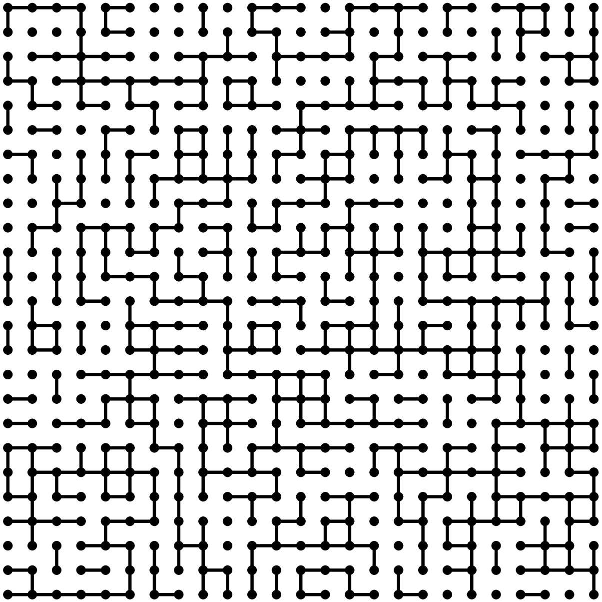

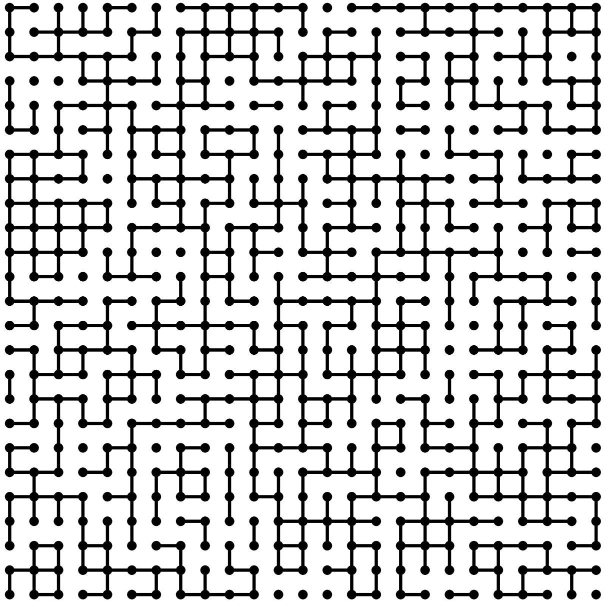

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