Matroids, the greedy algorithm, and the matroid polytope

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June 2, 2014
Spanning forests and -trees

Let $G = (V, E)$ be an undirected graph, and let $F \subseteq E$

- $F$ is a forest if $(V, F)$ does not contain any cycles.
- $F$ spans $G$ if $(V, F)$ and $G$ have the same number of components.
- $F$ is a tree if $(V, F)$ is a forest with exactly one component.

The maximum spanning forest problem

**Given:** A graph $G = (V, E)$, a weight function $w : E \to \mathbb{R}$.

**Find:** A spanning forest $F$ such that $w[F]$ is as large as possible.
Kruskal’s algorithm

Given are an undirected graph $G = (V, E)$ and a weight function $w : E \to \mathbb{R}$.

Kruskal’s algorithm

1. Sort the edges by weight, so that $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$.
2. $F \leftarrow \emptyset$, $i \leftarrow 1$
3. while $i < |E|$: 
   1. if $F \cup \{e_i\}$ is a forest, put $F \leftarrow F \cup \{e_i\}$
   2. $i \leftarrow i + 1$

Theorem

*Kruskal’s algorithm finds a maximum-weight spanning forest.*
A matroid is determined by a finite set $E$, the *ground set*, and a partition of the set of subsets of $E$ in *dependent* and *independent sets*.
Matroids

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**Definition (Matroid)**

A *matroid* is a pair $(E, \mathcal{I})$, where $E$ is a finite set, and $\mathcal{I} \subseteq 2^E$, such that:

1. $\emptyset \in \mathcal{I}$
2. if $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$
3. if $I, J \in \mathcal{I}$ and $|I| < |J|$, then $\exists e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$

**Example (The Fano Matroid)**

Let $E := \{a, b, c, d, e, f, g\}$ and let $\mathcal{I} := \{I \subseteq E | |I| \leq 3\} \setminus \{abc, cde, efa, adg, cfg, beg, bdf\}$.

Then $F_7 := (E, \mathcal{I})$ is the Fano matroid.
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**Example (Graphic matroid)**

Let \( G = (V, E) \) be an undirected graph and let

\[
\mathcal{I} := \{ F \subseteq E \mid (V, F) \text{ is a forest} \}.
\]

Then \( M(G) := (E, \mathcal{I}) \) is a graphic matroid.
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A matroid is a pair $(E, \mathcal{I})$, where $E$ is a finite set, and $\mathcal{I} \subseteq 2^E$, such that:

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**Example (Linear matroid)**

Let $\mathbb{F}$ be a field and let $E \subseteq \mathbb{F}^k$ be a finite set of vectors. Let

$$\mathcal{I} := \{F \subseteq E \mid F \text{ is linearly independent over } \mathbb{F}\}.$$ 

Then $M(E, \mathbb{F}) := (E, \mathcal{I})$ is a linear matroid.
The greedy algorithm

if $M = (E, \mathcal{I})$ is a matroid, then $F \subseteq E$ is a *basis* if $F$ is an inclusionwise maximal independent set.

The maximum-weight basis problem

**Given:** A matroid $M = (E, \mathcal{I})$, a weight function $w : E \rightarrow \mathbb{R}$.

**Find:** A basis $F$ such that $w[F]$ is as large as possible.

The greedy algorithm

1. Sort the edges by weight, so that $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$.
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   1. if $F \cup \{e_i\}$ is independent, put $F \leftarrow F \cup \{e_i\}$
   2. $i \leftarrow i + 1$
The greedy algorithm characterizes matroids

**Theorem**

Let $M = (E, I)$ be such that

1. $\emptyset \in I$, and
2. if $J \in I$ and $I \subseteq J$, then $I \in I$.

Then $M$ is a matroid if and only if the greedy algorithm finds a basis $B$ of maximum weight $w[B]$, for each weight function $w : E \to \mathbb{R}_+$. 

