The incipient infinite cluster for high-dimensional unoriented percolation

Remco van der Hofstad [*]
Antal A. Járai[†]

June 20, 2003

Abstract

We consider bond percolation on $\mathbb{Z}^d$ at the critical occupation density p_c for $d > 6$ in two different models. The first is the nearest-neighbor model in dimension $d \gg 6$. The second model is a "spread-out" model having long range parameterized by L in dimension $d > 6$. In the spread-out case, we show that the cluster of the origin conditioned to contain the site x weakly converges to an infinite cluster as $|x| \to \infty$ when $d > 6$ and L is sufficiently large. We also give a general criterion for this convergence to hold, which is satisfied in the case $d \gg 6$ in the nearest-neighbor model by work of Hara Hara00 [12].

The limiting object is the high-dimensional analogue of Kesten's incipient infinite cluster (IIC) in $d = 2$. We also investigate properties of the IIC such as bounds on the growth rate of the cluster that show its four-dimensional nature. The proofs of both the existence and of the claimed properties of the IIC use the lace expansion.

Finally, we give heuristics connecting the incipient infinite cluster to invasion percolation, and use this connection to support the well-known conjecture that for $d > 6$ the probability for invasion percolation to reach a site x is asymptotic to $c|x|^{-(d-4)}$ as $|x| \to \infty$.

1 Introduction and results

sec:intro

1.1 History

ssec:hist

For percolation models in any dimension $d \geq 2$, it is a well-known and partially confirmed conjecture that there is no infinite cluster at the critical point. On the other hand, for critical percolation restricted to a large box of radius n, one can find several 'macroscopic' clusters whose diameter is of order n. These large but sparse critical clusters, sometimes referred to as 'incipient infinite clusters', have a fractal dimension $d_f < d$. The validity of this picture is rigorously confirmed to a large extent by BCKS01 [8], whose main assumptions hold for $d = 2$ and are expected to hold when

[*]Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands. E-mail: rhofstad@win.tue.nl

[†]Centrum voor Wiskunde en Informatica, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands. E-mail: jarai@cwi.nl

$d \leq 6$, and by [2] in $d > 6$ whose main assumption is proved in [12] for the nearest-neighbor model and d sufficiently large, and in [13] for a sufficiently 'spread-out' model. (The term spread-out is explained in Section 1.2.)

Motivated by work in the physics literature, including the study of random walk on critical percolation clusters, Kesten [22] proposed to consider the following object. Condition the cluster of the origin in critical percolation to intersect the boundary of the box with radius n centered at the origin, and let $n \to \infty$. It is intuitively clear that in the weak limit an infinite cluster is obtained. Due to the absence of an infinite cluster at the critical point, the existence of the limit is not obvious. Kesten showed that in $d = 2$ the conditional distributions mentioned above do converge in the weak sense. He also showed that an alternative way of obtaining the limit is by taking $p > p_c$, conditioning on the cluster of the origin to be infinite, and letting $p \searrow p_c^+$. For brevity we refer to the infinite cluster obtained in this limit as the IIC.

In [23], the IIC served as a natural setting to study the asymptotic behavior of random walk on a critical percolation cluster. In addition, the IIC seems to be the unique object describing the 'microscopic view' of large critical clusters [2]. This is supported by the fact that natural procedures, different from the ones given by Kesten, also lead to the IIC. Namely, spanning clusters, the largest cluster in a finite box and the Chayes-Chayes-Durrett cluster, when viewed from a randomly picked site, all look asymptotically like the IIC [19]. Additional motivation to study the IIC comes from its close relationship with invasion percolation, a model we will introduce later on.

Lacking a general existence theorem for the IIC, its construction so far is limited to situations that are well understood. In particular, a construction exists only in cases where the absence of percolation at the critical point is rigorously known. So far, proofs of the latter are restricted to (a) $d = 2$ [20], [28], [21, Theorem 3.1], (b) high dimensions, that is $d \geq 19$ or $d > 6$ in a sufficiently 'spread-out' model [14], and (c) oriented percolation [7].

Recently, van der Hofstad, den Hollander and Slade [17] gave a construction of the IIC for so-called 'spread-out' oriented percolation in dimensions $d + 1 > 4 + 1$.

In this paper, we construct the IIC for unoriented nearest-neighbor percolation in dimension $d \gg 6$, and for sufficiently spread-out percolation in dimensions $d > 6$. The latter model is believed to be in the same universality class as the nearest-neighbor model. Let $\mathbb{P}_x$ denote the law of the critical configuration conditioned to contain a connection from 0 to x. We show that $\mathbb{P}_x$ converges weakly to a measure $\mathbb{P}_\infty$ as $|x| \to \infty$. Note that it is part of the statement that the limit does not depend on the direction or manner in which $|x| \to \infty$. We will comment on this later. We give an equivalent definition, where we first take $p < p_c$, define a measure $\mathbb{Q}_p$ and take its limit as $p \nearrow p_c^-$. This type of construction using subcritical approach to p_c is new. In view of the mentioned robustness, we expect both of our definitions to be equivalent to Kesten's.

We establish some geometric properties of the IIC measure $\mathbb{P}_\infty$. We obtain bounds on the IIC two-point function, showing that $\mathbb{P}_\infty(0 \longleftrightarrow y)$ is of order $|y|^{-(d-4)}$. This shows that the IIC is four-dimensional, as was conjectured by physicists [1, 3], and as is expected in view of the results of Aizenman [2] and Hara and Slade [16]. We also study the backbone of the IIC, which is defined as the set of sites from which there are disjoint connections to the origin and to infinity. We show that the backbone is two-dimensional, and that apart from small loops, it is a single path.

Our proofs are based on a modification of the Hara-Slade expansion for percolation [14]. Several variants of the lace expansion [9] have been used to analyze percolation and related models. In particular, the fact that the critical two-point function, $\mathbb{P}_{\text{cr}}(0 \longleftrightarrow x)$, is asymptotic to $c|x|^{-(d-2)}$ as $|x| \to \infty$ has been shown by these methods [13]. In analyzing the modified expansion, we heavily use this theorem as well as its analogue for the nearest-neighbor case Hara00 [12].

1.2 Main results

ssec:results

For general background on percolation, see Grim99 [11]. Our models are defined in terms of a function $D : \mathbb{Z}^d \to [0,1]$. Let $p \in [0, \|D\|_\infty^{-1}]$ be a parameter, where $\|\cdot\|_\infty$ denotes the supremum norm, so that $pD(x) \le 1$ for all x. We declare a bond $\{u,v\}$ to be *occupied* with probability $pD(v-u)$ and *vacant* with probability $1 - pD(v-u)$. The occupation status of all bonds are independent random variables. For the nearest-neighbor model, we take $D(x) = 1/(2d)$ for all x with $|x| = 1$, so that each bond is occupied with probability $p/(2d)$. A simple example of the spread-out model is

e:Dexample

$$D(x) = \begin{cases} \frac{1}{(2L+1)^d - 1} & 0 < \|x\|_\infty \le L \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

for which bonds are of the form $\{u,v\}$ with $0 < \|u-v\|_\infty \le L$, and bonds are occupied with probability $p/[(2L+1)^d - 1]$. Note that p is *not* a probability. We will always work at the percolation threshold $p = p_c$, which in this parametrization tends to 1, as either $d \to \infty$ or $L \to \infty$ (see HS95 [15] and HHS01a [13]).

In the spread-out case, our results hold for any function D that obeys the assumptions given in HHS01a [13, Definition 1.1]. These assumptions involve a parameter L, which serves to spread out the connections, and which will be taken large. In particular, they require that $\sum_{x\in\mathbb{Z}^d} D(x) = 1$, that $D(x) \le CL^{-d}$ for all x, and with σ defined by

e:def-sigma

$$\sigma^2 = \sum_{x\in\mathbb{Z}^d} |x|^2 D(x), \tag{1.2}$$

that $c_1 L \le \sigma \le c_2 L$. Here $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$. Finally, it is assumed that $D(x) = 0$ if $\|x\|_\infty > L$. The full details of the assumptions can be found in HHS01a [13]. The function in e:Dexample (1.1) does obey the assumptions.

The law of the configuration of occupied bonds (at the critical percolation threshold) is denoted by $\mathbb{P}$ with corresponding expectation denoted by $\mathbb{E}$. Given a configuration we say that x is connected to y, and write $x \longleftrightarrow y$, if there is a path of occupied bonds from x to y (or if $x = y$). We let $\mathcal{C}(x)$ denote the occupied cluster containing x. We may think of $\mathcal{C}(x)$ either as a set of sites or as a set of occupied bonds.

Let $\mathcal{F}$ denote the σ-algebra of events. A *cylinder event* is an event given by conditions on the states of finitely many bonds only. We denote the algebra of cylinder events by $\mathcal{F}_0$. We define

e:def-P_x

$$\mathbb{P}_x(F) = \mathbb{P}(F \,|\, 0 \longleftrightarrow x) = \frac{1}{\tau(x)} \mathbb{P}(F,\, 0 \longleftrightarrow x), \quad F \in \mathcal{F}, \tag{1.3}$$

where $\tau(x) = \mathbb{P}(0 \longleftrightarrow x)$. We prove the following theorem.

thm:IIC-lim

Theorem 1.1. *Let $d > 6$ and $p = p_c$. There is an $L_0 = L_0(d)$ such that for $L \ge L_0$ in the spread-out model, the limit*

e:IIC-lim

$$\mathbb{P}_\infty(F) = \lim_{|x|\to\infty} \mathbb{P}_x(F) \tag{1.4}$$

exists for any cylinder event F. Also, $\mathbb{P}_\infty$ extends uniquely from $\mathcal{F}_0$ to a probability measure on $\mathcal{F}$.

The reason why the limit does not depend on how $|x| \to \infty$ is roughly the following. The cylinder F depends on a finite set $\mathcal{W}$. The lace expansion is adapted to the event $\mathbb{P}(F, 0 \longleftrightarrow x)$ in such a way that only the part of the connection from 0 to x outside $\mathcal{W}$ is expanded. This writes $\mathbb{P}(F, 0 \longleftrightarrow x)$ as a convolution $\sum_y \Psi(y; F)\tau(x-y)$ plus a smaller order term. The main contribution to the sum comes from y near $\mathcal{W}$, and $\tau(x-y)$ is canceled by $\tau(x)$ in (1.3). This also leads to the formula $\mathbb{P}_\infty(F) = \sum_y \Psi(y; F)$.

To state our subcritical definition, for $p < p_c$ we define

$$\mathbb{Q}_p(F) = \frac{1}{\chi(p)} \sum_{x \in \mathbb{Z}^d} \mathbb{P}_p(F, 0 \longleftrightarrow x), \quad F \in \mathcal{F}, \tag{1.5}$$

where $\chi(p) = \sum_x \tau_p(x) = \mathbb{E}_p|C(0)|$ is the susceptibility. We prove the following theorem.

Theorem 1.2. *Let $d > 6$. Under the hypotheses of Theorem 1.1, the limit*

$$\mathbb{Q}_{p_c}(F) = \lim_{p \uparrow p_c} \mathbb{Q}_p(F) \tag{1.6}$$

exists for any cylinder event F and $\mathbb{Q}_{p_c} = \mathbb{P}_\infty$.

Not surprisingly, the proof of Theorem 1.1 relies on the asymptotic behavior of $\tau(x)$ for $|x|$ large. By [13, Theorem 1.2] there is an $L_0 = L_0(d)$ such that for $L \geq L_0$ there is a constant c depending on d and L such that

$$\tau(x) = \frac{c}{|x|^{d-2}}(1 + o(1)), \quad \text{as } |x| \to \infty. \tag{1.7}$$

We use the lace expansion and (1.7) to prove that the limit in (1.4) exists. In controlling the expansion we also need (1.8) below, which is a consequence of [13, Proposition 2.2]. Let $\tilde{\tau}(x) = p_c(D * \tau)(x)$. For $d > 6$, $\alpha > 0$, and L sufficiently large, depending on d and α, we have

$$\tau(x) \leq \frac{C}{(|x|+1)^{d-2}}, \qquad \tilde{\tau}(x) \leq \frac{C\beta}{(|x|+1)^{d-2}}, \tag{1.8}$$

where $\beta = L^{-2+\alpha}$, and C depends only on d and α. Here β provides the small parameter that ensures convergence of the expansion. We will fix an arbitrary, small value of α to apply (1.8).

Theorem 1.1 is similar to the existence statement of the IIC for spread-out oriented percolation above $4+1$ dimensions in [17], which we now describe in some detail. Fix D satisfying the above assumptions. Spread-out oriented percolation has vertices $\mathbb{Z}^d \times \mathbb{Z}_+$ and directed bonds $((x,n),(y,n+1))$, for $n \geq 0$ and $x, y \in \mathbb{Z}^d$. Similarly to unoriented percolation, a bond $((u,n),(v,n+1))$ is occupied with probability $pD(v-u)$ and vacant with probability $1-pD(v-u)$.

