LECTURE 4 — TOTAL UNIMODULARITY AND TOTAL DUAL INTEGRALITY

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In this lecture we give a complete answer to the following question: Let

\[ P = \{ x : Ax \leq b \} \]

be a rational polyhedron given by \( A \in \mathbb{Q}^{m \times n} \), \( b \in \mathbb{Q}^m \), and let

\[ P_I = \text{conv}(P \cap \mathbb{Z}^n) \]

its integral hull. Under which conditions on \( A \), \( b \) can we conclude that \( P \) coincides with \( P_I \); when is \( P \) integral? This is a wonderful situations what efficient algorithms is concerned for solving the integer program

\[ \max \{ c^T x : Ax \leq b, x \in \mathbb{Z}^n \} . \]

Then we simply solve the linear programming relaxation and we are sure that the optimal solution is (also) attained at an integral solution.

The first answer, totally unimodular matrices, will be independent of the right hand side \( b \). The second answer, totally dual integral systems, is more complicated and depends on \( A \) and \( b \).

1. Totally unimodular matrices

Reference: [S1, Chapter 19]

**Definition 1.1.** A matrix \( A \in \mathbb{Z}^{m \times n} \) is totally unimodular if the determinant of every square submatrix \( B \in \mathbb{Z}^{k \times k} \) equals either \(-1\), \(0\), or \(+1\).

Alternatively, by Cramer’s rule, \( A \in \mathbb{Z}^{m \times n} \) is totally unimodular if every nonsingular submatrix \( B \in \mathbb{Z}^{k \times k} \) has an integral inverse \( B^{-1} \in \mathbb{Z}^{k \times k} \). Recall: \( B^{-1} = \frac{1}{\det B} B^* \), where \( B^* \) is the adjugate matrix (transpose of the matrix of cofactors) of \( B \).

One important class of totally unimodular matrices are incidence matrices of directed graphs. These matrices underlie the fact network flow problems with integral, nonsplittable goods can be solved in polynomial time using, for example, linear programming.

**Lemma 1.2.** Let \( D = (V, A) \) be a directed graph. Its incidence matrix \( M \in \mathbb{R}^{V \times A} \) is given by

\[
M_{v,a} = \begin{cases} 
+1, & \text{if } a \text{ enters } v, \\
-1, & \text{if } a \text{ leaves } v, \\
0, & \text{otherwise}. 
\end{cases}
\]

The matrix \( M \) is totally unimodular.

Scribes: Joris Kinable, Maryam Steadie Seifi.
Proof. Let $B \in \mathbb{Z}^{k \times k}$ be a square submatrix of $M$. We show that $\det B = -1, 0, +1$ by induction on $k$.

If $k = 1$, then the statement holds immediately by definition of $M$.

For $k > 1$ we distinguish between three cases:

**Case 1:** $B$ has a complete zero column.

Then $\det B = 0$.

**Case 2:** $B$ has a column with exactly one non-zero element.

Then, $\det B = \det \left( \begin{pmatrix} \pm 1 & b^T \\ 0 & B' \end{pmatrix} \right) = \pm \det B'$ and $\det B' = -1, 0, +1$ by induction, since $B' \in \mathbb{Z}^{(k-1) \times (k-1)}$.

**Case 3:** $B$ has in every column exactly one $+1$ and exactly one $-1$.

Then by adding all the rows we obtain the zero vector. Hence, $\det B = 0$.

\[\square\]

**Theorem 1.3.** Let $A$ be a totally unimodular matrix. Then,

a) for every $b \in \mathbb{Z}^m$ we have

$$P = \{x : Ax \leq b\} = P_I = \text{conv}(P \cap \mathbb{Z}^n).$$

b) for every $b \in \mathbb{Z}^m$ and for every $c \in \mathbb{Z}^n$ we have that the primal and dual linear programs

$$\max \{c^T x : Ax \leq b\} = \min \{y^T b : y^T A = c^T, y \geq 0\}$$

both have integer optimal solutions $x \in \mathbb{Z}^n$ and $y \in \mathbb{Z}^m$, if the optimal values are finite.

