Arrangements, matroids and codes

third lecture

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1. Extended weight enumerator
2. Formulas for puncturing and shortening
3. Graph theory and colorings
4. Exercises
Extended weight enumerator
Let $G$ be the generator matrix of a linear $[n, k]$ code $C$ over $\mathbb{F}_q$

$\mathbb{F}_q$ is a subfield of $\mathbb{F}_{q^m}$

Consider the code $C \otimes \mathbb{F}_{q^m}$ over $\mathbb{F}_{q^m}$

by taking all $\mathbb{F}_{q^m}$-linear combinations of the codewords in $C$

This is called the extension code of $C$ over $\mathbb{F}_{q^m}$

$G$ is also a generator matrix for the extension code $C \otimes \mathbb{F}_{q^m}$

Hence $C \otimes \mathbb{F}_{q^m}$ has dimension $k$ over $\mathbb{F}_{q^m}$
A formalism for the weight enumerator

Remember:

**Definition**
For a subset $J$ of $[n] := \{1, 2, \ldots, n\}$ define

$$C(J) = \{c \in C : c_j = 0 \text{ for all } j \in J\}$$

$$l(J) = \dim C(J)$$

**Lemma**
Let $C$ be a linear code with generator matrix $G$
Let $J \subseteq [n]$ and $|J| = t$
$G_J$ is the $k \times t$ submatrix of $G$ existing of the columns of $G$ indexed by $J$
Let $r(J)$ be the rank of $G_J$
Then $l(J) = k - r(J)$
Dimension of $C(J)$ under extension

$l(J) = k - r(J)$ by a previous lemma
$r(J)$ is independent of the extension field $\mathbb{F}_{q^m}$

Therefore

$$\dim_{\mathbb{F}_q} C(J) = \dim_{\mathbb{F}_{q^m}} (C \otimes \mathbb{F}_{q^m})(J)$$

This motivates the usage of $T$ as a variable for $q^m$ in the next definition.
Extension of definition $B_t$

Remember:
Let $C$ be a linear code over $\mathbb{F}_q$

$$B_J = q^{l(J)} - 1$$
$$B_t = \sum_{|J|=t} B_J$$

Extend: Definition

$$B_J(T) = T^{l(J)} - 1$$
$$B_t(T) = \sum_{|J|=t} B_J(T)$$

Note that $B_J(q^m)$ is the number of nonzero codewords in $(C \otimes \mathbb{F}_{q^m})(J)$
Proposition on $B_t(T)$

**Proposition**

Let $C$ be an $\mathbb{F}_q$-linear code of dimension $k$

Let $d$ and $d \perp$ be the minimum distance of $C$ and $C \perp$, respectively

Let $J \subseteq [n]$ and $|J| = t$

Then

$$B_t(T) = \begin{cases} 
\binom{n}{t}(T^{k-t} - 1) & \text{for all } t < d \perp \\
0 & \text{for all } t > n - d
\end{cases}$$

**Proof**

Follows directly from the lemma on $l(J)$
Extension of definition $W_C(X, Y)$

Remember:

$$W_C(X, Y) = X^n + \sum_{t=0}^{n} B_t (X - Y)^t Y^{n-t}$$

Define the extended weight enumerator by

$$W_C(X, Y, T) = X^n + \sum_{t=0}^{n} B_t (T)(X - Y)^t Y^{n-t}$$
Relating $A_w(T)$ and $B_t(T)$’s

**Theorem**
The following holds:

$$W_C(X, Y, T) = \sum_{w=0}^{n} A_w(T) X^{n-w} Y^w$$

$$A_0(T) = 1, \text{ and } A_w(T) = \sum_{t=n-w}^{n} (-1)^{n+w+t} \binom{t}{n-w} B_t(T)$$

for $0 < w \leq n$ and

$$B_t(T) = \sum_{w=d}^{n-t} \binom{n-w}{t} A_w(T)$$

**Proof** is similar to the proof relating the $A_w$’s and $B_t$’s
A_w(T) of MDS codes

Proposition
The weight distribution of an MDS code of length \( n \) and dimension \( k \) is given by

\[
A_w(T) = \binom{n}{w}^{w-d} \sum_{j=0}^{w-d} (-1)^j \binom{w}{j} \left( T^{w-d+1-j} - 1 \right)
\]

for \( w \geq d = n - k + 1 \)