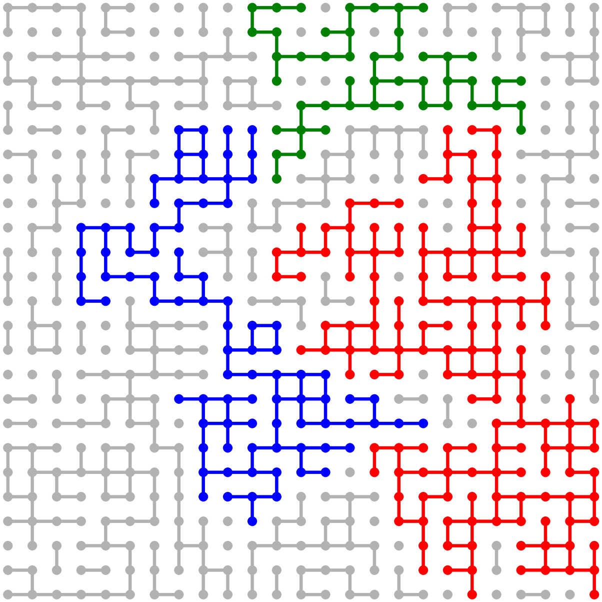
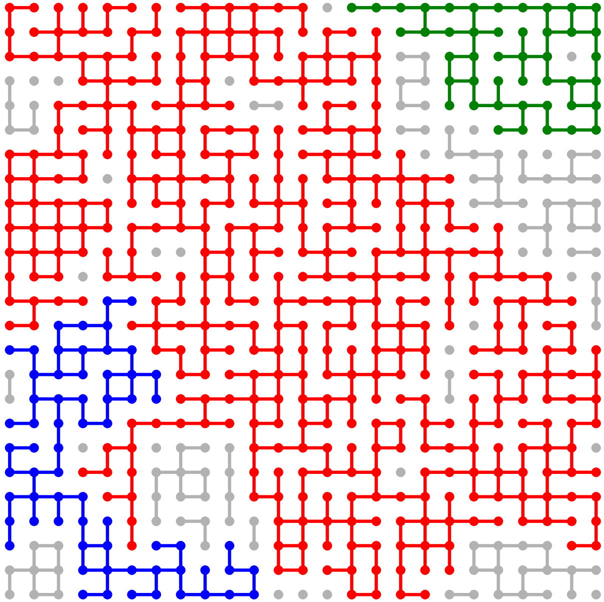
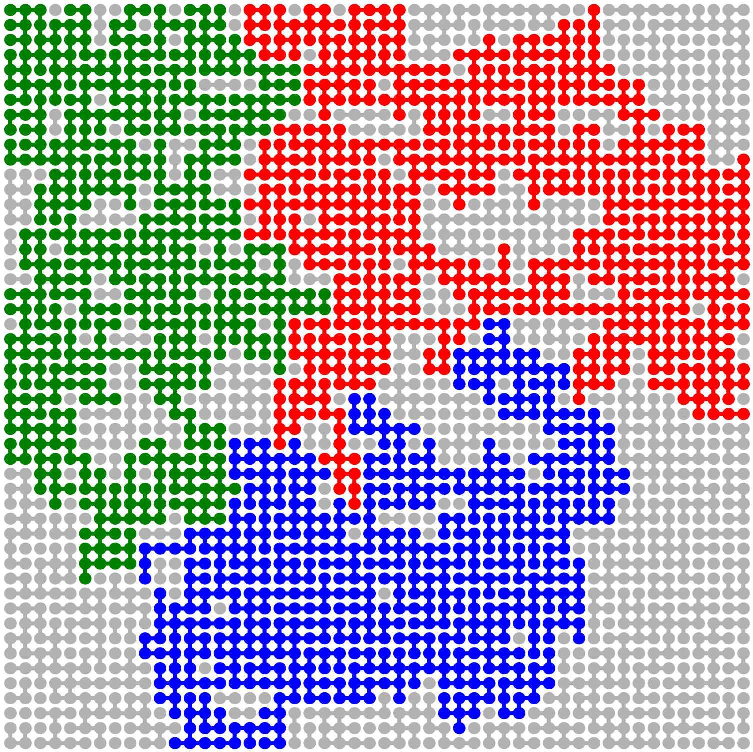
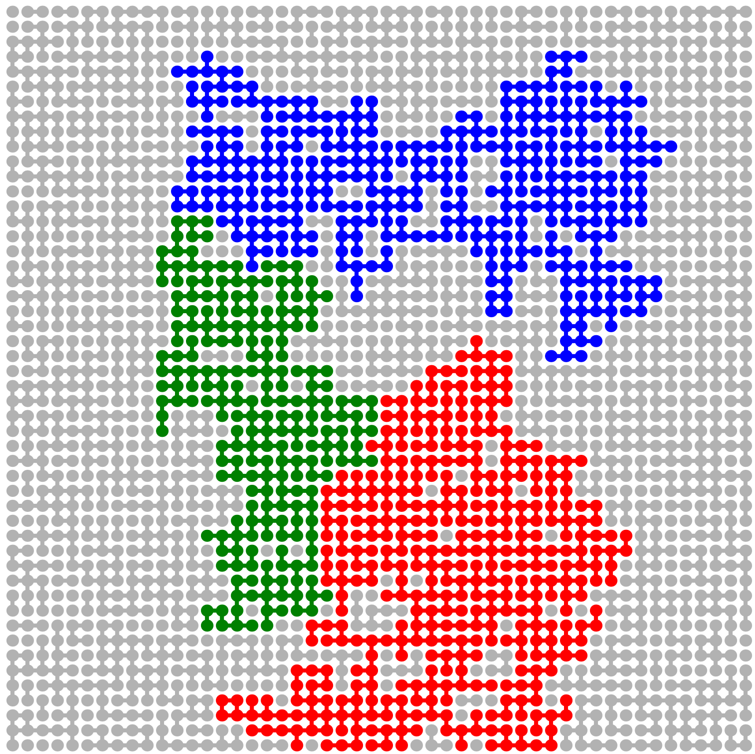