Proof outline: We first prove sufficiency, $\Rightarrow$. Suppose $M = (E, I)$ is not a matroid. Then there exist $I, J \in I$ such that $|I| < |J|$, but $\nexists e \in J \setminus I$ such that $I \cup \{e\} \in I$. Let $k := |I|$. Define $w : E \to \mathbb{R}_+$ by $w(e) := k + 2$ if $e \in I$, $w(e) := k + 1$ if $e \in J \setminus I$, and $w(e) := 0$ if $e \not\in J$. The greedy algorithm outputs $B \supseteq I$ with $w[B] = w[I] = k(k+2) < (k+1)(k+1) \leq w[J]$. So $B$ is not optimal.
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Matroids I
June 2, 2014 8 / 1
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The greedy algorithm characterizes matroids

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Then \( M \) is a matroid if and only if the greedy algorithm finds a basis \( B \) of maximum weight \( w[B] \), for each weight function \( w : E \rightarrow \mathbb{R}_+ \).

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The greedy algorithm characterizes matroids

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- Suppose $M = (E, \mathcal{I})$ is a matroid. Let $w : E \to \mathbb{R}_+$ be a weight function.
The greedy algorithm characterizes matroids

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**Proof outline:** We next prove necessity, ‘$\Rightarrow$’.

- Suppose $M = (E, \mathcal{I})$ is a matroid. Let $w : E \to \mathbb{R}_+$ be a weight function.
- Call an independent set $I \in \mathcal{I}$ *greedy* if there is a maximum-weight basis $B$ so that $I \subseteq B$. 
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Proof outline: We next prove necessity, ‘$\Rightarrow$’.

- Suppose $M = (E, \mathcal{I})$ is a matroid. Let $w : E \to \mathbb{R}_+$ be a weight function.
- Call an independent set $I \in \mathcal{I}$ greedy if there is a maximum-weight basis $B$ so that $I \subseteq B$.
- To prove: if $I$ is greedy, and $e$ attains the maximum in $\max\{w(e) \mid I \cup \{e\} \in \mathcal{I}, e \in E \setminus I\}$, then $I \cup \{e\}$ is greedy.
Let $E$ be a finite set, and let $\mathcal{A}$ be a finite set of subsets of $E$. A *transversal* of $\mathcal{A}$ is a set $F \subseteq E$ so that there exists an injection $\phi : F \to \mathcal{A}$ with $e \in \phi(e)$ for all $e \in F$.

**Example (Transversal matroids)**

Let $E$ be a finite set, and let $\mathcal{A}$ be a finite set of subsets of $E$. Put

$$\mathcal{I} := \{F \subseteq E \mid F \text{ is a transversal of } \mathcal{A}\}.$$  

Then $M(E, \mathcal{A}) := (E, \mathcal{I})$ is a *transversal matroid*. 

---
Let $D = (V, A)$ be a directed graph and let $S, T \subseteq V$. Then a subset $F \subseteq T$ is *linked* to $S$ in $D$ if there is a set of vertex-disjoint directed paths with starting points in $S$ and with endpoints $F$.

**Example (Gammoids)**

Let $D = (V, A)$ be a directed graph, and let $S, T \subseteq V$. Let

$$I := \{ F \subseteq T \mid F \text{ is linked to } S \text{ in } D \}.$$

Then $M(D, S, T) := (V, I)$ is a *gammoid*.
Definition

Let $\mathbb{K}$ be an extension field of $\mathbb{F}$. A set $\{x_1, \ldots, x_n\} \subseteq \mathbb{K}$ is \textit{algebraically dependent over} $\mathbb{F}$ if there exists a polynomial $p$ with coefficients in $\mathbb{F}$ such that $p(x_1, \ldots, x_n) = 0$.
Algebraic matroids

Definition
Let $K$ be an extension field of $F$. A set $\{x_1, \ldots, x_n\} \subseteq K$ is algebraically dependent over $F$ if there exists a polynomial $p$ with coefficients in $F$ such that $p(x_1, \ldots, x_n) = 0$.