Proof
Similar to the proof for \( A_w \)
**Proposition**

Let $C$ be a linear $[n, k]$ code over $\mathbb{F}_q$

Then

$$W_C(X, Y, q^m) = W_{C \otimes \mathbb{F}_{q^m}}(X, Y)$$

The number of codewords in $C \otimes \mathbb{F}_{q^m}$ of weight $w$ is equal to $A_w(q^m)$

**Proof**

Substituting $T = q^m$ in $B_t(T)$ gives $B_t(q^m)$ which is equal to the $B_t$ of $C \otimes \mathbb{F}_{q^m}$
Example

\[ G = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 1
\end{pmatrix} \]
Example, $q$ even

Let $C$ be the code with generator matrix $G$

We have seen for $q$ even that

$$W_C(X, Y) = X^7 + 7(q-1)X^3Y^4 + 7(q-1)(q-2)XY^6 + (q-1)(q-2)(q-4)Y^7$$

So

$$W_{C\otimes \mathbb{F}_{q^m}}(X, Y) =$$

$$X^7 + 7(q^m-1)X^3Y^4 + 7(q^m-1)(q^m-2)XY^6 + (q^m-1)(q^m-2)(q^m-4)Y^7$$

Therefore

$$W_C(X, Y, T) =$$

$$X^7 + 7(T-1)X^3Y^4 + 7(T-1)(T-2)XY^6 + (T-1)(T-2)(T-4)Y^7$$
Example, \( q \) even

Let \( C \) be the code with generator matrix \( G \).

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\]

So

\[
W_C \otimes \mathbb{F}_{q^m} (X, Y) = \\
X^7 + 7(q^m-1)X^3Y^4 + 7(q^m-1)(q^m-2)XY^6 + (q^m-1)(q^m-2)(q^m-4)Y^7
\]

Therefore

\[
W_C(X, Y, T) = \\
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$$W_{C \otimes F_{q^m}}(X, Y) =$$

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Therefore

$$W_C(X, Y, T) =$$

$$X^7 + 7(T - 1)X^3Y^4 + 7(T - 1)(T - 2)XY^6 + (T - 1)(T - 2)(T - 4)Y^7$$
Example, $q$ odd

For $q$ odd we have seen that

$$W_C(X, Y) = X^7 + 6(q - 1)X^3Y^4 +$$

$$3(q - 1)X^2Y^5 + (q - 1)(7q - 17)XY^6 + (q - 1)(q - 3)^2Y^7$$

So

$$W_C \otimes_{\mathbb{F}_q} F_{qm}(X, Y) = X^7 + 6(q^m - 1)X^3Y^4 +$$

$$3(q^m - 1)X^2Y^5 + (q^m - 1)(7q^m - 17)XY^6 + (q^m - 1)(q^m - 3)^2Y^7$$

Therefore

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\]

So

\[
W_{C \otimes \mathbb{F}_{q^m}}(X, Y) = X^7 + 6(q^m - 1)X^3Y^4 + 3(q^m - 1)X^2Y^5 + (q^m - 1)(7q^m - 17)XY^6 + (q^m - 1)(q^m - 3)^2Y^7
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Therefore

\[
W_C(X, Y, T) = X^7 + 6(T - 1)X^3Y^4 + 3(T - 1)X^2Y^5 + (T - 1)(7T - 17)XY^6 + (T - 1)(T - 3)^2Y^7
\]
MacWilliams identity of ext. wt. enum.

**Theorem**
Let $C$ be an $[n, k]$ code over $\mathbb{F}_q$
Then
\[
W_{C^\perp}(X, Y, T) = T^{-k}W_C(X + (q - 1)Y, X - Y, T)
\]

**Proof**
Substituting $T = q^m$ gives the MacWilliams identity for $C \otimes \mathbb{F}_{q^m}$
\[
W_{C^\perp}(X, Y, q^m) = q^{-mk}W_C(X + (q - 1)Y, X - Y, q^m)
\]
which holds for all $m$
Now $A_w(T)$ is a polynomial in $T$ with coefficient in $\mathbb{Z}$
Giving infinitely many identities for the weight distributions of
$C \otimes \mathbb{F}_{q^m}$ and $C^\perp \otimes \mathbb{F}_{q^m} = (C \otimes \mathbb{F}_{q^m})^\perp$
Proposition
Let $C$ be a linear $[n, k]$ code over $\mathbb{F}_q$
The following formula will be useful later in identifying the extended weight enumerator with the Tutte polynomial

$$W_C(X, Y, T) = \sum_{t=0}^{n} \sum_{|J|=t} T^{l(J)} (X - Y)^t Y^{n-t}$$

Proof
Use the description of $W_C(X, Y, T)$ in terms of the $B_t(T)$ and the definition of $B_t(T)$ in terms of the $l(J)$
Puncturing and shortening
There are several ways to get new codes from existing ones.