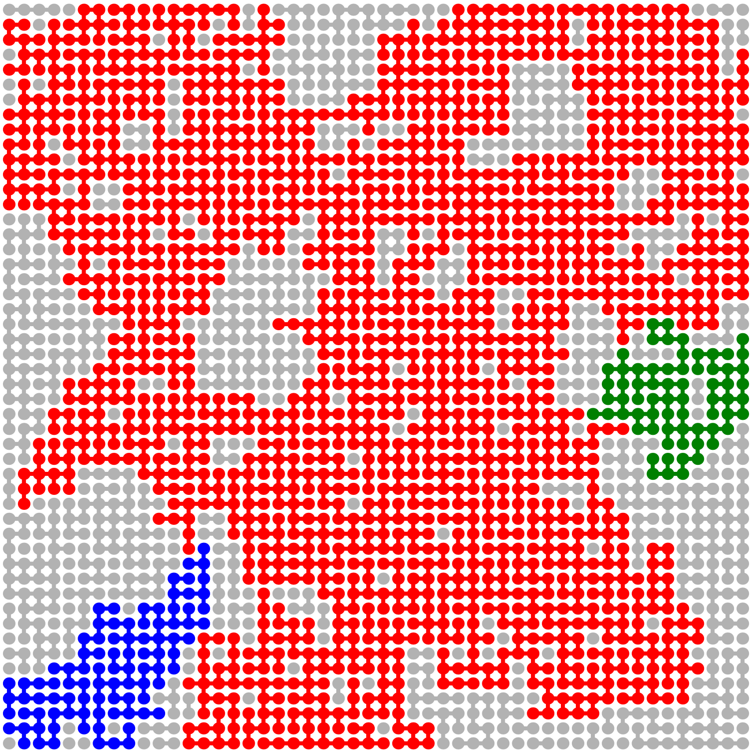
Critical percolation



Critical percolation



Critical percolation



Percolation in high d: critical exponents

Theorem 1. (AN84, BA91, HS90) For spread-out percolation with L sufficiently large and $d > 6$, or nearest-neighbour percolation for d sufficiently large,

$$\theta(p) \asymp (p - p_c), \qquad \chi(p) \asymp (p_c - p)^{-1} \qquad \mathbb{P}_{p_c}(|\mathcal{C}(0)| \geq n) \asymp n^{-1/2},$$

i.e., critical exponents β, γ, δ exist and take on tree values

$$\beta = 1, \qquad \gamma = 1, \qquad \delta = 2.$$

Many more results known in high dimension, shall discuss a few of those today.

Contrast: two-dimensions

Theorem 2. (Schramm00, Smirnov01, SW01, LSW02)
For site percolation on two-dimensional triangular lattice, critical exponents β, γ, δ exist in logarithmic sense, and take on values

$$\beta = \frac{5}{36}, \qquad \gamma = \frac{43}{18}, \qquad \delta = \frac{91}{5}.$$

Proof:
conformal invariance and Schramm/Stochastic Loewner Evolution (SLE).

Shall not discuss this exciting topic further today.

Percolation in high d: triangle condition

History of proof:

• Aizenman-Newman (1984) proved that $\gamma = 1, \delta = 2$;

• Barsky-Aizenman (1991) that $\beta = 1$,

both subject to geometric condition called

triangle condition.

Let

$$\tau_{p_c}(x, y) = \mathbb{P}_{p_c}(x \longleftrightarrow y).$$

Triangle condition:

$$\nabla(p_c) = \sum_{x,y} \tau_{p_c}(0, x)\tau_{p_c}(x, y)\tau_{p_c}(0, y) < \infty.$$

Triangle condition was proved by Hara-Slade (1990), using

lace expansion.