Define

$$\mathbb{P}_n(E) = \frac{1}{\tau_n} \sum_{x \in \mathbb{Z}^d} \mathbb{P}(E \cap \{(0,0) \longrightarrow (x,n)\}) \quad (E \in \mathcal{F}_0), \tag{1.9}$$

where $\tau_n = \sum_{x \in \mathbb{Z}^d} \tau_n(x)$ with $\tau_n(x) = \mathbb{P}((0,0) \longrightarrow (x,n))$, and

$$\mathbb{P}_\infty(E) = \lim_{n \to \infty} \mathbb{P}_n(E) \quad (E \in \mathcal{F}_0). \tag{1.10}$$

The main result in [17] is the proof that the limit in (1.10) exists. Furthermore, under some assumptions, it was shown to be equivalent to a second, perhaps more natural, construction. Define

$$\mathbb{Q}_n(E) = \mathbb{P}(E|(0,0) \longrightarrow n), \qquad \mathbb{Q}_\infty(E) = \lim_{n\to\infty} \mathbb{Q}_n(E) \quad (E \in \mathcal{F}_0), \tag{1.11}$$

where $\{(0,0) \longrightarrow n\} = \{(0,0) \longrightarrow (x,n) \;\text{ for some } x \in \mathbb{Z}^d\}$ is the survival event. It was shown that if there exists a constant $0 < B < \infty$ such that $\mathbb{P}((0,0) \longrightarrow n) = \frac{1}{Bn}[1 + o(1)]$, then the limit in (1.11) exists, and $\mathbb{Q}_\infty = \mathbb{P}_\infty$.

Finally, in [17], it was conjectured that for fixed $x \in \mathbb{Z}^d$, the measure $\mathbb{P}_n^{(x)}$ defined by

$$\mathbb{P}_n^{(x)}(E) = \frac{1}{\tau_n(x)} \mathbb{P}(E \cap \{(0,0) \longrightarrow (x,n)\}) \tag{1.12}$$

converges to the IIC measure $\mathbb{P}_\infty$. Proving this conjecture seems difficult, since we do not have a local central limit theorem describing the asymptotic behavior of $\tau_n(x)$ available. The definition of the IIC measure for unoriented percolation in (1.4) is closest in spirit to the definition in (1.12).

We now state some further remarks to Theorem 1.1.

Remark. (1) Later we formulate a similar theorem for the nearest-neighbor model under some assumptions (see Theorem 4.1 below). The assumptions hold when $d > d_0$, for some $d_0 \gg 6$ by [12]. Since the assumptions involve quantities that arise in the lace expansion, we defer the statement of the theorem until the expansion is introduced.

(2) Theorem 1.1 can be viewed as the $d > 6$ analogue of Kesten's result. We note that it is not hard to adapt the proof of [22, Theorem 3] to show that in $d = 2$ the limit in (1.4) coincides with Kesten's IIC measure. We expect this equivalence to hold in our case as well, however, conditioning on a connection to a point rather than to a box seems to be better suited for the lace expansion.

We need a few definitions in order to state the properties of $\mathbb{P}_\infty$. We say that the events $\{x_1 \longleftrightarrow y_1\}$ and $\{x_2 \longleftrightarrow y_2\}$ *occur disjointly*, if there exist bond disjoint occupied paths connecting x_1 to y_1 and x_2 to y_2. An infinite connected set of bonds has *a single end*, if any two infinite paths that remain inside the set have infinitely many points in common.

Theorem 1.3. *Let $d > 6$. There is $L_0(d)$, such that for $L \geq L_0$ the measure $\mathbb{P}_\infty$ has the following properties.*

(i) $\mathbb{P}_\infty(|\mathcal{C}(0)| = \infty) = 1$.

(ii) The cluster $\mathcal{C}(0)$ has a single end $\mathbb{P}_\infty$-a.s.

(iii) There are positive constants $c_1 = c_1(d,L)$ and $c_2 = c_2(d,L)$ such that for $|y| \geq 1$

$$\frac{c_1}{|y|^{d-4}} \leq \mathbb{P}_\infty(0 \longleftrightarrow y) \leq \frac{c_2}{|y|^{d-4}}. \tag{1.13}$$

(iv) There are positive constants $c_3 = c_3(d,L)$ and $c_4 = c_4(d,L)$ such that for $|y| \geq 1$

$$\frac{c_3}{|y|^{d-2}} \leq \mathbb{P}_\infty(0 \longleftrightarrow y \text{ and } y \longleftrightarrow \infty \text{ disjointly}) \leq \frac{c_4}{|y|^{d-2}}. \tag{1.14}$$

Remark. (1) Statements (i) and (ii) also hold in two dimensions, and in fact, should be valid in all dimensions. Statements (iii) and (iv) are only expected when $d > 6$. When $d = 2$, by [22, Theorem 8] and [24], on the triangular lattice,

$$\mathbb{P}_\infty(0 \longleftrightarrow y) = |y|^{-5/48+o(1)} \quad \text{as } |y| \to \infty. \tag{1.15}$$

In addition, by [22, Theorem 14] and [24]

$$\mathbb{P}_\infty(0 \longleftrightarrow y \text{ and } y \longleftrightarrow \infty \text{ disjointly}) = |y|^{-\lambda+o(1)}, \quad \text{as } |y| \to \infty, \tag{1.16}$$

where λ is the so-called "monochromatic 2-arm exponent", whose exact value is not known.

(2) We expect that the probabilities in (1.13) and (1.14) are in fact asymptotic to $c|y|^{-(d-2)}$ and $c|y|^{-(d-4)}$, respectively.

(3) Statement (iii) is reminiscent of the notion of stochastic dimension introduced in [6], although our setting is different due to the lack of translation invariance of the IIC.

(4) In [17], there is also a version of the fact that the IIC for oriented percolation above 4 + 1 dimensions has dimension 4.

1.3 Conjectures

In this section, we formulate some conjectures concerning the IIC.

We conjecture that we can alternatively obtain the IIC measure $\mathbb{P}_\infty$ by conditioning on the event $\{0 \longleftrightarrow \partial B(n)\}$ that the origin is connected to the boundary of the cube of width n. It is expected [11, Section 9.1] that $\mathbb{P}(0 \longleftrightarrow \partial B(n)) \asymp n^{-2}$ when $d > 6$, where $\asymp$ denotes that the left side is bounded above and below by multiples of the right side. Without a rigorous proof of such asymptotics, however, it seems difficult to prove the conjecture.

As indicated in Section 1.1, one reason to study the IIC is to be able to investigate random walk on infinite critical clusters. It has been conjectured long time ago by physicists [27] that when $d > 6$, random walk on a critical percolation cluster reaches a Euclidean distance of order $n^{1/6}$ in n steps. Below we explain how Theorem 1.3 supports this conjecture. See Barlow [4] for results on random walks on supercritical infinite percolation clusters.

Heuristically, in $d > 6$ the IIC is akin to the family tree of a critical branching process conditioned to survive. The backbone of the IIC is analogous to the unique infinite line of descent, and for the purposes of understanding the random walk, we will think of the backbone as a single path embedded into $\mathbb{Z}^d$ as a random walk path. Theorem 1.3 (iv) and its proof support this heuristic. Consider now simple random walk on the family tree of a critical branching process (conditioned to survive), with the root as starting point. The analysis in [23] shows that one can study the walk by controlling the times between successive returns to the infinite line of descent, and we expect that on the IIC return times to the backbone would behave similarly. In the branching process case, the distance of the walker from the root after n steps is of order $n^{1/3}$. Taking into account that distances are reduced by a square-root due to the embedding into $\mathbb{Z}^d$, this suggests that the typical displacement of random walk on the IIC after n steps is of order $n^{1/6}$.

Next we introduce the model for invasion percolation. For simplicity, we only define the model for a uniform step distribution D, such as the nearest-neighbor case or the case in (1.1). The bonds in these models are $\mathbb{B} = \{b = (u, v) : D(u - v) > 0\}$. We let $\{\omega(b)\}_{b\in\mathbb{B}}$ be a collection of i.i.d. uniform random variables. Given a random configuration ω, we define a random increasing sequence of subgraphs $G_0, G_1, \ldots$ as follows. We let G_0 be the graph with no edges, and the single vertex 0. We let $G_{i+1} = G_i \cup \{b_{i+1}\}$, where the edge b_{i+1} is obtained by taking the $b \notin G_i$ with minimal $\omega(b)$ and such that b has an end vertex in G_i. The invaded region is $\mathcal{S} = \cup_{i=0}^{\infty} G_i$.

It is well-known that the asymptotic behavior of invasion percolation is closely related to the incipient cluster. The heuristic behind this is that $\limsup_{i\to\infty} \omega(b_i) = p_c$, which is the critical percolation threshold in the model [10]. In other words, asymptotically the invasion process only accepts values from critical clusters. As mentioned earlier, critical clusters in $d > 6$ are four-dimensional, which leads to the well-known conjecture [26] that $\mathbb{P}(y \text{ is invaded}) \asymp |y|^{-(d-4)}$ when $d > 6$. Our results give non-rigorous support to this conjecture, as we outline below.

In [18] it is shown that in $d = 2$, conditioned on $v \in \mathcal{S}$, the invasion percolation neighborhood of v is asymptotically the same in probability as the neighborhood of the origin under the IIC measure, as $|v| \to \infty$. In other words, the two-dimensional IIC measure can be obtained by conditioning on $v \in \mathcal{S}$, shifting space by v, and taking the limit $|v| \to \infty$. The proof in [18] is intrinsically two-dimensional, but it is reasonable to believe that such a statement holds in general dimensions. In particular, when $d > 6$, note that if $v \in \mathcal{S}$ then v is connected to the origin inside $\mathcal{S}$, which is alike the construction in (1.3)–(1.4) (with v and 0 playing the roles of 0 and x).

Assuming that the IIC measure can be constructed by a limit of shifted versions of invasion percolation, we obtain from Theorem 1.3 that for any y with $|y| \geq 1$, and $|v|$ large

$$\mathbb{P}(v + y \text{ is invaded} \,|\, v \text{ is invaded}) \asymp |y|^{-(d-4)}. \tag{1.17}$$

We can stretch this heuristic a little further, and get

$$\mathbb{P}(y \text{ is invaded}) \asymp \mathbb{P}_\infty(0 \longleftrightarrow y) \asymp |y|^{-(d-4)}. \tag{1.18}$$

It would be of interest to study the connections between the IIC measure and invasion percolation in more detail, and to prove (1.17)-(1.18) in particular.

1.4 Organization

The remainder of the paper is organized as follows. In Section 2 we modify the Hara–Slade lace expansion to suit our needs. In Section 3 we give the proof of Theorem 1.1 assuming suitable bounds on the expansion. In Section 4 we prove these bounds, and state our theorem for the nearest-neighbor case. In Section 5 we prove Theorem 1.3.

2 The expansion

In order to facilitate the proof of (1.13), we prove the existence of $\lim_{|x|\to\infty} \mathbb{P}_x(F)$ not only for cylinder events but also for the event $\{0 \longleftrightarrow y\}$. We note that it is straightforward to generalize this to k-point function events $\{y_1, \ldots, y_{k-1} \in \mathcal{C}(0)\}$, but since we do not need the general case, we omit the details. The fact that $\{0 \longleftrightarrow y\}$ is not a cylinder event causes no extra difficulty in the expansion. However, some caution is necessary, since it is not a priori obvious that $\mathbb{P}_\infty(0 \longleftrightarrow y) = \lim_{|x|\to\infty} \mathbb{P}_x(0 \longleftrightarrow y)$, even if we know that the limit on the right-hand side exists. The equality of the two sides will be proved via an approximation by cylinder events.

For a set of sites A, we say that a bond lies in A if both of its endpoints are in A. We write $B(m)$ for the cube of width m centered around the origin. Throughout this section F will be a fixed event of one of the following two types.

(I) $F \in \mathcal{F}_0$, and F is determined by the states of bonds in $B(m)$, $1 \le m < \infty$, or

(II) $F = \{0 \longleftrightarrow w\}$, for some fixed $w \in \mathbb{Z}^d$.

For events of type (I), we fix m as the smallest integer for which F is determined by bonds in $B(m)$. In both cases, we define the set

$$\mathcal{W} = \mathcal{W}(F) = \begin{cases} B(m) & \text{if } F \in \mathcal{F}_0 \\ \{0, w\} & \text{if } F = \{0 \longleftrightarrow w\}. \end{cases} \tag{2.1}$$

Since F will be fixed, we suppress the dependence of $\mathcal{W}$ on F.