**Puncturing and shortening** codes give an alternative algorithm for finding the extended weight enumerator.

This algorithm is based on the Tutte-Grothendieck decomposition of matrices introduced by Brylawski.

Greene used this for the determination of the weight enumerator.
Puncturing and shortening

Let $C$ be a linear $[n, k]$ code and let $J \subseteq [n]$

Then the code $C$ punctured by $J$ is obtained by deleting all the coordinates indexed by $J$ from the codewords of $C$. $G_{[n] \setminus J}$ is a generator matrix of this punctured code.

The length of this punctured code is $n - |J|$ and its dimension is at most $k$.

If we puncture the code $C(J)$ by $J$, we get the code $C$ shortened by $J$. The length of this shortened code is $n - |J|$ and its dimension is $l(J)$.

The operations of puncturing and shortening a code are each other's dual. Puncturing a code $C$ by $J$ and then taking the dual gives the same code as shortening $C^\perp$ by $J$. 
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Generalizing definition $l(J)$

Let $\mathbb{F}$ be a field

Let $G$ be a $k \times n$ matrix over $\mathbb{F}$ possibly of rank smaller than $k$ and/or with zero columns

Define for $J \subseteq [n]$: 

$$l(J) = l(J, G) = k - r(G_J).$$

Define the extended weight enumerator of $G$ by

$$W_G(X, Y, T) = \sum_{t=0}^{n} \sum_{|J|=t} T^{l(J, G)} (X - Y)^t Y^{n-t}$$

This coincides with $W_C(X, Y, T)$ if $G$ is a generator matrix of $C$
Proposition

(i) $W_G(X, Y, T)$ is invariant under row-equivalence of matrices

(ii) Let $G'$ be a $l \times n$ matrix with the same row-space as $G$ then $W_G(X, Y, T) = T^{k-l}W_{G'}(X, Y, T)$

(iii) $W_G(X, Y, T)$ is invariant under permutation of the columns of $G$

(iv) $W_G(X, Y, T)$ is invariant under multiplying a column of $G$ with an element of $\mathbb{F}^*$

(v) If $G$ is the direct sum of $G_1$ and $G_2$, that is of the form

\[
\begin{pmatrix}
G_1 & 0 \\
0 & G_2
\end{pmatrix}
\]

then $W_G(X, Y, T) = W_{G_1}(X, Y, T) \cdot W_{G_2}(X, Y, T)$
Reduced matrix

Let $G$ be a $k \times n$ matrix with entries in $\mathbb{F}$
Suppose that the $j$-th column is not the zero vector
Then there exists a matrix $G'$ row-equivalent to $G$
such that the $j$-th column is of the form $(1, 0, \ldots, 0)^T$
Such a matrix is called reduced at the $j$-th column

$$G' = \begin{pmatrix}
1 & g'_{12} & \cdots & g'_{1n} \\
0 & g'_{22} & \cdots & g'_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & g'_{k2} & \cdots & g'_{kn}
\end{pmatrix}$$

In general, this reduction is not unique
Let $G$ be a matrix that is reduced at the $j$-th column $a$

$G \setminus a$ is the $k \times (n - 1)$ matrix with column $a$ removed from $G$

$G/a$ is the $(k - 1) \times (n - 1)$ with first row removed from $G \setminus a$

\[
G = \begin{pmatrix}
1 & & & \\
0 & & & \\
\vdots & & & \\
0 & & & \\
\end{pmatrix}
G \setminus a
\quad \text{and} \quad
G \setminus a = \begin{pmatrix}
g_{12} & \cdots & g_{1n} \\
\hline & & \\
G/a & & \\
\end{pmatrix}
\]

View $G \setminus a$ as $G$ punctured by $a$

and $G/a$ as $G$ shortened by $a$
Proposition

Let $G$ be a $k \times n$ matrix that is reduced at the $j$-th column $a$ Then

$$W_G = (X - Y)W_{G/a} + YW_{G\backslash a}$$

The $(X, Y, T)$ part in $W_G(X, Y, T)$ is omitted for clarity

Proof See notes
Graph theory and colorings
**Definition**

A graph $\Gamma$ is a pair $(V, E)$ where $V$ is a non-empty set and $E$ is a set disjoint from $V$. The elements of $V$ are vertices and members of $E$ are edges.