Percolation in high d: upper-critical dimension

Triangle condition can be expected to hold only when $d > 6$.

Reason: In high d, clusters look like trees.
Add geometry by embedding trees in $\mathbb{Z}^d$:

Branching Random Walk (BRW).

$$\begin{aligned} \tau_{p_c}(x, y) &\approx \text{expected \# of particles at } y \text{ from tree rooted at } x \\ &\approx |y - x|^{-(d-2)} = |y - x|^{-(d-2-\eta)}, \end{aligned}$$

with $\eta = 0$. Alternatively, when $x \longleftrightarrow y$, then

occupied path connecting x and y is like random walk path,

so that $\tau_{p_c}(x, y) \approx G(x, y) \approx |x - y|^{-(d-2)}$: random walk Green 's function.

Percolation in high d: upper-critical dimension

Triangle condition:

$$\begin{aligned}\nabla(p_c) &= \sum_{x,y} \tau_{p_c}(0,x)\tau_{p_c}(x,y)\tau_{p_c}(0,y) \\ &\approx \sum_{x,y} |x|^{-(d-2)}|y-x|^{-(d-2)}|y|^{-(d-2)} < \infty\end{aligned}$$

if and only if $d > 6$.

Denote one-arm exponent ρ_{ext} by

$$\mathbb{P}_{p_c}(0 \longleftrightarrow B_{\text{Eucl}}(R)) \asymp R^{-\rho_{\text{ext}}}.$$

Then, connection to BRW suggests that $\rho_{\text{ext}} = 2$.

$\eta = 0$ and $\rho_{\text{ext}} = 2$ imply that $d \geq 6$:

upper-critical dimension of percolation is $d_c = 6$.

Mean-field model of percolation on $\mathbb{Z}^d$ is

Percolation on tree or Branching random walk.

Large critical clusters

Central question:

What is structure of large critical clusters?

Here we can think of
- Dimension of large clusters;
- Local structure of large clusters.

Go by name of incipient infinite cluster (IIC), which is

infinite cluster that is on verge of appearing at criticality.

Question is how to define IIC.

2d-Incipient Infinite Cluster

Kesten (1986) has constructed IIC for percolation on $\mathbb{Z}^2$.
IIC describes local structure of large critical clusters.

Constructions Kesten:
(a) Condition 0 to be in the infinite component for $p > p_c$, and then take limit as $p \downarrow 0$.
(b) Condition on $0 \longleftrightarrow \partial B_n$ at $p = p_c$, and take limit as $n \to \infty$.

For events E, define

$$(\mathrm{IIC}) \qquad \mathbb{P}_\infty(E) = \lim_{p \downarrow p_c} \mathbb{P}_p(E | 0 \longleftrightarrow \infty).$$

Similar for Construction (b). RSW theory plays an important role.

Járai (03, 04) gives several more constructions for IIC.

Existence IIC for high-dimensional percolation

For cylinder events E, define

$$\text{(IIC)} \qquad \mathbb{P}_\infty(E) = \lim_{|x|\to\infty} \mathbb{P}_{p_c}(E|0 \longleftrightarrow x).$$

Theorem 3. **(vdH-Járai (03))** For spread-out percolation with L sufficiently large and $d > 6$, or nearest-neighbour percolation for d sufficiently large, the above limit exists for every cylinder event E. Moreover, $\mathbb{P}_\infty$ extends to a probability measure on full sigma-algebra of events, and $\mathbb{P}_\infty(|\mathcal{C}(0)| = \infty) = 1$.

Results make essential use of asymptotics critical two-point function

$$\tau(x) = \mathbb{P}_{p_c}(0 \longleftrightarrow x) \sim |x|^{-(d-2)},$$

proved in HHS(03) and Hara (08).

Properties IIC

Theorem 4. (vdH-Járai (03)) Under the conditions of Theorem 3, the IIC measure $\mathbb{P}_\infty$ satisfies that:

(i) The IIC has a single end $\mathbb{P}_\infty$-a.s.