**Proof.** a) Let

$$F = \{x : A'x = b'\}$$

be a minimal face of $P$. Here $A'x \leq b'$ is a subsystem of $Ax \leq b$ and $A' \in \mathbb{Z}^{k \times n}$ has full row rank $k$. Then (after permuting columns) $A'$ has the following form

$$A' = [U \ V], \quad \text{where } U \in \mathbb{Z}^{k \times k} \text{ is nonsingular, } V \in \mathbb{Z}^{k \times (n-k)}.$$

By the total unimodularity of $A$ it follows that $\det U = \pm 1$. Hence,

$$x = \begin{pmatrix} U^{-1}b' \\ 0 \end{pmatrix} \in F \cap \mathbb{Z}^n$$

As every minimal face of $P$ contains integer points, it implies that $P = P_I$. \[\square\]

b) follows from a), see [S1, Corollary 19.1a].

The following theorem gives several useful characterizations of totally unimodular matrices.

**Theorem 1.4.** Let $A \in \mathbb{Z}^{m \times n}$ be a matrix. The following three statements are equivalent:

a) $A$ is a totally unimodular matrix,

b) (Hoffman, Kruskal, 1956) For all $b \in \mathbb{Z}^m$ the polyhedron $P = \{x : Ax \leq b, x \geq 0\}$ is integral, i.e. $P = P_I$.

c) (Ghouila, Houri, 1962) Every collection of columns of $A$ can be split into two parts, so that adding the vectors in the first part minus adding the vectors in the second part gives a vector which has components $-1, 0, +1$ only.
See [S1, Corollary 19.2a, Theorem 19.3] for proofs.

The totally unimodularity of special kind of matrices imply several fundamental theorems in combinatorial optimization, namely integral min-max relations. One example is Birkhoff’s theorem:

**Theorem 1.5** (Birkhoff’s theorem). *Every doubly stochastic matrix is a convex combination of permutation matrices.*

Here a matrix \( A \in \mathbb{R}^{n \times n} \) is called doubly stochastic, if all its entries are nonnegative, \( A \geq 0 \), and every of its row sum and every of it column sum equals to 1. An integral doubly stochastic matrix is called a permutation matrix.

**Proof.** (see Problem 7 for some of the details) The incidence matrix of an undirected graph is totally unimodular if and only if the graph is bipartite. Let us consider the complete bipartite graph \( K_{n,n} = (V, E) \) and let \( M \in \{0,1\}^V \times E \) by its incidence matrix. Define the polytope \( P = \{ x \in \mathbb{R}^E : Mx \leq 1, x \geq 0 \} \).

Since \( M \) is totally unimodular, \( P = P_I \) because of the characterization of Hoffman, Kruskal. This now proves Birkhoff’s theorem because every \( x \in P \) corresponds to a doubly stochastic matrix and every vertex of \( P \) corresponds to a permutation matrix. \( \square \)

How difficult is it to recognize if a given matrix is totally unimodular? A deep result of Seymour from 1980 implies that one can check in polynomial time whether a matrix is totally unimodular or not. For time reasons we do not dare to give any details here and we refer to the book [S1, Chapter 20] by Schrijver who presents the algorithm. There is also an implementation available, the unimodularity library from Truemper:


The algorithm is based on Seymour’s decomposition theorem for regular matroids which says that network matrices are “building blocks” for totally unimodular matrices: Let \( D = (V, A) \) be a directed graph, and let \( T = (V, A_0) \) be a directed tree on \( V \). Then \( M \in \mathbb{R}^{A_0 \times A} \) is a network matrix which is componentwise defined by

\[
M_{a_0,(v,w)} = \begin{cases} +1, & \text{if the unique } v \rightarrow w \text{-path in } T \text{ passes } a_0 \text{ forwardly,} \\ -1, & \text{if the unique } v \rightarrow w \text{-path in } T \text{ passes } a_0 \text{ backwardly,} \\ 0, & \text{otherwise.} \end{cases}
\]