Edges are **incident** to one or two vertices, the **ends** of the edge.

If $u$ and $v$ are vertices that are incident with an edge, then they are called **neighbors** or **adjacent**.

Suppose that $V' \subseteq V$ and $E' \subseteq E$ and all the endpoints of $e'$ in $E'$ are in $V'$. Then $\Gamma' = (V', E')$ is called a **subgraph** of $\Gamma$. 
**Definition**

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Graph theory

A loop is an edge that is incident with exactly one vertex. Two edges are called parallel if they are incident with the same vertices.

The graph is called simple if it has no loops and no parallel edges.

Deleting loops and parallel edges from a graph gives a simple graph. There is a choice in the process of deleting parallel edges, but the resulting graphs are all isomorphic.

This simple graph is called the simplification $\tilde{\Gamma}$ of the graph $\Gamma$.
Graph colorings

Let $\Gamma = (V, E)$ be a graph
Let $K$ be a finite set and $k = |K|$
The elements of $K$ are called colors

A $k$-coloring of $\Gamma$ is a map $\gamma : V \rightarrow K$
such that $\gamma(u) \neq \gamma(v)$ for all distinct adjacent vertices $u$ and $v$ in $V$

So vertex $u$ has color $\gamma(u)$ and
all other adjacent vertices have a color distinct from $\gamma(u)$

Let $P_\Gamma(k)$ be the number of $k$-colorings of $\Gamma$
Then $P_\Gamma$ is called the chromatic polynomial of $\Gamma$
Example: disconnected graph

If the graph $\Gamma$ has no edges and $v$ vertices then

$$P_\Gamma(k) = k^v$$

since it is equal to the number of all maps from $V$ to $K$

In particular there is no map from $V$ to an empty set in case $V$ is nonempty.

So the number of 0-colorings is zero for every graph.
Let $K_n$ be the **complete graph** on $n$ vertices in which every pair of two distinct vertices is connected by exactly one edge.

Then there is no $k$ coloring if $k < n$

Now let $k \geq n$

Take an enumeration of the vertices

Then there are $k$ possible choices of a color of the first vertex

Now suppose by induction that we have a coloring of the first $i$ vertices then there are $k - i$ possibilities to color the next vertex

since the $(i + 1)$-th vertex is connected to the first $i$ vertices

Hence

$$P_{K_n}(k) = k(k - 1) \cdots (k - n + 1)$$

So $P_{K_n}(k)$ is a polynomial in $k$ of degree $n$
Proposition
Let $\Gamma = (V, E)$ be a graph
Then $P_\Gamma(k)$ is a polynomial in $k$

Proof
Let $\gamma : V \rightarrow K$ be a $k$-coloring of $\Gamma$ with exactly $i$ colors

Let $\sigma$ be a permutation of $K$
Then $\sigma \circ \gamma$ is also a $k$-coloring of $\Gamma$ with exactly $i$ colors

Two such colorings are called equivalent

Then $k(k - 1) \cdots (k - i + 1)$ is the number of colorings in the equivalence class of a given $k$-coloring of $\Gamma$ with exactly $i$ colors
Proposition
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Proof
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Two such colorings are called equivalent

Then $k(k - 1) \cdots (k - i + 1)$ is the number of colorings in the equivalence class of a given $k$-coloring of $\Gamma$ with exactly $i$ colors
Proof: \( P_\Gamma(k) \) is a polynomial

Let \( m_i \) be the number of equivalence classes of colorings with exactly \( i \) colors of the set \( K \).

Let \( v = |V| \).