Example (Algebraic matroids)
Let $K$ be an extension field of $F$, and let $E \subseteq K$ be finite. Let

$$I := \{F \subseteq E \mid F \text{ algebraically independent over } F\}$$

Then $M(E, F) := (E, I)$ is an algebraic matroid.
Definition

Let $H := \{ z \in \mathbb{C} \mid \Re(z) > 0 \}$. A complex polynomial $p$ in $n$ variables has the half-plane property if $p(x_1, \ldots, x_n) \neq 0$ for all $x_1, \ldots, x_n \in H$. 

Theorem

Let $p = \sum_{F \subseteq E} p_F x_F$ be a homogeneous complex polynomial. If $p$ has the half-plane property, then $\{ F \subseteq E \mid p_F \neq 0 \}$ is the set of bases of a matroid on $E$. 

... so these are the half-plane-property (HPP) matroids.
'Half-plane property' matroids

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Let \( \{ x_e \mid e \in E \} \) be variables. For \( F \subseteq E \), we write \( x^F := \prod_{e \in F} x_e \).
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The matroid polytope

If $A \subseteq E$, then its incidence vector $x^A \in \mathbb{R}^E$ is determined by

$$x_e^A = \begin{cases} 1 & \text{if } e \in A \\ 0 & \text{if } e \notin A \end{cases}$$

**Definition (Matroid polytope)**

Let $M = (E, \mathcal{I})$ be a matroid. The matroid polytope is

$$P(M) := \text{conv.hull}\{x^I \mid I \in \mathcal{I}\}.$$ 

The rank of $F \subseteq E$ in $M = (E, \mathcal{I})$ is $r_M(F) := \max\{|I| \mid I \in \mathcal{I}, I \subseteq F\}$.

**Theorem**

$$P(M) = \{x \in \mathbb{R}^E \mid x[F] \leq r_M(F) \text{ for all } F \subseteq E, \ x \geq 0\}$$
Theorem

\[ P(M) = \{ x \in \mathbb{R}^E \mid x[F] \leq r_M(F) \text{ for all } F \subseteq E, \ x \geq 0 \} \]

Proof outline: It suffices to prove that for any \( w : E \to \mathbb{R} \), the problem

\[ \max \{ w^T x \mid x \in P(M) \} \]

has an optimal solution \( x^* = x^I \), where \( I \) is an independent set of \( M \).

- Let \( f_1, f_2, \ldots, f_m \) be the elements of \( E \) as chosen by the greedy algorithm.
- Let \( F_i := \{ e \in E \mid r_M\{f_1, \ldots, f_i, e\} = r_M\{f_1, \ldots, f_i\} \} \).
- Let \( p = \max \{ i \mid w(f_i) > 0 \} \), and put \( I := \{ f_1, \ldots, f_p \} \).
- If \( x \in P(M) \), then

\[
w^T x \leq \sum_{i=1}^{p} u_i x[F_i] \leq \sum_{i=1}^{p} u_i r_M(F_i) \leq \sum_{i=1}^{p} w(f_i) = w^T x^I
\]

for an appropriate choice of \( u_i \geq 0 \). So \( x^I \) is an optimal solution.
Some proof details

We choose $u_i := w(f_i) - w(f_{i+1})$ for $i = 1, \ldots, p - 1$, $u_p := w(f_p)$.

- note: $r(F_i) = i$ for each $i$
- $\sum_{i=1}^{p} u_i r_M(F_i) = \sum_{i=1}^{p} w(f_i)$
- if $x \in P(M)$, then $x[F_i] \leq r(F_i)$ by definition of $P(M)$, hence

$$\sum_{i=1}^{p} u_i x[F_i] \leq \sum_{i=1}^{p} u_i r_M(F_i)$$

- to prove $w^T x \leq \sum_{i=1}^{p} u_i x[F_i]$, we need to argue for each $e$ that

$$w(e) \leq \sum_{i=k}^{p} u_i = w(f_k)$$

where $k := \min\{i \mid e \in F_i\}$. But if $e \in F_k \setminus F_{k-1}$, then $w(f_k) \geq w(e)$.