We carry out an expansion of $\mathbb{P}(F, 0 \longleftrightarrow x)$. For this we need to modify the first step of the Hara–Slade expansion [14]. First we need some more notation.

Definition 2.1. (i) Two paths are called *disjoint* if they have no bond in common. The sites x and y are *doubly-connected* if there exist two disjoint occupied paths from x to y. We denote this event by $x \Longleftrightarrow y$. A set of sites A is connected to y if there exist $x \in A$ such that $x \longleftrightarrow y$, and it is doubly-connected to y if there exist $x_1, x_2 \in A$ such that $x_1 \longleftrightarrow y$ and $x_2 \longleftrightarrow y$ via disjoint paths. We denote by $x \overset{A}{\longleftrightarrow} y$ the event that every occupied path from x to y passes through some site in A.

(ii) We let $\tilde{\mathcal{C}}^{\{u,v\}}(x)$ denote the cluster of x remaining after the bond $\{u,v\}$ is declared to be vacant, that is

$$\tilde{\mathcal{C}}^{\{u,v\}}(x) = \{y \in \mathbb{Z}^d : x \longleftrightarrow y \text{ via a path not using } \{u,v\}\}. \tag{2.2}$$

We let

$$\mathcal{C}(\mathcal{W}) = \{y \in \mathbb{Z}^d : \mathcal{W} \longleftrightarrow y\} = \bigcup_{w \in \mathcal{W}} \mathcal{C}(w). \tag{2.3}$$

Similarly to (2.2), we denote by $\tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W})$ the set of sites that remain connected to $\mathcal{W}$ after declaring the bond $\{u,v\}$ to be vacant, that is

$$\tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W}) = \{y \in \mathbb{Z}^d : \mathcal{W} \longleftrightarrow y \text{ via a path not using } \{u,v\}\} = \bigcup_{w \in \mathcal{W}} \tilde{\mathcal{C}}^{\{u,v\}}(w). \tag{2.4}$$

(iii) We write (u,v) for a *directed* bond from u to v. The directed bond (u,v) is *pivotal* for the connection *from x to y* (in short for $x \longrightarrow y$) if a directed occupied path from x to y using the directed bond (u,v) exists when (u,v) is declared to be occupied, but not if (u,v) is declared to be vacant. That is, if $u \in \tilde{\mathcal{C}}^{\{u,v\}}(x)$, $v \in \tilde{\mathcal{C}}^{\{u,v\}}(y)$ and $y \notin \tilde{\mathcal{C}}^{\{u,v\}}(x)$. Similarly, (u,v) is pivotal for $\mathcal{W} \longrightarrow x$ if $u \in \tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W})$, $v \in \tilde{\mathcal{C}}^{\{u,v\}}(x)$ and $x \notin \tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W})$.

(iv) We define the *two-point function*

$$\tau(x,y) = \mathbb{P}(x \longleftrightarrow y). \tag{2.5}$$

By translation invariance, we have that $\tau(x,y) = \tau(y-x)$, where $\tau(x) = \tau(0,x)$. Given a set of sites $A \subset \mathbb{Z}^d$ we say that x is connected to y in A if there is an occupied path from x to y consisting of bonds with both endpoints in A. The *restricted two-point function* is defined by

$$\tau^A(x,y) = \mathbb{P}(x \longleftrightarrow y \text{ in } \mathbb{Z}^d \setminus A). \tag{2.6}$$

In other words, $\tau^A(x,y)$ is the probability that x is connected to y via a path not using any bonds touching the set A.

We are ready to give the expansion. The event $\{0 \longleftrightarrow x\}$ implies the event $\{\mathcal{W} \longleftrightarrow x\}$. We distinguish between the cases when there is no pivotal bond for the latter event, and when there is one. The first case is equivalent to the occurrence of $0 \longleftrightarrow x$, $\mathcal{W} \Longleftrightarrow x$. In the second case, let (u,v) denote the first pivotal bond for the event $\mathcal{W} \longrightarrow x$ as the connection is traversed from $\mathcal{W}$ to x. Since $0 \longleftrightarrow x$, the bond (u,v) is also pivotal for $0 \longrightarrow x$. We note that there may be other pivotal bonds for $0 \longrightarrow x$ that precede (u,v), but we ignore those. We can write

$$\begin{aligned}\mathbb{P}(F,\, 0 \longleftrightarrow x) &= \mathbb{P}(F,\, 0 \longleftrightarrow x,\, \mathcal{W} \Longleftrightarrow x) \\ &\quad + \sum_{(u,v)} \mathbb{P}\begin{pmatrix} F,\, 0 \longleftrightarrow u,\, \mathcal{W} \Longleftrightarrow u,\, (u,v) \text{ is occupied} \\ \text{and pivotal for } \mathcal{W} \longrightarrow x \end{pmatrix},\end{aligned} \tag{2.7}$$

where the summation is over all directed bonds. We introduce the notation

$$\pi^{(0)}(x;F) = \mathbb{P}(F,\, 0 \longleftrightarrow x,\, \mathcal{W} \Longleftrightarrow x). \tag{2.8}$$

Next we rewrite the summand in the second term of (2.7) by conditioning on the bond cluster $\tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W})$. As a result of this, we will get that the right-hand side of (2.7) equals

$$\pi^{(0)}(x;F) + \sum_{(u,v)} p_{uv} \mathbb{E}\Big(I[F,\, 0 \longleftrightarrow u,\, \mathcal{W} \Longleftrightarrow u] \tau^{\tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W})}(v,x)\Big), \tag{2.9}$$

where $p_{uv} = p_c D(v-u)$ and $I[\cdot]$ denotes the indicator of an event. Before proving (2.9), we note that $\tau^{\tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W})}(v,x)$ is the random variable obtained from the function $\tau^A(v,x)$, defined for a deterministic set A, by substituting the random set $\tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W})$ for A. In other words, the law of $\tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W})$ is governed by the expectation $\mathbb{E}$ in (2.9), whereas the value of the restricted two-point function in (2.9), given the value of $\tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W})$, is determined by a *second* expectation, that is implicitly present through the definition of the restricted two-point function. The second expectation is "nested" inside the first.

The proof of (2.9) requires a few definitions. For a configuration ω and a (possibly random) set of sites S, we let ω_S denote the configuration obtained by declaring all bonds with both endpoints in $\mathbb{Z}^d \setminus S$ to be vacant. By the event 'F occurs on S' we mean $\{\omega : \omega_S \in F\}$. The event in the summand of (2.7) can be rewritten as the intersection of three events, namely

- F, $0 \longleftrightarrow u$, $\mathcal{W} \Longleftrightarrow u$ occur on $\tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W})$
- $\{u,v\}$ is occupied
- $v \longleftrightarrow x$ in $\mathbb{Z}^d \setminus \tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W})$.

One can prove, for example via an application of Lemma 2.4 in [16], that the probability of the intersection is

$$p_{uv}\mathbb{E}\Big(I[F,\, 0 \longleftrightarrow u,\, \mathcal{W} \Longleftrightarrow u \text{ occur on } \tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W})]\tau^{\tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W})}(v,x)\Big). \tag{2.10}$$

The expression in (2.10) can be simplified by noting that the words 'occur on $\tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W})$' can be omitted, since the restricted two-point function is 0 whenever the required connections exist, but do not occur on $\tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W})$. Recalling the definition in (2.8), it follows that the right-hand side of (2.7) equals (2.9). We note that the summation in (2.9) is over *all bonds*, and it includes bonds with $u = x$ and $v \in \mathcal{W}$, despite the fact that (u, v) did not correspond to a pivotal bond in these cases. The identity still holds, since $\tau^{\tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W})}(v,x) = 0$.

We write

$$\tau^{\tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W})}(v,x) = \tau(v,x) - \Big[\tau(v,x) - \tau^{\tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W})}(v,x)\Big]. \tag{2.11}$$

We replace the restricted two-point function in (2.9) by the difference above. To write the result concisely, we define

$$\psi^{(0)}(v;F) = \sum_u p_{uv}\pi^{(0)}(u;F) = \sum_u p_{uv}\mathbb{P}(F,\, 0 \longleftrightarrow u,\, \mathcal{W} \Longleftrightarrow u), \tag{2.12}$$

and

$$R^{(0)}(x;F) = \sum_{(u,v)} p_{uv}\mathbb{E}\Big(I[F,\, 0 \longleftrightarrow u,\, \mathcal{W} \Longleftrightarrow u]\Big\{\tau(v,x) - \tau^{\tilde{\mathcal{C}}^{\{u,v\}}(\mathcal{W})}(v,x)\Big\}\Big). \tag{2.13}$$

Then the expression in (2.9) equals

$$\pi^{(0)}(x;F) + \sum_v \psi^{(0)}(v;F)\,\tau(v,x) - R^{(0)}(x;F), \tag{2.14}$$

assuming that the expressions for the second and third term converge.

The Hara–Slade expansion gives an expression for $\tau(v,x) - \tau^A(v,x)$, with A a deterministic set of sites as input. We will take the result of this part of the expansion without change from [14], and substitute it into (2.13). This leads to a further expansion of the remainder term $R^{(0)}(x;F)$ in (2.14). The proposition below summarizes the resulting expression for $\mathbb{P}(F, 0 \longleftrightarrow x)$. The statement of the proposition involves events $E'(v,x;A)$ defined in [14]. We repeat the definitions for the reader. Let

$$E'(v,x;A) = \left\{ \begin{matrix} v \xleftrightarrow{\;A\;} x \text{ and there is no pivotal bond } (u_1,v_1) \\ \text{for the connection } v \longleftrightarrow x \text{ such that } v \xleftrightarrow{\;A\;} u_1 \end{matrix} \right\}. \tag{2.15}$$

The key identity behind the further expansion of $\tau(v,x) - \tau^A(v,x)$ is

$$\tau(v,x) - \tau^A(v,x) = \mathbb{P}(E'(v,x;A)) + \sum_{(u_1,v_1)} p_{u_1v_1}\mathbb{E}\Big(I[E'(v,u_1;A)]\tau^{\tilde{\mathcal{C}}^{\{u_1,v_1\}}(v)}(v_1,x)\Big), \tag{2.16}$$

which is obtained similarly to (2.7) and (2.9), and is proved in [14]. Now an application of (2.11) to $A = \tilde{\mathcal{C}}^{\{u_1,v_1\}(v)}(v_1,x)$ allows one to iterate the procedure. The final result can be described as follows. Given bonds $(u_0,v_0),(u_1,v_1),\dots$ let

$$\tilde{\mathcal{C}}_0 = \tilde{\mathcal{C}}^{\{u_0,v_0\}}(\mathcal{W}) \text{ and } \tilde{\mathcal{C}}_j = \tilde{\mathcal{C}}^{\{u_j,v_j\}}(v_{j-1}) \text{ for } j \geq 1. \tag{2.17}$$

Let

$$I_j = I[E'(v_{j-1},u_j;\tilde{\mathcal{C}}_{j-1})], \quad j \geq 1. \tag{2.18}$$

Proposition 2.2. *If the expressions in* (2.19), (2.20), (2.21) *and* (2.22) *below converge, then for* $N \geq 0$ *and* $p = p_c$

$$\begin{aligned}
&\mathbb{P}(F,\, 0 \longleftrightarrow x) \\
&\quad = \sum_{n=0}^{N} (-1)^n \, \pi^{(n)}(x;F) + \sum_{n=0}^{N} (-1)^n \sum_v \psi^{(n)}(v;F)\, \tau(v,x) + (-1)^{N+1} \, R^{(N)}(x;F).
\end{aligned} \tag{2.19}$$

Here $\pi^{(0)}(x;F)$ *is given by* (2.8), *and for* $n \geq 1$,

$$\begin{aligned}
\pi^{(n)}(x;F) = &\sum_{(u_0,v_0)} p_{u_0 v_0} \cdots \sum_{(u_{n-1},v_{n-1})} p_{u_{n-1}v_{n-1}} \mathbb{E}_0 (I[F,\, 0 \longleftrightarrow u_0, \mathcal{W} \Longleftrightarrow u_0] \\
&\times \mathbb{E}_1 (I_1 \, \mathbb{E}_2 (I_2 \, \cdots \mathbb{E}_{n-1} (I_{n-1} \, \mathbb{E}_n (I[E'(v_{n-1}, x; \tilde{\mathcal{C}}_{n-1})]) \,) \cdots) \,) \,).
\end{aligned} \tag{2.20}$$

Also, for $n \geq 0$,

$$\psi^{(n)}(v;F) = \sum_u p_{uv} \pi^{(n)}(u;F) \tag{2.21}$$

and

$$\begin{aligned}
R^{(N)}(x;F) = &\sum_{(u_0,v_0)} p_{u_0 v_0} \cdots \sum_{(u_N,v_N)} p_{u_N v_N} \mathbb{E}_0 \Big(I[F,\, 0 \longleftrightarrow u_0, \mathcal{W} \Longleftrightarrow u_0] \\
&\times \mathbb{E}_1 \Big(I_1 \, \mathbb{E}_2 \Big(I_2 \, \cdots \mathbb{E}_N \Big(I_N \cdot \Big\{ \tau(v_N, x) - \tau^{\tilde{\mathcal{C}}_N}(v_N, x) \Big\} \Big) \cdots \Big) \Big) \Big).
\end{aligned} \tag{2.22}$$

Remark. (1) The expectations in the above expressions are nested in the sense explained after (2.9). That is, the law of the set $\tilde{\mathcal{C}}_k$ is governed by the expectation $\mathbb{E}_k$, and $\tilde{\mathcal{C}}_k$ is a deterministic set with respect to the nested expectation $\mathbb{E}_{k+1}$.