(ii) There are positive constants $c_1 = c_1(d, L)$ and $c_2 = c_2(d, L)$ such that for $|y| \geq 1$

$$\frac{c_1}{|y|^{d-4}} \leq \mathbb{P}_\infty(0 \longleftrightarrow y) \leq \frac{c_2}{|y|^{d-4}}.$$

(iii) There are positive constants $c_3 = c_3(d, L)$ and $c_4 = c_4(d, L)$ such that for $|y| \geq 1$

$$\frac{c_3}{|y|^{d-2}} \leq \mathbb{P}_\infty(0 \longleftrightarrow y \text{ and } y \longleftrightarrow \infty \text{ disjointly}) \leq \frac{c_4}{|y|^{d-2}}.$$

(Indicates that IIC backbone is 2-, and IIC 4-dimensional).

Alternative definition IIC

For cylinder events E, define

$$\mathbb{Q}_\infty(E) = \lim_{p \uparrow p_c} \frac{1}{\chi(p)} \sum_{x \in \mathbb{Z}^d} \mathbb{P}_p(E \cap \{0 \longleftrightarrow x\}).$$

Theorem 5. (vdH-Járai (03)) Under the conditions of Theorem 3, the above limit exists, and $\mathbb{Q}_\infty = \mathbb{P}_\infty$.

Proof

Proof relies on various forms of lace expansion for two-point function $\tau(x) = \mathbb{P}_{p_c}(0 \longleftrightarrow x)$.

Lace expansion can be used to show that when E only depends on bonds within cube of width m,

$$\mathbb{P}_{p_c}(E \cap \{0 \longleftrightarrow x\}) = \psi(E; y) + \sum_y \pi(E; y)\tau_{p_c}(x - y).$$

Lace expansion coefficients $\psi(E; y), \pi(E; y)$ are small when y is large, so we can divide by $\tau_{p_c}(x)$ and take limit using $\tau_{p_c}(x) = c|x|^{-(d-2)}(1 + o(1))$ to get

$$\mathbb{P}_\infty(E) = \lim_{|x|\to\infty} \mathbb{P}_{p_c}(E|0 \longleftrightarrow x) = \sum_y \pi(E; y).$$

Next lecture:

Random walks on high-dimensional IIC.

Lecture 3:

Finite-size scaling in high-dimensional percolation

Percolation on tori

Random subgraph of finite tori

$$\mathbb{T}_{r,n} = (\mathbb{V}, \mathbb{B}) \qquad \text{where} \qquad \mathbb{V} = \{0, \ldots, r-1\}^n.$$

Two extreme examples:
- High-dimensional tori with finite-range bond set;
- Hypercube $\mathbb{Q}_n = \{0, 1\}^n$.

For n**-cube**, *bond* (u, v) is pair $u, v \in \mathbb{Q}_n$ satisfying $\#\{i : u_i \neq v_i\} = 1$.

Bonds independently

occupied	with probab. p,
vacant	with probab. $1 - p$.

Goal: Study structure largest critical clusters as number of vertices grows large.

Phase transition on $\mathbb{Z}^d$

The critical value $p_c(\mathbb{Z}^d)$ for percolation on $\mathbb{Z}^d$ is

$$p_c(\mathbb{Z}^d) = \inf\{p : \mathbb{P}_p(0 \longleftrightarrow \infty) > 0\}$$

Let $\mathcal{C}(x)$ be the cluster of x, and $\chi(p) = \mathbb{E}_p|\mathcal{C}(0)|$.
Aizenman and Barsky (87) or Menshikov (86):

$$p_c(\mathbb{Z}^d) = \sup\{p : \chi(p) < \infty\}.$$

Central question:

What is structure of large critical clusters?

Here we think of
- Size of large clusters;
- Structure of large clusters.

Mean-field model percolation on finite graph is

percolation on complete graph.

Erdős-Rényi random graph

Erdős-Rényi random graph is random subgraph of complete graph on V vertices where each of $\binom{V}{2}$ edges is occupied with probab. p.