Again we refer to the book by Schrijver [S1, Chapter 19.4, Chapter 20] for the exact statement and to the original paper by Seymour to its proof which is about 50 pages long. The result can be roughly stated as follows:

a) Every totally unimodular matrix can be “essentially decomposed” into network matrices.

b) One can solve the problem “Given a matrix \( A \), is it totally unimodular?” in polynomial time.
2. Total dual integrality

Reference: [S1, Chapter 22.1, 22.3]

**Definition 2.1.** Let $A \in \mathbb{Q}^{m \times n}$ be a rational matrix and let $b \in \mathbb{Q}^n$ be a rational vector. The system of linear inequalities $Ax \leq b$ is called totally dual integral if for all $c \in \mathbb{Z}^n$ the dual linear program

$$\min\{y^T b : y^T A = c^T, y \geq 0\}$$

has an integer solution $y \in \mathbb{Z}^n$ whenever the optimum is not $-\infty$.

In Problem 8 we shall see that the total dual integrality is a property of a system $Ax \leq b$ but not a property of the polyhedron $P = \{x : Ax \leq b\}$ alone. It can happen that two systems $Ax \leq b$ and $A'x \leq b'$ define the same polyhedron whereas the first one is TDI but the second one is not.

The following theorem gives a complete characterization for the condition $P = P_I$.

**Theorem 2.2.** The rational polyhedron $P$ is integral if and only if there is a totally dual integral system $Ax \leq b$ with $P = \{x : Ax \leq b\}$ and $b \in \mathbb{Z}^n$.

For a proof see [S1, Corollary 22.6a].

One nice application of total dual integrality is the following inequality description of the matching polytope of an undirected graph $G = (V, E)$, see [S2, Chapter 25]. The matching polytope of $G$ is the convex hull of incidence vectors $x \in \{0, 1\}^E$ of all perfect matchings of $G$. Then, the following systems is totally dual integral and determines the matching polytope:

- i) $x_e \geq 0$, for each $e \in E$,
- ii) $\sum_{e \in E} x_e \leq 1$ for all $v \in V$,
- iii) $\sum_{e \in E[U]} x_e \leq \frac{1}{2}|U|$ for each $U \subseteq V$ where $|U|$ is odd.

3. Problems

**Problem 7. (Totally unimodular matrices)**

a) Is the following matrix totally unimodular?

$$\begin{pmatrix}
1 & -1 & 0 & 0 & -1 \\
-1 & 1 & -1 & 0 & 0 \\
0 & -1 & 1 & -1 & 0 \\
0 & 0 & -1 & 1 & -1 \\
-1 & 0 & 0 & -1 & 1
\end{pmatrix}$$

b) Show that the incidence matrix of an undirected graph $G = (V, E)$ is totally unimodular if and only if $G$ is a bipartite graph.

c) Apply b) to the complete bipartite graph $K_{n,n}$ to derive Birkhoff’s theorem: Every doubly stochastic matrix is a convex combination of permutation matrices.

A matrix $A \in \mathbb{R}^{n \times n}$ is called doubly stochastic when $A \geq 0$ and when the sum of every row and every column equals to one. A permutation matrix is a doubly stochastic matrix which only has integral entries.

**Problem 8. (Totally dual integral systems)**
a) [KV, Exercise 5.8]
Consider the following two systems of linear inequalities
\[
\begin{pmatrix}
1 & 1 \\
1 & 0 \\
1 & -1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\leq
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\leq
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\]
which define the same polyhedron. Show that the first system is totally dual integral whereas the second one is not.

b) [KV, Exercise 5.9]
Let \(a \in \mathbb{Z}^n \setminus \{0\}\) be a nonzero integral vector and let \(b \in \mathbb{Q}\) be a rational number. Show that the inequality \(a^T x \leq b\) is totally dual integral if and only if the entries of \(a\) are relatively prime, i.e \(\gcd(a_1, \ldots, a_n) = 1\).