(2) By definition, $\pi^{(n)}(x;F)$ and $\psi^{(n)}(x;F)$ are non-negative. The bounds we prove in Section 4 will imply that they are finite, justifying the steps leading to (2.19).

Proposition 2.2 constitutes the lace expansion for $\mathbb{P}(F,\, 0 \longleftrightarrow x)$. In Section 3, we state bounds on the expansion that will allow us to prove Theorem 1.1 and Theorem 1.2, and we prove the bounds in Section 4. The proof of Theorem 1.3 follows in Section 5.

3 Existence of the IIC

In Section 4 below, we will prove the following proposition. In its statement, we write $\beta = L^{-2+\alpha}$ (recall (1.8)).

Proposition 3.1. *Fix* $d > 6$ *and* $p = p_c$. *There is an* $L_0 = L_0(d)$, *such that*
(a) for $L \geq L_0$, *and any event* F *of type (I) or type (II), there is a* $K = K(F,L,d)$ *and* $C = C(F,L,d)$ *such that*

$$\sum_{n=0}^{\infty} \pi^{(n)}(x;F) \leq K I_{\mathcal{W}}(x) + \frac{C\beta^2}{(|x|+1)^{2(d-2)}}, \tag{3.1}$$

where $I_{\mathcal{W}}(x) = 1$ *if* $x \in \mathcal{W}$ *and* $I_{\mathcal{W}}(x) = 0$ *otherwise.*

(b) for any $x \in \mathbb{Z}^d$

$$\lim_{N \to \infty} R^{(N)}(x;F) = 0. \tag{3.2}$$

The proof of Proposition 3.1 is in Section 4 below, and is an adaptation of the proof of Proposition 1.8 in [13].

Proof of Theorem 1.1 subject to Proposition 3.1.
By Proposition 3.1(a) and (2.21), the expressions

$$\Pi(x;F) = \sum_{n=0}^{\infty} (-1)^n \, \pi^{(n)}(x;F), \qquad \Psi(x;F) = \sum_{n=0}^{\infty} (-1)^n \, \psi^{(n)}(x;F) \tag{3.3}$$

converge, and we have

$$|\Pi(x;F)| \leq \frac{C}{(|x|+1)^{2(d-2)}}, \qquad |\Psi(x;F)| \leq \frac{C}{(|x|+1)^{2(d-2)}}, \tag{3.4}$$

where C denotes a constant depending on F, L, and d. By part (b) of the Proposition, we may take the limit $N \to \infty$ in (2.19) to obtain

$$\mathbb{P}(F,\, 0 \longleftrightarrow x) = \Pi(x;F) + \sum_{y \in \mathbb{Z}^d} \Psi(y;F)\, \tau(x-y). \tag{3.5}$$

Therefore, dividing (3.5) by $\tau(x)$ gives

$$\mathbb{P}_x(F) = \frac{\Pi(x;F)}{\tau(x)} + \sum_{y \in \mathbb{Z}^d} \Psi(y;F) \frac{\tau(x-y)}{\tau(x)}. \tag{3.6}$$

By (1.7) and (3.4) we have

$$\lim_{|x| \to \infty} \frac{\Pi(x;F)}{\tau(x)} = 0. \tag{3.7}$$

In the second term of (3.6), we split the sum according to whether $|x-y| \leq \frac{1}{2}|x|$ or $|x-y| \geq \frac{1}{2}|x|$. In the former case, we use $|y| \geq \frac{1}{2}|x|$, (3.4) and $d > 4$ to obtain the bound

$$\sum_{y: |x-y| \leq \frac{1}{2}|x|} |\Psi(y;F)\tau(x-y)| \leq \frac{C}{|x|^{2(d-2)}} \sum_{y: |x-y| \leq \frac{1}{2}|x|} \frac{1}{(|x-y|+1)^{d-2}} \leq \frac{C}{|x|^{2(d-2)}} |x|^2 = o(\tau(x)). \tag{3.8}$$

Therefore, we are left to deal with $|x-y| \geq \frac{1}{2}|x|$. In this regime, $\tau(x-y)/\tau(x)$ is uniformly bounded, and converges to 1 for every fixed y. Moreover, $\Psi(y;F)$ is absolutely summable, hence by dominated convergence, we have

$$\mathbb{P}_\infty(F) = \lim_{|x| \to \infty} \mathbb{P}_x(F) = \sum_{y \in \mathbb{Z}^d} \Psi(y;F) = p_c \sum_{y \in \mathbb{Z}^d} \Pi(y;F), \tag{3.9}$$

where the last step follows from $\sum_v D(v-u) = 1$. This completes the proof of the first statement of Theorem 1.1. The second statement follows from Kolmogorov's extension theorem. $\square$

Remark. It is not apparent from the proof that the assumption $d > 6$ was necessary. The role of this condition is hidden in the proof of the bounds on $\pi^{(n)}$, and in many parts of this paper does not show up directly. Although the bound on $\pi^{(n)}$ becomes summable when $d > 4$, in order to prove it, the condition $d > 6$ is essential. In fact, the expansion is expected to diverge at p_c when $d \leq 6$. The role of $d > 6$ will be clear in Section 5.

We conclude this section by showing that for any $y \in \mathbb{Z}^d$

$$\mathbb{P}_\infty(0 \longleftrightarrow y) = \lim_{|x|\to\infty} \mathbb{P}_x(0 \longleftrightarrow y). \tag{3.10}$$

Since Proposition 3.1 holds for events of type (II), the existence of the limit on the right-hand side follows by the argument we have just given. We write

$$F = \{0 \longleftrightarrow y\}, \qquad F_n = \{0 \longleftrightarrow y \text{ inside } B(n)\}, \tag{3.11}$$

so that $\mathbb{P}_x(F) = \mathbb{P}_x(F_n) + \mathbb{P}_x(F \setminus F_n)$. We also define the event

$$G_n(y, z; x) = \Big\{y \longleftrightarrow x \text{ and } z \longleftrightarrow \mathbb{Z}^d \setminus B(n) \text{ disjointly}\Big\}. \tag{3.12}$$

We claim that for $\|y\|_\infty < n < \|x\|_\infty$ we have

$$(F \setminus F_n) \cap \{0 \longleftrightarrow x\} \subset G_n(0, y; x) \cup G_n(y, 0; x). \tag{3.13}$$

Indeed, when the left-hand event occurs, y is connected to 0, but not inside $B(n)$. Fix an occupied path γ that realizes this connection, and also another occupied path γ' from x to γ, with P denoting the point where γ' reaches γ. Since γ starts and ends in $B(n)$, but leaves $B(n)$, it can be subdivided into three pieces: γ_1 = the part from 0 to the first point outside $B(n)$, γ_2 = the part from y to the first point outside $B(n)$ and γ_3 = the piece between γ_1 and γ_2. If $P \in \gamma_1$ then $G_n(0, y; x)$ occurs, if $P \in \gamma_2$ then $G_n(y, 0; x)$ occurs, while if $P \in \gamma_3$ then both of them occur, which implies (3.13). By the BK inequality [11, Section 2.3], we have

$$\mathbb{P}(F \setminus F_n,\, 0 \longleftrightarrow x) \le \mathbb{P}(0 \longleftrightarrow x)\mathbb{P}(y \longleftrightarrow \mathbb{Z}^d \setminus B(n)) + \mathbb{P}(y \longleftrightarrow x)\mathbb{P}(0 \longleftrightarrow \mathbb{Z}^d \setminus B(n)). \tag{3.14}$$

Hence,

$$\mathbb{P}_x(F \setminus F_n) \le \mathbb{P}(y \longleftrightarrow \mathbb{Z}^d \setminus B(n)) + \mathbb{P}(0 \longleftrightarrow \mathbb{Z}^d \setminus B(n)) \frac{\tau(x-y)}{\tau(x)}. \tag{3.15}$$

Since there is no percolation at p_c, and since y is fixed, by choosing n large, the right-hand side of (3.15) can be made arbitrarily small, uniformly in x with $\|x\|_\infty > n$. This implies that

$$\lim_{|x|\to\infty} \mathbb{P}_x(F) = \lim_{n\to\infty} \lim_{|x|\to\infty} \mathbb{P}_x(F_n) = \lim_{n\to\infty} \mathbb{P}_\infty(F_n) = \mathbb{P}_\infty(F), \tag{3.16}$$

where the second equality holds, because F_n is a cylinder event. This establishes (3.10). □

3.1 Susceptibility definition of $\mathbb{P}_\infty$

Proof Theorem 1.2. We start by noting that the above proof applies verbatim, once we check the convergence of $\mathbb{Q}_p(F)$ towards a limiting measure $\mathbb{Q}_{p_c}(F)$. The latter is established using the lace expansion in a similar way as above. We now give the details.

The lace expansion is valid for all $p \le p_c$, and yields that

$$\mathbb{P}_p(F, 0 \longleftrightarrow x) = \Pi_p(x; F) + \sum_v \Psi_p(v; F)\, \tau_p(v, x), \tag{3.17}$$

where we write the subscript explicitly to emphasize the dependence on $p \le p_c$. Note that Π_p and Ψ_p satisfy the bound (3.4) uniformly in $p \le p_c$.

Fix $p < p_c$ and sum out over x to obtain

$$\sum_{x\in\mathbb{Z}^d} \mathbb{P}_p(F, 0 \longleftrightarrow x) = \sum_{x\in\mathbb{Z}^d} \Pi_p(x;F) + \chi(p) \sum_{v\in\mathbb{Z}^d} \Psi_p(v;F). \tag{3.18}$$

Thus,

$$\mathbb{Q}_p(F) = \sum_{v\in\mathbb{Z}^d} \Psi_p(v;F) + \frac{\sum_{x\in\mathbb{Z}^d} \Pi_p(x;F)}{\chi(p)}. \tag{3.19}$$

When $p \uparrow p_c$, we obtain that

$$\mathbb{Q}_{p_c}(F) = \sum_{v\in\mathbb{Z}^d} \Psi_{p_c}(v;F), \tag{3.20}$$

since the first term converges when $p \uparrow p_c$, and the numerator of the second term is uniformly bounded when $p \le p_c$, whereas $\chi(p) \uparrow \infty$. We complete the proof that $\mathbb{P}_\infty = \mathbb{Q}_{p_c}$ by noting that for any cylinder event F, (3.9) implies $\mathbb{P}_\infty(F) = \sum_{v\in\mathbb{Z}^d} \Psi_{p_c}(v;F) = \mathbb{Q}_{p_c}(F)$. □

4 Bounds on the lace expansion

We now obtain diagrammatic bounds on $\pi^{(n)}(x)$, using the method of [14]. In fact, our argument only differs from theirs in estimating the contribution to (2.20) involving $\tilde{\mathcal{C}}_0$, which is special in our case. A key ingredient in the diagrammatic estimates is the following consequence of the BK inequality [11, Section 2.3]. Let $V_1, \dots, V_n$ be sets of lattice paths, and let E_i be the event that at least one of the paths in V_i is occupied. The event $E_1 \circ \cdots \circ E_n$ represents the event that there exist pairwise bond-disjoint occupied paths $\omega_i \in V_i$, $i = 1, \dots, n$. Then the inequality

$$\mathbb{P}(E_1 \circ \cdots \circ E_n) \le \mathbb{P}(E_1)\,\mathbb{P}(E_2)\,\dots\mathbb{P}(E_n) \tag{4.1}$$

follows from the BK inequality and the fact that the E_i are increasing events.