Phase transition: (Erdős and Rényi (60))
For $p = (1+\varepsilon)/V$, largest component is
(a) $\Theta_{\mathbb{P}}(\log V)$ for $\varepsilon < 0$;
(b) $\Theta_{\mathbb{P}}(V)$ for $\varepsilon > 0$;

Scaling window: (Bollobás (84) and Łuczak (90))
For $p = (1/V)(1+\lambda/V^{1/3})$, largest component is $\Theta_{\mathbb{P}}(V^{2/3})$, with expected cluster size $\Theta(V^{1/3})$.

Extensions:
- Aldous (97): Weak convergence of ordered clusters;
- Nachmias and Peres (05): Diameter of large clusters is $\Theta_{\mathbb{P}}(V^{1/3})$.

High-dimensional tori

Torus in $\mathbb{Z}^d$ and $d > 6$:
Aizenman (1997), in combination with Hara(03), HHS(01), show that largest intersection of critical cluster with cube of width r is r^4.

Conjecture Aizenman:
Changes to $r^{2d/3} = V^{2/3}$ for periodic boundary conditions (= percolation on torus).

Conclusion: Scaling largest critical component is sensitive to boundary conditions.

Volume largest critical cluster

Define largest connected component for percolation on torus by

$$|\mathcal{C}_{\max}| = \max_{v} |\mathcal{C}_{\mathbb{T}}(v)|,$$

where $|\mathcal{C}_{\mathbb{T}}(v)|$ is size connected component of v, and $V = r^d$ is volume torus of width r.

Theorem 1. (Heydenreich+vdH 07+09) For $d \gg 6$ and nearest-neighbor bonds, or for $d > 6$ and sufficiently spread-out bonds, there exists $b > 0$ such that for all $\omega \geq 1$

$$\mathbb{P}_{p_c(\mathbb{Z}^d)}\Big(\frac{1}{\omega}V^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega V^{2/3}\Big) \geq 1 - \frac{b}{\omega} \qquad \text{as } r \to \infty.$$

Moreover, $|\mathcal{C}_{\max}|V^{-2/3}$ is not concentrated.

Non-concentration of largest cluster is hallmark of critical behavior.

Related results

Theorem 2. (Heydenreich+vdH (09)) Under conditions of Theorem 1, for every $m = 1, 2, \ldots$ there exist constants $b_1, \ldots, b_m > 0$, s.t. for all $\omega \geq 1$, and all $i = 1, \ldots, m$,

$$\mathbb{P}_{p_c(\mathbb{Z}^d)}\Big(\omega^{-1} V^{2/3} \leq |\mathcal{C}_{(i)}| \leq \omega V^{2/3}\Big) \geq 1 - \frac{b_i}{\omega} \qquad \text{as } r \to \infty.$$

Theorem 3. (Heydenreich+vdH (09)) Under conditions of Theorem 1, for every $m = 1, 2, \ldots$ there exist constants $c_1, \ldots, c_m > 0$, s.t. for all $\omega \geq 1$, and all $i = 1, \ldots, m$,

$$\mathbb{P}_{p_c(\mathbb{Z}^d)}\Big(\omega^{-1} V^{1/3} \leq \mathrm{diam}(\mathcal{C}_{(i)}) \leq \omega V^{1/3}\Big) \geq 1 - \frac{c_i}{\omega^{1/3}} \qquad \text{as } r \to \infty.$$

Lazy random walks mixing time

Transition probabilities lazy simple random walk on graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

$$p(x,y) = \begin{cases} 1/2 & \text{if } x = y; \\ \frac{1}{2\deg(x)} & \text{if } (x,y) \in \mathcal{E}; \\ 0 & \text{otherwise.} \end{cases}$$

Stationary distribution on finite graph is $\pi(x) = \deg(x)/(2|\mathcal{E}|)$.
Mixing time of lazy simple random walk is

$$T_{\mathrm{mix}}(\mathcal{G}) = \min\left\{n \colon \|\mathsf{p}^n(x,\cdot) - \pi(\cdot)\|_{\mathrm{TV}} \leq 1/4 \text{ for all } x \in \mathcal{V}\right\},$$