Then \( P_\Gamma(k) \) is equal to

\[
m_1 k + \ldots + m_i k(k - 1) \cdots (k - i + 1) + \ldots + m_v k(k - 1) \cdots (k - v + 1)
\]
A graph $\Gamma = (V, E)$ is called **bipartite** if $V$ is the disjoint union of two nonempty sets $M$ and $N$ such that the ends of an edge are in $M$ and in $N$

Hence no two points in $M$ are adjacent and no two points in $N$ are adjacent. Let $m$ and $n$ be integers such that $1 \leq m \leq n$

The **complete bipartite graph** $K_{m,n}$ is the graph on a set of vertices $V$ that is the disjoint union of two sets $M$ and $N$ with $|M| = m$ and $|N| = n$ and such that every vertex in $M$ is connected with every vertex in $N$ by a unique edge.
A graph $\Gamma = (V, E)$ is called \textbf{bipartite} if $V$ is the disjoint union of two nonempty sets $M$ and $N$ such that the ends of an edge are in $M$ and in $N$

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The \textbf{complete bipartite graph} $K_{m,n}$ is the graph on a set of vertices $V$ that is the disjoint union of two sets $M$ and $N$ with $|M| = m$ and $|N| = n$ and such that every vertex in $M$ is connected with every vertex in $N$ by a unique edge
Let $\Gamma = (V, E)$ be a graph
Let $e$ be an edge that is incident to the vertices $u$ and $v$

Then the deletion $\Gamma \setminus e$ is the graph with vertices $V$ and edges $E \setminus \{e\}$

The contraction $\Gamma / e$ is the graph obtained by identifying $u$ and $v$ and deleting $e$

Notice that the number of $k$-colorings of $\Gamma$ does not change by deleting loops and a parallel edge
Hence the chromatic polynomial of $\Gamma$ and $\tilde{\Gamma}$ are the same
Deletion and contraction

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Deletion and contraction

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Notice that the number of $k$-colorings of $\Gamma$ does not change
by deleting loops and a parallel edge
Hence the chromatic polynomial of $\Gamma$ and $\tilde{\Gamma}$ are the same
Proposition
Let $\Gamma = (V, E)$ be a simple graph
Let $e$ be an edge of $\Gamma$
Then
$$P_\Gamma(k) = P_{\Gamma \setminus e}(k) - P_{\Gamma/e}(k)$$
for all positive integers $k$

Proof
Define $\Gamma/e$ formally as follows
Let $\tilde{u} = \tilde{v} = \{u, v\}$, and $\tilde{w} = \{w\}$ if $w \neq u$ and $w \neq v$
Let $\tilde{V} = \{\tilde{w} : w \in V\}$
Then $\Gamma/e$ is the graph $(\tilde{V}, E \setminus \{e\})$
where an edge $f \neq e$ is incident with $\tilde{w}$ in $\Gamma/e$
if $f$ is incident with $w$ in $\Gamma$
Proof of deletion-contraction formula

Let $u$ and $v$ be the vertices of $e$
Then $u \neq v$, since the graph is simple

Let $\gamma$ be a $k$-coloring of $\Gamma \setminus e$
Then $\gamma$ is also a coloring of $\Gamma$ if and only if $\gamma(u) \neq \gamma(v)$

If $\gamma(u) = \gamma(v)$, then consider the induced map $\tilde{\gamma}$ on $\tilde{V}$
defined by $\tilde{\gamma}(\tilde{u}) = \gamma(u)$ and $\tilde{\gamma}(\tilde{w}) = \gamma(w)$ if $w \neq u$ and $w \neq v$
The map $\tilde{\gamma}$ gives a $k$-coloring of $\Gamma/e$

Conversely, every $k$-coloring of $\Gamma/e$ gives a $k$-coloring $\gamma$
of $\Gamma \setminus e$ such that $\gamma(u) = \gamma(v)$
Therefore

$$P_{\Gamma \setminus e}(k) = P_{\Gamma}(k) + P_{\Gamma/e}(k)$$
Exercises
Exercises third lecture

1. Compute the extended weight enumerator of the binary simplex code $S_3(2)$ and its dual Hamming code $H_3(2)$
2. Compute the extended weight enumerators of the ternary simplex code $S_3(3)$ and its dual the ternary Hamming code $H_3(3)$
3. Compute the extended weight enumerators of the $n$-fold repetition code and its dual
4. Compute the extended weight enumerators of all codes of length at most 5 using the puncturing-shortening formula
5. Give the complexity of the computation of the extended weight enumerator of code by means of the puncturing-shortening formula as a function of the length $n$ and dimension $k$ of the code
6. Compute the chromatic polynomial of $K_{3,3}$
7. Compute the chromatic polynomials of all simple graphs on at most 4 points by using the deletion-contraction formula