Throughout this section, C and $C_{\mathcal{W}}$ denote generic constants that depend on d, L, and in the second case also on $\mathcal{W}$. Their value may change from line to line. Using (4.1), we immediately obtain the estimate

$$\pi^{(0)}(x;F) \le \mathbb{P}\Big(\cup_{w\in\mathcal{W}} (0 \longleftrightarrow x) \circ (w \longleftrightarrow x)\Big) \le \sum_{w\in\mathcal{W}} \tau(x)\tau(x-w). \tag{4.2}$$

To bound $\pi^{(n)}$ for $n \ge 1$, we estimate the nested expectation in (2.20) from the inside out (right to left). Apart from the contribution coming from $\mathbb{E}_0$, these bounds are identical to those in [14], and we do not give all the details. For the innermost expectation $\mathbb{E}_n$, we first observe that whenever $E'(v_{n-1}, x; \tilde{\mathcal{C}}_{n-1})$ occurs, there exist $w_n \in \tilde{\mathcal{C}}_{n-1}$ and $t \in \mathbb{Z}^d$ with four disjoint paths realizing the connections $v_{n-1} \longleftrightarrow t$, $t \longleftrightarrow w_n$, $w_n \longleftrightarrow x$, $t \longleftrightarrow x$. Applying the BK inequality gives

$$\mathbb{E}_n(I[E'(v_{n-1}, x; \tilde{\mathcal{C}}_{n-1})]) \le \sum_{t, w_n\in\mathbb{Z}^d} I[w_n \in \tilde{\mathcal{C}}_{n-1}]\tau(t-v_{n-1})\tau(w_n-t)\tau(x-w_n)\tau(x-t). \tag{4.3}$$

The indicator $I[w_n \in \tilde{\mathcal{C}}_{n-1}]$ is a random variable for the expectation $\mathbb{E}_{n-1}$ that must be treated in conjunction with the event $E'(v_{n-2}, u_{n-1}; \tilde{\mathcal{C}}_{n-2})$, when $n \geq 2$. It can be shown (see [14, Lemma 2.5] or [25, Lemma 5.5.8] for details) that for $1 \leq i \leq n-1$ we have

$$\begin{aligned} &\mathbb{E}_i\Big(I[E'(v_{i-1}, u_i; \tilde{\mathcal{C}}_{i-1})]I[w_{i+1} \in \tilde{\mathcal{C}}_i]\Big) \\ &\quad \leq \sum_{w_i, z_i, t} I[w_i \in \tilde{\mathcal{C}}_{i-1}]\tau(w_{i+1} - z_i)\tau(u_i - w_i)\tau(w_i - t) \\ &\quad \times \Big(\tau(t - v_{i-1})\tau(z_i - t)\tau(u_i - z_i) + \tau(z_i - v_{i-1})\tau(t - z_i)\tau(u_i - t)\Big). \end{aligned} \tag{4.4}$$

Finally, the expectation $\mathbb{E}_0$ is estimated using

$$\begin{aligned} &\mathbb{E}_0\Big(I[F,\, 0 \longleftrightarrow u_0,\, \mathcal{W} \Longleftrightarrow u_0]I[w_1 \in \tilde{\mathcal{C}}_0]\Big) \\ &\quad \leq \mathbb{E}_0\Big(\sum_{w \in \mathcal{W}} I[(0 \longleftrightarrow u_0) \circ (w \longleftrightarrow u_0)]I[w_1 \in \tilde{\mathcal{C}}_0]\Big) \\ &\quad \leq C_{\mathcal{W}} \sum_{w \in \mathcal{W}} \sum_{z_0} \tau(u_0)\tau(z_0 - w)\tau(u_0 - z_0)\tau(w_1 - z_0), \end{aligned} \tag{4.5}$$

for some constant $C_{\mathcal{W}}$, where the second step is proved as follows. On the event inside the expectation, there are disjoint connections $0 \longleftrightarrow u_0$ and $w \longleftrightarrow u_0$, and we also have a connection $\mathcal{W} \longleftrightarrow w_1$. We distinguish between three cases in how the latter can happen. One of them is that there is a z_0 on the path from w to u_0 with disjoint connections $w \longleftrightarrow z_0$, $z_0 \longleftrightarrow u_0$, $z_0 \longleftrightarrow w_1$ and $0 \longleftrightarrow u_0$. The probability of this is bounded above by the right hand side of (4.5) with $C_{\mathcal{W}} = 1$. The second case is when the roles of 0 and w are interchanged, and z_0 lies on the path from 0 to u_0. The upper bound in this case has the factors $\tau(z_0)\tau(u_0 - w)$ instead of the factors $\tau(u_0)\tau(z_0 - w)$ in (4.5). Since $\mathcal{W}$ is bounded, the former is bounded above by $C_{\mathcal{W}}$ times the latter for some constant $C_{\mathcal{W}}$, leading to an upper bound of the required form. The third case is when there is a $z_0 \in \mathcal{W}$, and disjoint connections $0 \longleftrightarrow u_0$, $w \longleftrightarrow u_0$ and $z_0 \longleftrightarrow w_1$. In this case, we use $w, z_0 \in \mathcal{W}$ and the inequalities $\tau(u_0 - w) \leq C_{\mathcal{W}}\tau(u_0 - z_0)$ and $1 \leq C_{\mathcal{W}}\tau(z_0 - w)$ to arrive at an upper bound of the required form, after extending the summation over z_0 from $\mathcal{W}$ to $\mathbb{Z}^d$. We note that the $w = 0$ term of our bound corresponds with the bound proved in [14, Proposition 2.4].

We now recast the bounds on $\pi^{(n)}(x; F)$ in a more convenient form. We define

$$A^{(0)}(x, y; F) = \sum_{w \in \mathcal{W}} \sum_{a,b \in \mathbb{Z}^d} \tau(a)\tau(b - w)\tau(a - b)\tilde{\tau}(x - a)\tau(y - b), \tag{4.6}$$

$$A_1(u, v, x, y) = \tau(u - v) \sum_{a,b \in \mathbb{Z}^d} \tau(u - a)\tau(v - b)\tau(a - b)\tau(y - a)\tilde{\tau}(x - b), \tag{4.7}$$

$$A_2(u, v, x, y) = \tau(y - u) \sum_{a,b \in \mathbb{Z}^d} \tau(u - a)\tau(v - a)\tau(a - b)\tau(v - b)\tilde{\tau}(x - b), \tag{4.8}$$

$$A^{(i)}(u, v, x, y) = A_1(u, v, x, y) + A_2(u, v, x, y) \quad (i \geq 1), \tag{4.9}$$

$$A^{(\mathrm{end})}(u, v, x, y) = \tau(u - v)\tau(x - v)\tau(y - u). \tag{4.10}$$

The above quantities are depicted in Figure 1. The statement of the bound involves the function

$$M^{(0)}(x, y; F) = \sum_{w \in \mathcal{W}} \tau(x)\tau(y - w), \tag{4.11}$$

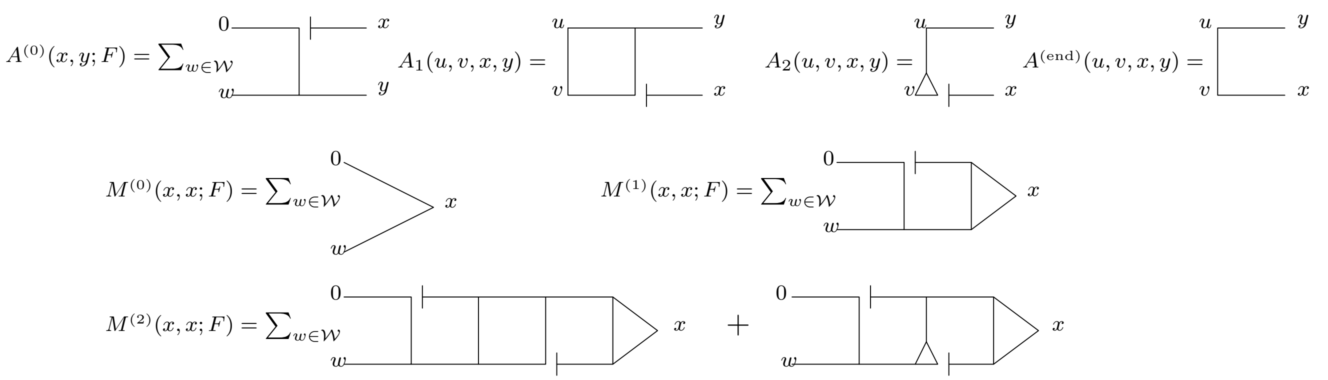

Figure 1: Diagrams representing the quantities in (4.6)–(4.10) and diagrams for the bounds on $\pi^{(0)}(x;F)$, $\pi^{(1)}(x;F)$ and $\pi^{(2)}(x;F)$. Lines ending with a horizontal bar represent $\tilde\tau$-lines, and are due to pivotal bonds.

and the functions $M^{(N)}$ defined for $N \geq 1$ by

$$M^{(N)}(x,y;F) = \sum_{u_1,v_1,\dots,u_N,v_N\in\mathbb{Z}^d} A^{(0)}(u_1,v_1;F) \prod_{i=1}^{N-1} A^{(i)}(u_i,v_i,u_{i+1},v_{i+1}) A^{(\mathrm{end})}(u_N,v_N,x,y). \tag{4.12}$$

For $N = 1$, the empty product over i is interpreted as 1. Note that the dependence on the event F only resides in the first factor $A^{(0)}$.

As in Proposition 2.4 of [14], it follows from (4.2)–(4.5) that we have

$$0 \leq \pi^{(N)}(x;F) \leq C_{\mathcal{W}}\, M^{(N)}(x,x;F), \quad N \geq 0. \tag{4.13}$$

In the proof of (3.1) and later in the paper we use the following observation. For $x \neq 0$, by the BK-inequality, we have

$$\tau(x) = \mathbb{P}\Big(\bigcup_y \Big\{\{0,y\}\text{occ.}\Big\} \circ \Big\{y \longleftrightarrow x\Big\}\Big) \leq \sum_y p_c D(y)\tau(x-y) = \tilde\tau(x). \tag{4.14}$$

We start by proving the bound in (3.1) for the $n = 0$ term. We claim that

$$M^{(0)}(x,x;F) \leq K I_{\mathcal{W}}(x) + \frac{C\beta^2}{(|x|+1)^{2(d-2)}}. \tag{4.15}$$

Indeed,

$$M^{(0)}(x,x;F) = \sum_{w\in\mathcal{W}} \tau(x)\tau(x-w) = \sum_{w\in\mathcal{W}} \tau(w)[\delta_{x,w} + \delta_{x,0}] + \sum_{w\in\mathcal{W}} \tau(x)\tau(x-w) I[x \neq 0, w]. \tag{4.16}$$

Clearly, the first term is bounded by $KI_{\mathcal{W}}(x)$, with $K = \sum_{w\in\mathcal{W}} \tau(w)$, while by (4.14) and (1.8), the second term is bounded by $C_{\mathcal{W}}\beta^2(|x|+1)^{-2(d-2)}$ for some constant $C_{\mathcal{W}}$. This proves the claim in Proposition 3.1 for the $n = 0$ term.

For the case $n \geq 1$, let $A^{(0)}(x,y) = \tau(x)\tau(y)$. Define $M^{(n)}(x,y)$ by replacing $A^{(0)}(x,y;F)$ by $A^{(0)}(x,y)$ in (4.12). Then there exists a $C_{\mathcal{W}}$ such that

$$A^{(0)}(x,y;F) \leq C_{\mathcal{W}} A^{(0)}(x,y), \quad x,y \in Z^d, \tag{4.17}$$

so that also

$$M^{(n)}(x,x;F) \le C_{\mathcal{W}} M^{(n)}(x,x), \quad n \ge 1. \tag{4.18}$$

It is shown in [13, (4.72)] that

$$M^{(n)}(x,x) \le \frac{(C\beta)^n}{(|x|+1)^{2(d-2)}}, \quad n \ge 1, \tag{4.19}$$

Where C does not depend on L, and hence on β. It is also shown in [13, (4.72)] that when $n=1$ and $x \neq 0$, the factor β can actually be replaced by β^2. This proves Proposition 3.1 (a).

For part (b), we recall from Proposition 2.4 of [14] that for $N \ge 1$ the expansion remainder term $R^{(N)}(x;F)$ of (2.22) obeys

$$0 \le R^{(N)}(x;F) \le C_{\mathcal{W}} \sum_{u\in\mathbb{Z}^d} M^{(N)}(u,u;F)\tilde{\tau}(x-u) \le C_{\mathcal{W}} \sum_{u\in\mathbb{Z}^d} M^{(N)}(u,u)\tilde{\tau}(x-u). \tag{4.20}$$

Recalling (4.20), (1.8) and (4.19), the claim follows from and the fact

$$\sup_{x\in\mathbb{Z}^d} \sum_u \frac{1}{(|u|+1)^{2(d-2)}(|x-u|+1)^{d-2}} < \infty, \tag{4.21}$$

which holds whenever $d>4$. This completes the proof of Proposition 3.1.