Theorem 4. (Heydenreich+vdH (09)) Under conditions of Theorem 1,

$$\mathbb{P}_{p_c(\mathbb{Z}^d)}\Big(\omega^{-1}V \leq T_{\mathrm{mix}}(\mathcal{C}_{(i)}) \leq \omega V\Big) \geq 1 - \frac{d_i}{\omega^{1/34}} \qquad \text{as } r \to \infty.$$

Random graph asymptotics

Results confirm that large critical percolation clusters on high-dimensional tori behave similarly as those on the Erdős-Rényi random graph:

random graph asymptotics.

Investigated similarities: cluster sizes, diameter, mixing times.

Would be of interest to investigate this further:
- Scaling limit same as one in Aldous (97)?
- Super critical phase same as one for Erdős-Rényi random graph?

Results apply more generally, e.g., to other high-dimensional tori as studied in (BCHSS05a,b).
Gives yet another motivation for choice of critical value in (BCHSS05a,b).

Key ingredients in proof

(1) Coupling relating $|\mathcal{C}(v)|$, size of cluster of v, on torus and on $\mathbb{Z}^d$.

(2) Lace expansion results for high-dimensional tori in BCHSS05a,b, which establish

• random graph scaling around internal critical value on torus defined by

$$\chi_{\mathbb{T}}(p_c(\mathbb{T}_{r,d})) = \lambda V^{1/3},$$

where $\chi_{\mathbb{T}}(p)$ is the expected cluster size on torus.

• an excellent control over various quantities (such as expected cluster size) for subcritical p.

(3) Proof completed using mean-field results for high-dimensional percolation on $\mathbb{Z}^d$ in HS90, Hara 95, HHS03, Hara 06, proved using lace expansion.

Need results on expected cluster size $\gamma = 1$, correlation length $\nu = 1/2$, and critical two-point function $\eta = 0$.

Coupling of connected components on torus and $\mathbb{Z}^d$

Proposition 1. (Heydenreich+vdH (07)) There exists a coupling $\mathbb{P}_{\mathbb{Z},\mathbb{T}}$ such that $\mathbb{P}_{\mathbb{Z},\mathbb{T}}$-almost surely,

$$|\mathcal{C}_{\mathbb{T}}(0)| \leq |\mathcal{C}_{\mathbb{Z}}(0)|.$$

Consequently, $\chi_{\mathbb{T}}(p) \leq \chi_{\mathbb{Z}}(p)$.

Also some form of lower bound is given.

Proof upper bound Theorem 1

Let $Z_{\geq k}$ denote the number of vertices in clusters of size at least $k,$ i.e.,

$$Z_{\geq k} = \sum_{v \in \mathbb{T}_{r,d}} \mathbb{1}_{\{|\mathcal{C}(v)| \geq k\}}.$$

Intimate relation between $Z_{\geq k}$ and $|\mathcal{C}_{\max}|$:

$$\{|\mathcal{C}_{\max}| \geq k\} = \{Z_{\geq k} \geq k\}.$$

Proofs BCHSS05a,b, vdHH07,09 center around analysis of $k \mapsto Z_{\geq k}$.

Lace expansion results in high dimensions

Hara-Slade (90) and Aizenman-Newman (84) show, under conditions Theorem 1, that $\delta = 2$, i.e.,

$$\mathbb{P}_{p_c(\mathbb{Z}^d)}(|\mathcal{C}_{\mathbb{Z}}(v)| \geq k) \leq bk^{-1/2}.$$

Then,

$$\mathbb{P}_p(|\mathcal{C}_{\max}| \geq k) = \mathbb{P}_p(Z_{\geq k} \geq k) \leq \frac{\mathbb{E}_p[Z_{\geq k}]}{k} = \frac{V}{k}\mathbb{P}_p(|\mathcal{C}_{\mathbb{T}}(v)| \geq k).$$

By Proposition 1 and $\delta = 2$,

$$\mathbb{P}_{p_c(\mathbb{Z}^d)}(|\mathcal{C}_{\max}| \geq \omega V^{2/3}) \leq \frac{V}{\omega V^{2/3}}\mathbb{P}_{p_c(\mathbb{Z}^d)}(|\mathcal{C}_{\mathbb{Z}}(v)| \geq \omega V^{2/3}) \leq \frac{b}{\omega^{3/2}},$$

which proves (slightly better) upper bound Theorem 1.

BCHSS approach to criticality

Recall $\mathbb{E}_p|\mathcal{C}(0)| = \Theta(V^{1/3})$ for random graph and $p = \frac{1}{V}$, where V is volume graph.

Suggests us to define critical threshold as

$$p_c(\mathbb{T}_{r,n}) = \min\{p : \chi(p) \geq \lambda V^{1/3}\},$$

for some $\lambda > 0$, where $\chi_{\mathbb{T}}(p) = \mathbb{E}_p|\mathcal{C}_{\mathbb{T}}(0)|$ is expected cluster size.

Does this bring us closer to critical value?
Is this indeed natural?

H+vdH (07+09):

$$|p_c(\mathbb{T}_{r,n}) - p_c(\mathbb{Z}^d)| = O(V^{-1/3}).$$

Example: n-cube

Write $\mathcal{C}_{\max}$ for largest connected component, and take $p = (1+\varepsilon)/n$.

Ajtai, Komlós and Szemerédi (82):

$$|\mathcal{C}_{\max}| = O(\varepsilon 2^n) \quad \text{for } \varepsilon > 0, \qquad |\mathcal{C}_{\max}| = o(2^n) \quad \textbf{a.a.s.} \quad \text{for } \varepsilon < 0.$$

Bollobás, Kohayakawa and Łuczak (92):

$$|\mathcal{C}_{\max}| = (2\log 2)\frac{n}{\varepsilon^2} \quad \textbf{a.a.s.} \quad \text{for } \varepsilon \leq -(\log n)^2/(\log\log n)\sqrt{n},$$

while

$$|\mathcal{C}_{\max}| = 2\varepsilon 2^n\big(1 + o(1)\big) \quad \textbf{a.a.s.} \quad \text{for } \varepsilon \geq 60(\log n)^3/n.$$

Transition is extremely sharp for n large, critical value close to $1/n$.

A hierarchy of phase transitions on n-cube

Theorem 5. (BCHSS, HS05,06) There exist rational numbers a_i with $a_1 = a_2 = 1, a_3 = 7/2$, s.t., for every $s \geq 1$, if

$$p = \sum_{i=1}^{s} a_i n^{-i} + \delta n^{-s},$$

with $\delta < 0$, then, as $n \to \infty$,

$$|\mathcal{C}_{\max}| \leq (2 \log 2) n^{2s-1} |\delta|^{-2} [1 + o(1)] \qquad \text{a.a.s.},$$

while if $\delta > 0$, then, as $n \to \infty$, there exist $0 < c_1 < c_2 < \infty$, such that

$$c_1 \delta^{-1} n^{-(s-1)} 2^n \leq |\mathcal{C}_{\max}| \leq c_2 \delta^{-1} n^{-(s-1)} 2^n \qquad \text{a.a.s.}$$

Extension of AKS to all powers of $1/n$. Exponential jump for all s!

Critical window

Theorem 6 (Scaling window).
If $p = p_c(1+\varepsilon)$, with $|\varepsilon| \leq \Lambda V^{-1/3}$, then there exist $b_1 = b_1(\Lambda) > 0$ such that

$$\mathbb{P}_p\Big(\omega^{-1} V^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega V^{2/3}\Big) \geq 1 - \frac{b_1}{\omega}.$$

Tightness of scaled largest cluster size when $p = p_c(1 + O(V^{-1/3}))$.

H+vdH (07+09): $|\mathcal{C}_{\max}| V^{-2/3}$ is non-degenerate.
Non-degeneracy is hallmark of critical behavior, so this gives (yet another) justification for choice $p_c(\mathbb{T}_{r,n})$.