We end this section by stating a theorem that provides a sufficient condition for the conclusions of Theorem 1.1 in a more general setting.

Theorem 4.1. *Let $d>6$ and $p=p_c$. Assume that there exists a $c>0$ such that*

$$\tau(x) = \frac{c}{|x|^{d-2}}(1+o(1)) \qquad \text{as } |x| \to \infty. \tag{4.22}$$

Assume also that there exists $q>d$ and $C>0$, such that

$$\sum_{n=0}^{\infty} M^{(n)}(x,x) \le \frac{C}{(|x|+1)^q}. \tag{4.23}$$

Then the limit

$$\mathbb{P}_\infty(F) = \lim_{|x|\to\infty} \mathbb{P}_x(F) \tag{4.24}$$

exists for any cylinder event F, and also for $F=\{0 \longleftrightarrow y\}$. Also, $\mathbb{P}_\infty$ extends uniquely from $\mathcal{F}_0$ to a probability measure on $\mathcal{F}$.

Proof. The proof is essentially the same as the proof of Theorem 1.1. By (4.23) we get that $M^{(N)}(x,x)$ is summable in N, therefore, by (4.18) and (4.13), $\Pi(x;F)$ and $\Psi(x;F)$ are well defined, and (3.5) holds. By (4.23) and $q>d-2$, the term $\Pi(x;F)/\tau(x)$ in (3.6) is an error term. We use (4.23) and $q>d$ to see that the sum over y with $|x-y| \le \frac{1}{2}|x|$ leads to an error term by adapting the argument in (3.8). We use the summability of the bound (4.23) in x, as well as (4.22) to see that the sum over y with $|x-y| \ge \frac{1}{2}|x|$ converges by dominated convergence to the right-hand side of (3.9). This completes the proof. □

Corollary 4.2. *There exists $d_0>6$ such that for $d>d_0$ in the nearest-neighbor model the limit in (4.24) exists, and defines a measure $\mathbb{P}_\infty$.*

Proof. The conditions of Theorem 4.1 hold for the nearest-neighbor percolation in $d \ge d_0$ for some $d_0>6$ by [12]. □

5 Properties of the IIC

We now proceed to prove Theorem 1.3. We will separate the proof into three parts, and first prove Theorem 1.3(i-ii).

Proof of Theorem 1.3(i-ii). To prove Theorem 1.3(i), we note that

$$\mathbb{P}_\infty(|\mathcal{C}(0)| = \infty) = \lim_{n\to\infty} \mathbb{P}_\infty(0 \longleftrightarrow \mathbb{Z}^d \setminus B(n)) = \lim_{n\to\infty} \lim_{|x|\to\infty} \mathbb{P}_x(0 \longleftrightarrow \mathbb{Z}^d \setminus B(n)) = 1, \tag{5.1}$$

where in the second step we used that $\{0 \longleftrightarrow \mathbb{Z}^d \setminus B(n)\}$ is a cylinder event, since D has finite range. This completes the proof of Theorem 1.3(i).

To show that the IIC has one end, let $F_n(y_1, y_2)$ denote the event that $y_1, y_2 \in \mathcal{C}(0)$ and there are disjoint paths from y_1 and y_2 to $\mathbb{Z}^d \setminus B(n)$. We have, using the definition of G_n in (3.12),

$$F_n(y_1, y_2) \cap \{0 \longleftrightarrow x\} \subset G_n(y_1, y_2; x) \cup G_n(y_2, y_1; x), \tag{5.2}$$

which as in the proof of (3.10) implies $\lim_{n\to\infty} \mathbb{P}_\infty(F_n(y_1, y_2)) = 0$. This means that there are no disjoint infinite paths starting at y_1 and y_2 $\mathbb{P}_\infty$-a.s. Since y_1 and y_2 are arbitrary, this proves the claim in Theorem 1.3(ii).

Proof of Theorem 1.3(iii). In the proof of Theorem 1.3(iii), we need the description of $\mathbb{P}_\infty$ using the lace expansion.

We first prove the upper bound in Theorem 1.3(iii). We use that by (3.10)

$$\mathbb{P}_\infty(0 \longleftrightarrow y) = \lim_{|x|\to\infty} \frac{\mathbb{P}(0 \longleftrightarrow y, x)}{\tau(x)}. \tag{5.3}$$

The tree-graph inequality [5] gives

$$\mathbb{P}(0 \longleftrightarrow y, x) \le \sum_{a\in\mathbb{Z}^d} \tau(a)\tau(x-a)\tau(y-a). \tag{5.4}$$

This yields

$$\mathbb{P}_\infty(0 \longleftrightarrow y) \le \limsup_{|x|\to\infty} \sum_{a\in\mathbb{Z}^d} \tau(a)\tau(y-a)\frac{\tau(x-a)}{\tau(x)}. \tag{5.5}$$

We again split the sum into $|x-a| \le \frac{1}{2}|x|$ and $|x-a| \ge \frac{1}{2}|x|$. In the former case, we use that $|a| \ge \frac{1}{2}|x|$, and since $|y| \ll |x|$, we can assume that $|y-a| \ge \frac{1}{4}|x|$. We then use (1.8), to obtain the bound

$$\begin{aligned}\sum_{a:|x-a|\le\frac{1}{2}|a|} \tau(a)\tau(y-a)\tau(x-a) &\le \frac{C}{|x|^{2(d-2)}} \sum_{a:|x-a|\le\frac{1}{2}|x|} \frac{1}{(|x-a|+1)^{d-2}} \\ &\le \frac{C}{|x|^{2(d-2)}}|x|^2 = o(\tau(x)).\end{aligned} \tag{5.6}$$

Therefore, we are left to deal with $|x-a| \ge \frac{1}{2}|x|$. In this regime, $\tau(x-a)/\tau(x)$ is uniformly bounded, and converges to 1 for every fixed a. Moreover, $\tau(a)\tau(y-a)$ is absolutely summable in a, so that by dominated convergence,

$$\mathbb{P}_\infty(0 \longleftrightarrow y) \le \sum_{a\in\mathbb{Z}^d} \tau(a)\tau(y-a) = (\tau * \tau)(y). \tag{5.7}$$

The upper bound now follows from (1.7), together with the convolution bound in [13, Proposition 1.7], which shows that $(\tau * \tau)(y) \leq c\beta(|y|+1)^{-(d-4)}$.

For the lower bound, we use the lace expansion in (3.9) to obtain with $F = \{0 \longleftrightarrow y\}$

$$\mathbb{P}_\infty(0 \longleftrightarrow y) = p_c \sum_{u\in\mathbb{Z}^d} \Pi(u;F). \tag{5.8}$$

We will use the bounding diagrams in the previous section. We use that $\mathcal{W} = \{0,y\}$ for $F = \{0 \longleftrightarrow y\}$. Moreover, when $0 \longleftrightarrow y$,

$$\Big\{\{0,y\} \Longrightarrow u\Big\} = \{0 \longleftrightarrow u\} \circ \{y \longleftrightarrow u\}. \tag{5.9}$$

Thus, we can replace the bounds in (4.5) by

$$\begin{aligned} &\mathbb{E}_0\Big(I[F,\, 0 \longleftrightarrow u_0,\, \mathcal{W} \Longleftrightarrow u_0]I[w_1 \in \tilde{\mathcal{C}}_0]\Big) \\ &\quad \leq \sum_{z_0} \left[\tau(u_0)\tau(z_0-y)\tau(u_0-z_0)\tau(w_1-z_0) + \tau(u_0-y)\tau(z_0)\tau(u_0-z_0)\tau(w_1-z_0)\right]. \end{aligned} \tag{5.10}$$

We now investigate the arising diagrams in some detail.

The lemma below is a minor modification of [13, Lemma 4.1], and allows us to prove the necessary bounds on $M^{(N)}(x,x;F)$. In its statement, we use symmetrized bounds to allow for a simpler proof. Also, we use the definition

$$T(x,y) = \sum_{u,v\in\mathbb{Z}^d} \frac{1}{(|u-v|+1)^{d-2}(|y-u|+1)^{d-2}(|x-v|+1)^{d-2}}, \qquad \bar{T} = \sup_{x,y\in\mathbb{Z}^d} T(x,y). \tag{5.11}$$

We note that $T(x,y) < \infty$ precisely when $d > 6$. Thus, here we indeed use that $d > 6$.

Lemma 5.1. *Fix $d > 6$. Suppose that there exists a K_0 such that*

$$A^{(0)}(x,y;F) \leq K_0 \sum_{w\in\mathcal{W}} \left\{\frac{1}{(|x|+1)^{d-2}(|y-w|+1)^{d-2}} + \frac{1}{(|y|+1)^{d-2}(|x-w|+1)^{d-2}}\right\}, \tag{5.12}$$

and suppose that $A^{(i)}$ for $i \geq 1$ and $A^{(\mathrm{end})}$ satisfy

$$A^{(*)}(u,v,x,y) \leq \frac{K_*}{(|u-v|+1)^{d-2}(|y-u|+1)^{d-2}(|x-v|+1)^{d-2}} \tag{5.13}$$

with $K_* > 0$. Then there is a C depending only on d, such that for $N \geq 1$

$$\begin{aligned} M^{(N)}(x,y;F) \leq (C\bar{T})^N \Big(\prod_{i=0}^{N-1} K_i\Big) K_{\mathrm{end}} \\ \times \sum_{w\in\mathcal{W}} \left\{\frac{1}{(|x|+1)^{d-2}(|y-w|+1)^{d-2}} + \frac{1}{(|y|+1)^{d-2}(|x-w|+1)^{d-2}}\right\}. \end{aligned} \tag{5.14}$$

Proof. The proof is a minor adaptation of the proof of [13, Lemma 4.1] and is by induction on N. To deal with the fact that $M^{(N)}$ is not defined literally by a convolution of $M^{(N-1)}$ with $A^{(\mathrm{end})}$, we proceed as follows. Let $\tilde{M}^{(N)}$ be the quantity defined by replacing $A^{(\mathrm{end})}$ by $A^{(N)}$ in the definition of $M^{(N)}$. Because all the constituent factors in the definitions of $M^{(N)}$ and $\tilde{M}^{(N)}$ obey bounds of the same form, it suffices to prove that $\tilde{M}^{(N)}$ obeys (5.14) with $K_{\rm end}$ replaced by K_N. We prove this by induction on N.

For $x, y \in \mathbb{Z}^d$, let

$$S(x,y;F) = \sum_{w\in\mathcal{W}} \sum_{u,v\in\mathbb{Z}^d} \frac{1}{(|u|+1)^{d-2}(|w-v|+1)^{d-2}(|u-v|+1)^{d-2}} \times \frac{1}{(|y-u|+1)^{d-2}(|x-v|+1)^{d-2}}. \tag{5.15}$$

By definition, and using (5.12)–(5.13),

$$\tilde{M}^{(1)}(x,y;F) \le K_0 K_1 \Big[S(x,y;F) + S(y,x;F)\Big]. \tag{5.16}$$

Therefore, for the case $N=1$, it suffices to show that

$$S(x,y;F) \le C\bar{T} \sum_{w\in\mathcal{W}} \frac{1}{(|x-w|+1)^{d-2}(|y|+1)^{d-2}}. \tag{5.17}$$

Before proving this, we show that (5.17) also allows us to complete the induction. Indeed, assume that $N \ge 2$, and the inductive hypothesis that $\tilde{M}^{(N-1)}$ obeys (5.14) with $K_{\rm end}$ replaced by K_{N-1} and N replaced by $N-1$ on the right side. Then by (4.12) and (5.13),

$$\tilde{M}^{(N)}(x,y;F) \le (C\bar{T})^{N-1} \Big(\prod_{i=0}^{N} K_i \Big) \Big[S(x,y;F) + S(y,x;F)\Big], \tag{5.18}$$

and (5.14) follows from (5.17). To prove (5.17), we write $S(x,y;F) \le \sum_{i=1}^4 S_i(x,y;F)$, with $S_i(x,y;F)$ defined to be the contribution to $S(x,y;F)$ arising from each of the four cases below.

Case 1. $|w-v| \ge |x-v|$ and $|u| \ge |y-u|$. This implies $|w-v| \ge |x-w|/2$ and $|u| \ge |y|/2$, so that

$$\begin{aligned} S_1(x,y;F) &\le \sum_{w\in\mathcal{W}} \frac{C}{(|x-w|+1)^{d-2}(|y|+1)^{d-2}} \\ &\qquad \times \sum_{u,v} \frac{1}{(|y-u|+1)^{d-2}(|u-v|+1)^{d-2}(|x-v|+1)^{d-2}} \\ &\le \bar{T} \sum_{w\in\mathcal{W}} \frac{C}{(|x-w|+1)^{d-2}(|y|+1)^{d-2}}. \end{aligned} \tag{5.19}$$

Case 2. $|w-v| \ge |x-v|$ and $|u| \le |u-y|$. This implies $|w-v| \ge |x-w|/2$ and $|u-y| \ge |y|/2$. Then

$$\begin{aligned} S_2(x,y;F) &\le \sum_{w\in\mathcal{W}} \frac{C}{(|x-w|+1)^{d-2}(|y|+1)^{d-2}} \\ &\qquad \times \sum_{u,v} \frac{1}{(|u|+1)^{d-2}(|u-v|+1)^{d-2}(|x-v|+1)^{d-2}} \\ &\le \bar{T} \sum_{w\in\mathcal{W}} \frac{C}{(|x-w|+1)^{d-2}(|y|+1)^{d-2}}. \end{aligned} \tag{5.20}$$

Cases 3 and 4 are when $|w-v| \leq |x-v|$, and either $|u| \leq |u-y|$ or $|u| > |u-y|$, respectively, and are similar. Adding the contributions in the four cases yields (5.17) and completes the proof. $\square$

Lemma 5.1 immediately yields that since $\mathcal{W} = \{0, y\}$, we have

$$M^{(N)}(x, x; F) \leq C^N \beta^{1\vee N} \frac{1}{(|x|+1)^{d-2}} \frac{1}{(|x-y|+1)^{d-2}}, \qquad N \geq 1. \tag{5.21}$$

We next argue that the bound in (5.21) can be improved to

$$M^{(N)}(x, x; F) \leq C^N \beta^{3\vee N} \frac{1}{(|x|+1)^{d-2}} \frac{1}{(|x-y|+1)^{d-2}} + C\beta^2 \frac{1}{(|y|+1)^{d-2}} \delta_{1,N}[\delta_{0,x} + \delta_{y,x}], \qquad N \geq 1. \tag{5.22}$$

For $N \geq 3$ there is no difference, so we restrict attention to $N = 1$ and $N = 2$.

We start with $N = 2$, which is easiest. The bound on $M^{(2)}(x, x; F)$ is equal to the one in Figure 1, where the sum over $w \in \mathcal{W}$ is replaced by $w = y$. In general, for any diagram, the power of β is equal to the number of lines with a non-trivial displacement, where lines corresponding to $\tilde{\tau}$ always receive a β (recall (1.8)). For $N = 2$, there are at least two factors of β due to the lines with factors $\tilde{\tau}$. Since $y \neq 0$, we also must have that either one of the lines starting from 0 or y, or the line between the end-points of those two lines must have non-trivial displacement. This gives the third factor β, as required.

For $N = 1$, the above reasoning yields a single extra factor of β, and we need to find one more. See Figure 2 for the diagram $M^{(N)}(x, x; F)$ when $F = \{0 \longrightarrow y\}$, and for the labels of the different lines. Thus, we have that a, b or c has a non-trivial displacement. We will now list the possibilities is the different cases. We will call a line non-trivial when it corresponds to a factor $\tilde{\tau}$, or when it corresponds to a factor $\tau(s)$ with $s \neq 0$. When a is non-trivial, and b, c are trivial, then the least number of non-trivial lines occurs when e, f, g are trivial. In all other cases there are at least three non-trivial lines giving a factor β^3. In the remaining case, the bound we obtain is $\tau(y)\tau(x-y)\tilde{\tau}(x-y)$. Since $y \neq 0$, we thus obtain three powers of β unless when $x = y$, in which case we end up with $\tau(y)\tilde{\tau}(0)\delta_{y,x}$.

In a similar fashion, when b is non-trivial, we obtain terms with three powers of β except for the term $\tau(y)\tilde{\tau}(0)\delta_{0,x}$, since only the role of 0 and y are interchanged. When c is non-trivial, then we obtain the least powers of β when a, b, d, e, f, g are all trivial, in which case we end up with $\tau(y)\tilde{\tau}(y)\delta_{0,x}$. All other cases receive three factors of β. By (1.8), we obtain (5.22).

Thus, for $F = \{0 \longleftrightarrow y\}$, we obtain by summing over $N \geq 1$ and $x \in \mathbb{Z}^d$ and using [13, Proposition 1.7 (i)] that

$$\sum_{N\geq 1} \sum_{x\in\mathbb{Z}^d} M^{(N)}(x, x; F) \leq C\beta^3 \frac{1}{(|y|+1)^{d-4}} + C\beta^2 \frac{1}{(|y|+1)^{d-2}}. \tag{5.23}$$

When $y > L$, we can combine the two bounds as

$$\sum_{N\geq 1} \sum_{x\in\mathbb{Z}^d} M^{(N)}(x, x; F) \leq C\beta^3 \frac{1}{(|y|+1)^{d-4}}. \tag{5.24}$$

We will see that for large L, this is smaller than the leading term due to $N = 0$.

We next investigate $N = 0$, and use (5.9) to obtain

$$\pi^{(0)}(u; F) = \mathbb{P}((0 \longleftrightarrow u) \circ (u \longleftrightarrow y)) \geq \sum_{v} \mathbb{P}((u, v) \text{ occ. and piv. for } 0 \longleftrightarrow y). \tag{5.25}$$

$$M^{(1)}(x,x;F) =$$

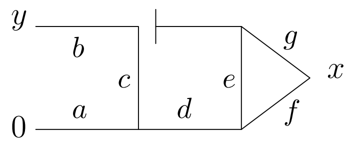

Figure 2: The diagrams representing $M^{(1)}(x,x;F)$ together with the labels of the different lines. Lines ending with a horizontal bar represent $\tilde{\tau}$-lines, and are due to pivotal bonds.

We will evaluate this expression in more detail now.

Similarly to the discussion leading to (2.9), we can write

$$\mathbb{P}((u,v) \text{ is occ. and piv. for } 0 \longleftrightarrow y) = p_c D(v-u)\mathbb{E}\left(I[0 \longleftrightarrow u]\, \tau^{\tilde{\mathcal{C}}^{(u,v)}(0)}(v,y)\right). \tag{5.26}$$

The right-hand side can be rewritten as

$$\mathbb{P}(0 \longleftrightarrow u) p_c D(v-u)\tau(y-v) - \mathbb{E}_0\Big(I[0 \longleftrightarrow u]\mathbb{P}_1(v \xrightarrow{\tilde{\mathcal{C}}^{(u,v)}(0)} y)\Big). \tag{5.27}$$

By the BK-inequality, for $A \subset \mathbb{Z}^d$ we have

$$\begin{aligned}
\mathbb{P}(v \xrightarrow{A} y) &= \mathbb{P}\Big(\bigcup_{w\in A} \{v \longleftrightarrow w\} \circ \{w \longleftrightarrow y\}\Big) \\
&\le \sum_{w\in\mathbb{Z}^d} I[w \in A]\mathbb{P}(\{v \longleftrightarrow w\} \circ \{w \longleftrightarrow y\}) \\
&\le \sum_{w\in\mathbb{Z}^d} I[w \in A]\tau(v-w)\tau(y-w).
\end{aligned} \tag{5.28}$$

Therefore, taking $A = \tilde{\mathcal{C}}^{(u,v)}(0) \subset \mathcal{C}(0)$,

$$\mathbb{P}_1(v \xrightarrow{\tilde{\mathcal{C}}^{(u,v)}(0)} y) \le \sum_{w\in\mathbb{Z}^d} I[w \in \mathcal{C}(0)]\tau(v-w)\tau(y-w). \tag{5.29}$$

Substituting this into (5.27) yields

$$\begin{aligned}
&\mathbb{P}((u,v) \text{ is occ. and piv. for } 0 \longleftrightarrow y) \\
&\quad \ge p_c D(v-u)\tau(u)\tau(y-v) - \sum_{w\in\mathbb{Z}^d} \mathbb{P}(0 \longleftrightarrow u,w)\tau(v-w)\tau(y-w).
\end{aligned} \tag{5.30}$$

The tree-graph bound [5] implies that

$$\mathbb{P}(0 \longleftrightarrow u,w) \le \sum_{z\in\mathbb{Z}^d} \tau(z)\tau(w-z)\tau(u-z). \tag{5.31}$$

Therefore,

$$\begin{aligned}&\mathbb{P}((u,v) \text{ is occ. and piv. for } 0 \longleftrightarrow y)\\ &\qquad \geq p_c D(v-u)\tau(u)\tau(y-v) - p_c D(v-u) \sum_{w,z\in\mathbb{Z}^d} \tau(z)\tau(w-z)\tau(u-z)\tau(v-w)\tau(y-w).\end{aligned} \tag{5.32}$$

We conclude that

$$\begin{aligned}\sum_u \pi^{(0)}(u;F) \geq& \sum_{(u,v)} p_c D(v-u)\tau(u)\tau(y-v) \\ &- \sum_{(u,v)} p_c D(v-u) \sum_{w,z\in\mathbb{Z}^d} \tau(z)\tau(w-z)\tau(u-z)\tau(v-w)\tau(y-w)\\ =& (\tau*\tilde{\tau})(y) - (\tau * g * \tau)(y),\end{aligned} \tag{5.33}$$

where

$$g(x) = \tau(x)(\tau*\tilde{\tau})(x). \tag{5.34}$$

We first investigate g in more detail. We claim that

$$g(x) \leq C\beta\delta_{0,x} + \frac{B\beta^2}{(|x|+1)^{2d-6}}. \tag{5.35}$$

To see (5.35), we note that by (4.14),

$$\tau(x) = \delta_{0,x} + [1-\delta_{0,x}]\tau(x) \leq \delta_{0,x} + \tilde{\tau}(x). \tag{5.36}$$

We use this for the first factor of τ in (5.34) to see that

$$g(x) \leq \delta_{0,x}(\tau*\tilde{\tau})(0) + \tilde{\tau}(x)(\tilde{\tau}*\tau)(x). \tag{5.37}$$

Since $(\tau*\tilde{\tau})(0) \leq C\beta$, we obtain that the first term is bounded by $C\beta\delta_{0,x}$. By [13, Proposition 1.7(i)], the second term is bounded by $\frac{C\beta^2}{(|x|+1)^{2d-6}}$. This proves (5.35).

Thus, by (5.35), g is summable when $d>6$. We finally write

$$(\tau*\tilde{\tau})(y) - (\tau*g*\tau)(y) = \Big(\tau * \Big[\tau*[p_cD-g]\Big]\Big)(x), \tag{5.38}$$

and investigate $\tau*[p_cD-g]$ in more detail. We use the convolution bound in [13, Proposition 1.7(ii)], but to be able to do that, we need an improvement on (1.7). By [13, Theorem 1.2], we have that the constant c in (1.7) equals $a_dA\sigma^{-2}$, where a_d is an absolute constant independent of β, $A = 1+O(\beta)$, while σ^2 is defined in (1.2), and that $o(1)$ is bounded by

$$O\Big(\frac{L^{\epsilon_2}}{(|x|\vee 1)^{\epsilon_2-\alpha}}\Big) + O\Big(\frac{L^2}{(|x|\vee 1)^{2-\alpha}}\Big), \tag{5.39}$$

where the exponent α is the same as the one appearing in the definition of β and $\epsilon_2 = \min\{d-6, 2\}$. Then, [13, Proposition 1.7(ii)] together with (5.35) and (5.39) yields that

$$(\tau*[D-g])(x) = \frac{a_dA\sum_y[p_cD(y)-g(y)]}{\sigma^2(|x|+1)^{d-2}} + e(x), \tag{5.40}$$

where

$$e(x) = \frac{O\Big(BC\beta^3\Big)}{(|x|+1)^{d-2+\epsilon_2-\alpha}}. \tag{5.41}$$

We further note that

$$\sum_y [p_c D(y) - g(y)] = 1 + O(\beta), \tag{5.42}$$

and that the factors $\frac{a_d A}{\sigma^2(|x|+1)^{d-2}}$ are almost $\tau(x)$. Thus, we finally end up with

$$\mathbb{P}_\infty(0 \longleftrightarrow y) \geq (1+O(\beta))(\tau * \tau)(y) - C\beta^3 \frac{1}{(|y|+1)^{d-4}}. \tag{5.43}$$

We complete the proof by obtaining a lower bound on $(\tau * \tau)(y)$ for $|y| > L$, by estimating

$$\begin{aligned}(\tau * \tau)(y) &= \sum_z \tau(z)\tau(y-z) \geq \sum_{\frac{1}{4}|y| \leq |z|, |y-z| \leq 4|y|} \tau(z)\tau(y-z) \\ &\geq \frac{C'}{\sigma^4(|y|+1)^{2d-4}} \sum_{\frac{1}{4}|y| \leq |z|, |y-z| \leq 4|y|} = \frac{C'}{\sigma^4(|y|+1)^{d-4}},\end{aligned} \tag{5.44}$$

since, when $\frac{1}{4}|y| \leq |z| \leq 4|y|$ and $|y| > L$, we have that $\tau(z) \geq C\sigma^{-2}(|y|+1)^{-(d-2)}$.

Thus, when we pick $\alpha > 0$ sufficiently small so that $\beta^3 = L^{-6+3\alpha} \ll \sigma^4 = CL^4$, we finally end up with

$$\mathbb{P}_\infty(0 \longleftrightarrow y) \geq (1 - o(1)) \frac{C}{\sigma^4(|y|+1)^{d-4}}, \tag{5.45}$$

where $o(1)$ tends to 0 when $L \to \infty$. This completes the proof of Theorem 1.3(iii).

Proof of Theorem 1.3(iv). To prove (1.14) we first show that

$$\mathbb{P}_\infty((0 \longleftrightarrow y) \circ (y \longleftrightarrow \infty)) = \lim_{|x|\to\infty} \mathbb{P}_x((0 \longleftrightarrow y) \circ (y \longleftrightarrow x)). \tag{5.46}$$

Denote the events on the left- and right-hand sides by F and F_x, respectively. (We suppress the dependence on y.) Also, define the events

$$\begin{aligned} F_k &= \{0 \longleftrightarrow y \text{ inside } B(k)\} \circ \{y \longleftrightarrow \infty\}, \qquad k \geq 1, \\ F_{n,k} &= \{0 \longleftrightarrow y \text{ inside } B(k)\} \circ \{y \longleftrightarrow \mathbb{Z}^d \setminus B(n)\}, \qquad n \geq k \geq 1, \\ F_{x,k} &= \{0 \longleftrightarrow y \text{ inside } B(k)\} \circ \{y \longleftrightarrow x\}, \qquad k \geq 1. \end{aligned} \tag{5.47}$$

For $\|y\|_\infty < n < \|x\|_\infty$ we have $F_{n,k} \supset F_{x,k}$. Moreover,

$$(F_{n,k} \setminus F_{x,k}) \cap \{0 \longleftrightarrow x\} \subset G_n(0, y; x), \tag{5.48}$$

where $G_n(0,y;x)$ is introduced in (3.12). This implies that for any k we have

$$|\mathbb{P}_x(F_{n,k}) - \mathbb{P}_x(F_{x,k})| \leq \mathbb{P}(y \longleftrightarrow \mathbb{Z}^d \setminus B(n)). \tag{5.49}$$

Therefore, since $F_{n,k}$ is a cylinder event, we have

$$\mathbb{P}_\infty(F_k) = \lim_{n\to\infty} \mathbb{P}_\infty(F_{n,k}) = \lim_{n\to\infty} \lim_{|x|\to\infty} \mathbb{P}_x(F_{n,k}) = \lim_{|x|\to\infty} \mathbb{P}_x(F_{x,k}). \tag{5.50}$$

Similarly, we have $F_{x,k} \subset F_x$ and $F_x \setminus F_{x,k} \subset G_k(y,0;x)$, which implies

$$|\mathbb{P}_x(F_x) - \mathbb{P}_x(F_{x,k})| \leq \mathbb{P}(0 \longleftrightarrow \mathbb{Z}^d \setminus B(k)) \frac{\tau(x-y)}{\tau(x)}. \tag{5.51}$$

Using (5.50), this implies

$$\mathbb{P}_\infty(F) = \lim_{k\to\infty} \mathbb{P}_\infty(F_k) = \lim_{k\to\infty} \lim_{|x|\to\infty} \mathbb{P}_x(F_{x,k}) = \lim_{|x|\to\infty} \mathbb{P}_x(F_x). \tag{5.52}$$

The upper bound in (1.14) now follows from the BK inequality and (1.8), since

$$\lim_{|x|\to\infty} \mathbb{P}_x((0 \longleftrightarrow y) \circ (y \longleftrightarrow x)) \leq \limsup_{|x|\to\infty} \tau(y) \frac{\tau(x-y)}{\tau(x)} = \tau(y) \leq \frac{C}{|y|^{d-2}}. \tag{5.53}$$

For the lower bound we argue similarly to part (iii) of the theorem. We have

$$\begin{aligned} \mathbb{P}((0 \longleftrightarrow y) \circ (y \longleftrightarrow x)) &\geq \sum_v \mathbb{P}((y,v) \text{ is occupied and pivotal for } 0 \longrightarrow x) \\ &\geq \sum_v p_c D(v-y) \Big\{ \tau(y)\tau(x-v) - \sum_{w,z} \tau(z)\tau(w-z)\tau(y-z)\tau(v-w)\tau(x-w) \Big\}. \end{aligned} \tag{5.54}$$

We now divide by $\tau(x)$, and take the limit $|x| \to \infty$. The contribution of the first term is

$$\tau(y) \lim_{|x|\to\infty} \frac{\tilde{\tau}(x-y)}{\tau(x)} = p_c \tau(y). \tag{5.55}$$

The second term is

$$\sum_{w,z} \tau(z)\tau(w-z)\tau(y-z)\tilde{\tau}(y-w) \frac{\tau(x-w)}{\tau(x)}. \tag{5.56}$$

We split the summation into two parts according to whether $|w-x| \leq |x|/2$ or $\geq |x|/2$. In the first case, we apply (1.7) to get $\tau(x) \geq C(|x|+1)^{-(d-2)}$, and we apply (1.8) to the remaining factors. Then we use that $|x-w| \leq |x|/2$ and $|y| \ll |x|$ imply $|w-y| \geq |x|/4$, and therefore we can bound $(|w-y|+1)^{-(d-2)}$ by $C(|x|+1)^{-(d-2)}$. This gives

$$\begin{aligned} &\sum_{w,z:|x-w|\leq|x|/2} \tau(z)\tau(y-z)\tau(w-z)\tilde{\tau}(y-w) \frac{\tau(x-w)}{\tau(x)} \\ &\qquad \leq \sum_{w,z} \frac{C\beta}{(|z|+1)^{d-2}(|y-z|+1)^{d-2}(|w-z|+1)^{d-2}(|x-w|+1)^{d-2}}. \end{aligned} \tag{5.57}$$

Next we use that either $|z| \geq |y|/2$ or $|y-z| \geq |y|/2$ to bound one of the first two factors above by $C(|y|+1)^{-(d-2)}$. This shows that the right-hand side of (5.57) is bounded above by

$$\frac{C\beta(T(y,x) + T(0,x))}{(|y|+1)^{d-2}}. \tag{5.58}$$

An application of the convolution bound in [13, Proposition 1.7] shows that when $d > 6$, we have $T(y,x) \leq C(|x-y|+1)^{-(d-6)}$, and hence both $T(y,x)$ and $T(0,x)$ go to 0 as $x \to \infty$.

In the case $|w-x| \geq |x|/2$, we use that $\tau(w-x)/\tau(x)$ is uniformly bounded, and converges to 1 for every fixed w, z as $|x| \to \infty$. Also, the sum of the remaining part over w, z is bounded by $C\beta/(|y|+1)^{d-6}$, again by an application of [13, Proposition 1.7]. Therefore, by the dominated convergence theorem and (5.46),

$$\mathbb{P}_\infty((0 \longleftrightarrow y) \circ (y \longleftrightarrow \infty)) \geq p_c \tau(y) - (\tau * g)(y). \tag{5.59}$$

As in part (iii), we see that the second term contains an extra factor of β, hence we get the lower bound $c|y|^{-(d-2)}$. $\square$

We also expect the conclusions of Theorem 1.3 to hold for the nearest-neighbor model under conditions alike Theorem 4.1, but we refrain from stating it.

Acknowledgements

The discussions for this article were initiated during an extensive visit of RvdH to the University of British Columbia, Vancouver, Canada. This work was supported in part by NSERC of Canada. The work of AAJ was also supported by a Postdoctoral Fellowship from the Pacific Institute for the Mathematical Sciences. The work of RvdH was carried out in part also at Delft University of Technology, Delft, The Netherlands. The work of AAJ was carried out in part also at the University of British Columbia, Vancouver, Canada. We thank Gordon Slade for stimulating discussions at the start of the project.

References

[1] A. Aharony, Y. Gefen and A. Kapitulnik. Scaling at the percolation threshold above six dimensions. *J. Phys. A.* **17** L 197, (1984)

[2] M. Aizenman. On the number of incipient spanning clusters. *Nuclear Phys. B*, **485** 551–582, (1997).

[3] S. Alexander, G. Grest, H. Nakanishi and T. Witten Jr. Branched polymer approach to the structure of lattice animals and percolation clusters. *J. Phys. A.* **17** L 185, (1984).

[4] M.T. Barlow. Random walks on supercritical percolation clusters. Preprint (2003).

[5] M. Aizenman and C.M. Newman. Tree graph inequalities and critical behavior in percolation models. *J. Stat. Phys.*, **36**:107–143, (1984).

[6] I. Benjamini, H. Kesten, Y. Peres and O. Schramm. Geometry of the Uniform Spanning Forest: Transitions in Dimensions 4, 8, 12, ... Preprint, arXiv:math.PR/0107140, (2002).

[7] C.E. Bezuidenhout and G.R. Grimmett. The critical contact process dies out. *Ann. Probab.*, **18** 1462–1482, (1990).

[8] C. Borgs, J.T. Chayes, H. Kesten and J. Spencer. The birth of the infinite cluster cluster: finite-size scaling in percolation. *Commun. Math. Phys.* **224** 153–204, (2001).

BS85
[9] D.C. Brydges and T. Spencer. Self-avoiding walk in 5 or more dimensions. *Commun. Math. Phys.*, **97** 125–148, (1985).

CCN85
[10] J.T. Chayes, L. Chayes and C.M. Newman. Stochastic geometry of invasion percolation. *Commun. Math. Phys.*, **101** 383–407, (1985).

Grim99
[11] G. Grimmett. *Percolation.* Springer, Berlin, 2nd edition, (1999).

Hara00
[12] T. Hara. Critical two-point functions for nearest-neighbour high-dimensional self-avoiding walk and percolation. In preparation.

HHS01a
[13] T. Hara, R. van der Hofstad and G. Slade. Critical two-point functions and the lace expansion for spread-out high-dimensional percolation and related models. *Ann. Probab.* **31** 349–408, (2003).

HS90
[14] T. Hara and G. Slade. Mean-field critical behaviour for percolation in high dimensions. *Commun. Math. Phys.*, **128** 333–391, (1990).

HS95
[15] T. Hara and G. Slade. The self-avoiding-walk and percolation critical points in high dimensions. *Combinatorics, Probability and Computing*, **4**:197–215, (1995).

HS00
[16] T. Hara and G. Slade. The scaling limit of the incipient infinite cluster in high-dimensional percolation. I. Critical exponents. *J. Statist. Phys.*, **99** 1075–1168, (2000).

HofHolS01
[17] R. van der Hofstad, F. den Hollander and G. Slade. Construction of the incipient infinite cluster for spread-out oriented percolation above $4+1$ dimensions. *Commun. Math. Phys.* **231** 435–461, (2002).

Jara01
[18] A.A. Járai. Invasion percolation and the incipient infinite cluster in 2D. To appear in *Commun. Math. Phys.*

Jara02
[19] A.A. Járai. Incipient infinite percolation clusters in 2D. *Ann. Probab.* **31** 444–485, (2003).

Kest80
[20] H. Kesten. The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. *Commun. Math. Phys.* **74** 41–59, (1980).

Kestenbook
[21] H. Kesten. *Percolation theory for mathematicians.* Birkäuser, Boston, (1982).

Kest86a
[22] H. Kesten. The incipient infinite cluster in two-dimensional percolation. *Probab. Theory Related Fields*, **73** 369–394, (1986).

Kest86b
[23] H. Kesten. Subdiffusive behavior of random walk on a random cluster. *Ann. Inst. Henri Poincaré*, **22** 425–487, (1986).

LSW02
[24] G.F. Lawler, O. Schramm and W. Werner. One-arm exponent for critical 2D percolation. *Electron. J. Probab.*, **7**, 13 pp., electronic. (2002).

MS93
[25] N. Madras and G. Slade. *The self-avoiding walk.* Birkhäuser, Boston, (1993).

NS96
[26] C. M. Newman and D. L. Stein. Ground-state structure in a highly disordered spin-glass model. *J. Statist. Phys.* **82** 1113–1132, (1996).

RT83
[27] R. Rammal and G. Toulouse. Random walk on fractal structures and percolation clusters. *J. Physique Lett.* **44** L13–L22, (1983).

Russo81
[28] L. Russo. On the critical percolation probabilities. *Z. Wahrsch. Verw. Gebiete*, **56** 229–237, (1981).